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ON STATISTICAL BOUNDEDNESS IN PARTIAL METRIC SPACES

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Abstract: Here in this paper, we have an idea of statistical boundedness for a partial metric space (X, φ) where φ is partial metric on X and establish the relationship between statistical convergence and statistical boundedness. Besides this, statistical analogue of monotone convergence theorem of reals has been established.

Keywords and Phrases: Statistical convergence, partial metric space, statistical boundedness.

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1. Introduction

Although the credit of introducing the notion of statistical convergence was given independently to Fast [7] and Steinhaus [27], but the initial idea of this notion, i.e., "almost convergence" was given by Zygmund [31] in 1935 in his book "Trignometric Series". After these studies, Schoenberg [26] studied this concept as a summability method. Presently, this field has become a main choice of many researchers. The concept of statistical convergence has been extended for arbitrary metric space and this has provided a general framework for summability in metric spaces. In order to get a deep insight into statistical convergence, one may refer to [3-7, 9-11, 13-14, 16-17, 22-24, 28-30] where many more references can be found. The reader may also refer to the recent textbooks [1] and [18] for functional analysis, summability theory, sequence spaces and related topics.

S. Matthews [15] in 1994, originally introduced the notion of partial metric which is a generalized version of metric. In this generalization, each object does not necessarily to have zero distance from itself, i.e., $\varphi(u,u)$ may be non zero. An open ball/open sphere for a partial metric space (X,φ) defined by Matthews [15] as $B^{\varphi}_{\varepsilon}(u) = \{v \in X : \varphi(u,v) < \varepsilon\}$ for $\varepsilon > 0$ and $u \in X$. Unlike metric spaces, some open balls in partial metric space may be empty. For example, if $\varphi(u,u) > 0$, then $B^{\varphi}_{\varphi(u,u)}(u) = \emptyset$.

Simon O. Neil [19] stepped into partial metric space from a new perspective and allow to take negative values for the partial metric function. He slightly modified the definition of open ball to make it (always) non-empty. For any $u \in X$ and $\varepsilon \in \mathbb{R}$ $(\varepsilon > 0)$, he defined open ball as $B^{\varphi}_{\varepsilon}(u) = \{v \in X : \varphi(u,v) < \varphi(u,u) + \varepsilon\}$. Simon studied (X,φ) not as set X with a partial metric function φ, but (using the metric d induced on X, by the partial metric φ of X) as a bi-topological space $(X, T[\varphi], T[d])$ where $T[\varphi]$ and $T[d]$ are partial metric topology and metric topology respectively. Simon admitted that though it is difficult to accept that bi-topological approach is correct context to view partial metric space and we certainly far away from generalizations. In his work, he obtained very surprisingly and new results.

Many more researchers have explored the above concept to enrich the theory of sequence spaces, for instance one may refer to [2, 12, 15, 19, 21].

Before proceeding further, we recall some definitions and notations, which will be frequently used throughout the paper.

Definition 1.1. ([20]) *The natural density $\delta(A)$ of $A \subset \mathbb{N}$ is defined as*

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} card(\{m \in A : m \leq n\}),$$

provided the limit exists, and $\delta(A) = 0$ for finite A. Also $\delta(\mathbb{N} - A) = 1 - \delta(A)$.

Definition 1.2. ([8]) *A property P for (u_m) is said to be hold good for almost all m written as a.a. m. if the set of indices m, where property fails to hold for u_m has natural density 0, i.e., if $\delta(\{m \in \mathbb{N} : u_m$ does not satisfy property $P\}) = 0$, then we say property P hold for a.a. m.*

Definition 1.3. ([25]) *A sequence (u_m) is statistically convergent to $u_0 \in \mathbb{R}$ if for*

$\varepsilon > 0$,

$$\delta(\{m \in \mathbb{N} : |u_m - u_0| \geq \varepsilon\}) = 0, \ i.e., |u_m - u_0| < \varepsilon \ a.a. \ m.,$$

i.e., $\lim_{n\to\infty} \frac{1}{n} card(\{m \leq n : |u_m - u_0| \geq \varepsilon\}) = 0$. *And* u_0 *is referred as statistical limit of* (u_m). *We write* $u_m \to u_0(S)$ *and by* $S(c)$ *we denote the set of all statistically convergent real sequences.*

Theorem 1.4. ([25]) *A real sequence* (u_m) *is statistically convergent to* $u_0 \in \mathbb{R}$ *iff there exists* $T = \{m_1 < m_2 < ... < m_n...\} \subset \mathbb{N}$ *such that* $\delta(T) = 1$ *and* $\lim_{n\to\infty} u_{m_n} = u_0$.

Fridy and Orhan [10], added the idea of statistical boundedness to the theory of statistical convergence as follow:

Definition 1.5. ([10]) *A sequence* (u_m) *is statistically bounded if* $\delta(\{m \in \mathbb{N} : |u_m| > M\}) = 0$, *for some* $M > 0$, *i.e.,* (u_m) *is bounded for a.a. m.*

Theorem 1.6. ([3]) *A sequence* (u_m) *is statistically bounded if and only if there exists* $T = \{m_1 < m_2 < ... < m_n...\} \subset \mathbb{N}$ *such that* $\delta(T) = 1$ *and* $\{u_{m_n}\}_{n\in\mathbb{N}}$ *is bounded.*

Definition 1.7. ([17]) *Let* $X \neq \emptyset$. *A function* $\varphi : X \times X \to \mathbb{R}$ *satisfying the following*

(φ_1) $0 \leq \varphi(u, u) \leq \varphi(u, v)$

(φ_2) $\varphi(u, u) = \varphi(u, v) = \varphi(v, v) \iff u = v$

(φ_3) $\varphi(u, v) = \varphi(v, u)$

(φ_4) $\varphi(u, v) \leq \varphi(u, w) + \varphi(w, v) - \varphi(w, w)$ *for all* $u, v, w \in X$, *is said to be a partial metric on* X *and* (X, φ) *is called a partial metric space*

The following points are worth observing for a partial metric space (X, φ):

(1) $\varphi(u, u)$ is not necessarily zero for $u \in X$.

(2) Every metric is partial metric, but a partial metric need not be a metric (because for a metric d, it is necessary that $d(u, u) = 0$).

Definition 1.8. (Subspace) *Let* (X, φ) *and* (Y, φ') *be two partial metric spaces. We say* (Y, φ') *is partial metric subspace of* (X, φ) *if*

(i) $Y \subset X$

(ii) φ' *is a restriction of* φ *from* X *to* Y*, i.e.,* $\varphi'(u,v) = \varphi(u,v)$ *for* $u, v \in Y$*.*

In the present paper, we are going to explore partial metric space (X, φ) with reference to only its partial metric φ (without using bi-topological approach) without rely upon induced metric d. Here we are considering the sequences from a partial metric space (X, φ) where X is an arbitrary set. Whenever we use or turn to statistical concepts for real line, we use them by quoting in **usual sense**. In this work, we are following the approach of Matthews [15], i.e., considering $\varphi(u,u) \geq 0$ for all $u \in X$. An open sphere centred on $u \in X$ and radius $\varepsilon(> 0)$ is taken as $B^{\varphi}_{\varepsilon}(u) = \{v \in X : \varphi(v,u) < \varphi(u,u) + \varepsilon\}$.

Throughout the paper, by (X, φ) we mean a partial metric space with abbreviation *p.m.s.*

2. Statistical convergence and statistical boundedness in (X, φ)

In this section, we have the concept of boundedness, statistical boundedness with reference to *p.m.s.* and establish the relation between boundedness, convergence, statistical convergence and statistical boundedness. Apart this, it is investigated that boundedness and statistical boundedness are the same thing for bounded partial metric space.

Definition 2.1. *A sequence* (u_m) *is said to be bounded if there exists* $u \in X$ *such that*

$$u_m \in B^{\varphi}_{M}(u), \text{ i.e.}, \varphi(u_m, u) < \varphi(u,u) + M \text{ for all } m \geq 1 \text{ and for some } M > 0.$$

Definition 2.2. *A sequence* (u_m)*, in p.m.s.* (X, φ) *is said to be convergent to* $u \in X$ *if for every* $\varepsilon > 0$*, there exists a positive integer* N *such that* $u_m \in B^{\varphi}_{u}(\varepsilon)$ *for all* $m \geq N$*, i.e.,* $\varphi(u_m, u) < \varphi(u,u) + \varepsilon$ *for all* $m \geq N$*, i.e.,* $|\varphi(u_m, u) - \varphi(u,u)| < \varepsilon$ *for all* $m \geq N$*. In other words,* $\lim_{m\to\infty} \varphi(u_m, u) = \varphi(u,u)$*.*

It is to be noted, in view of axiom (φ_1) of *p.m.s.*, $|\varphi(u_m, u) - \varphi(u,u)|$ and $\varphi(u_m, u) - \varphi(u,u)$ are the same thing.

Definition 2.3. ([21]) *A sequence* (u_m) *in p.m.s.* (X, φ) *is statistically convergent to some* $u \in X$ *if for* $\varepsilon > 0$*,*

$$\delta(\{m \in \mathbb{N} : |\varphi(u_m, u) - \varphi(u,u)| \geq \epsilon\}) = 0.$$

Definition 2.4. *Let* (X, φ) *be a p.m.s. and* (u_m) *be a sequence in* X*. We say* (u_m) *is statistically bounded if there exist some* $u \in X$ *and* $M > 0$ *such that*

$$\delta(\{m \in \mathbb{N} : |\varphi(u_m, u) - \varphi(u,u)| \geq M\}) = 0.$$

Definition 2.5. *A p.m.s. X is said to be bounded if $d(X) < \infty$, that is, diameter of X is finite. In other words, X is bounded if*

$$\sup_{u,v\in X}\{\varphi(u,v)-\varphi(u,u)\}<\infty, \text{ i.e., } \sup_{u,v\in X}|\varphi(u,v)-\varphi(u,u)|<\infty.$$

Theorem 2.6. *In a p.m.s. (X, φ), every convergent sequence is bounded.*
Proof. The proof is a routine verification and hence left for reader. However, converse need not be true.

For this, let $X = \mathbb{R}$ with partial metric φ on X, as $\varphi(u, v) = |u - v|$; $u, v \in X$. Consider (z_m) as

$$z_m = \begin{cases} a & \text{if m is even} \\ 2a & \text{otherwise} \end{cases} \qquad m \in \mathbb{N} \text{ where } a \in \mathbb{R}(a > 0) \text{ is fixed .}$$

For any $u \in R$, let $M = 2a + |u|$. This yields $\varphi(z_m, u) \leq \varphi(u, u) + M$ for all $m \geq 1$. Hence (z_m) is bounded. It is easy to see $z_m \nrightarrow a, 2a$ as $m \to \infty$ (one may verify by choosing ε, $0 < \varepsilon < a$).

Theorem 2.7. *In a p.m.s. (X, φ), every bounded sequence is statistically bounded. Converse may not be true, in general.*
Proof. The proof follows in the light of null natural density of empty set.

In order to show that the converse part is not true in general, we discuss the following:

Example 1. let us take $X = [0, \infty)$ and partial metric as $\varphi(u, v) = \max\{u, v\}$; $u, v \in [0, \infty)$. Consider a sequence (u_m) defined as

$$u_m = \begin{cases} m & \text{if m is a perfect square} \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $(u_m) = (1, 0, 0, 4, 0, 0, 0, 0, 9, 0, \ldots)$. Let, if possible, (u_m) is bounded. Then there exist some $u \in X$ and choose $M \geq 2$ such that $p(u_m, u) < \varphi(u, u) + M$ for all $m \geq 1$, i.e., $\max\{u_m, u\} < u + M$ $\forall\, m \geq 1$. Let $t = [u + M]$, where $[\cdot]$ denotes the greatest integer function. Clearly t is an integer ≥ 1. Setting $p = t^2$. Now p is a perfect square and so we have, $\varphi(u_p, u) < u + M$, i.e., $\max\{u_p, u\} < u + M$. Thus $p < u + M$, i.e., $t^2 < t$, a contradiction as $t \geq 1$.

In order to see (u_m) is statistically bounded, let $u \in X$ arbitrary but fixed.

$$\begin{aligned}\{m \in \mathbb{N} : \varphi(u_m, u) - \varphi(u, u) > M\} &= \{m \in \mathbb{N} : \varphi(u_m, u) - u > M\} \\ &= \{m \in \mathbb{N} : \max\{u_m, u\} - u > M\} \\ &\subset \{1, 4, 9, ...\}.\end{aligned}$$

Since set of squares has null natural density, so $\delta(\{m \in \mathbb{N} : \varphi(u_m, u) - \varphi(u, u) > M\}) = 0$. Thus (u_m) is statistically bounded.

Theorem 2.8. *In a p.m.s. (X, φ), statistically convergence implies statistically boundedness. Converse may not hold, in general.*

Proof. Let (u_m) be statistically convergent to $u \in X$. Then for $\varepsilon > 0$, we have $\delta(\{m \in \mathbb{N} : \varphi(u_m, u) - \varphi(u, u) > \varepsilon\}) = 0$. The result follows by taking sufficiently large $M(> 0)$ and in view of fact that

$$\{m \in \mathbb{N} : \varphi(u_m, u) > \varphi(u, u) + M\} \subseteq \{m \in \mathbb{N} : \varphi(u_m, u) > \varphi(u, u) + \varepsilon\}$$

For reverse implication, we consider the example $(z_m) = \{2a, a, 2a, a, \ldots\}$ of Theorem 2.6. Since (z_m) is bounded, so it is statistically bounded (in view of Theorem 2.7). However (z_m) is not statistically convergent to a (or $2a$) as set of odd (even) natural numbers has non-zero natural density.

Theorem 2.9. *If (u_m) is a sequence in X converging to $u \in X$, then (u_m) is statistically convergent to u.*

Proof. Since every finite subset of $\mathbb{N}$ has null natural density, the proof is trivial.

Converse of above may not be true, i.e., there are statistical convergent sequence in (X, φ), which are not convergent.

For this, consider the sequence (u_m) of example cited in Theorem 2.7. Suppose, if possible, (u_m) is convergent to some $u \in X$, ($u \in X$ arbitrary but fixed). Then for $\varepsilon > 0$, there exists positive integer N such that $\varphi(u_m, u) < \varphi(u, u) + \varepsilon$ for all $m \geq N$, i.e., $\max\{u_m, u\} < u + \varepsilon$ for all $m \geq N$. For sufficiently large $m(\geq N)$ and to be perfect square, we have $\max\{m, u\} < u + \varepsilon$, a contradiction, as $u \in X$ is fixed. This implies that (u_m) is not convergent.

However, (u_m) is statistically convergent to 0, because

$$\begin{aligned}\{m \in \mathbb{N} : |\varphi(u_m, 0) - \varphi(0, 0)| > \varepsilon\} &= \{m \in \mathbb{N} : \max\{u_m, 0\} > \varepsilon\} \\ &\subseteq \{1, 4, 9, \ldots\}.\end{aligned}$$

Theorem 2.10. *Let (X, φ) be a p.m.s. with $X \neq \emptyset$. Then X is bounded iff the set of all bounded sequences coincides with the set of statistical bounded sequences.*

Proof. Let (X, φ) be a bounded *p.m.s.* and (u_m) be any sequence in X. Let $M = sup_{u,v \in X}\{\varphi(u, v) - \varphi(u, u)\}$. Then for $u \in X$ arbitrary, but fixed we have $\varphi(u, v) - \varphi(u, u) < M$ for all $v \in X$. In particular, $\varphi(u_m, u) - \varphi(u, u) < M$ for all $m \geq 1$ and hence (u_m) is bounded, i.e, every sequence in (X, φ) is bounded. In particular, every statistically bounded sequence is bounded. The other part holds in the light of Theorem 2.7.

Conversely, let if possible, X is not bounded. Then

$$sup_{v\in X}\{\varphi(u_0,v)-\varphi(u_0,u_0)\}=\infty, \text{ for some } u_0\in X.$$

This implies that for each $n\in\mathbb{N}$, there exists $(u_n)\in X$ such that $\varphi(u_0,u_n)$ $-\varphi(u_0,u_0)>n$, i.e., $\varphi(u_0,u_n)>n+\varphi(u_0,u_0)$ for $n\in\mathbb{N}$. Take

$$v_n=\begin{cases} u_n & \text{if n is a perfect square} \\ u_0 & \text{if n is not a perfect square,}\end{cases}$$

i.e., $\langle v_n\rangle=\{u_1,u_0,u_0,u_4,u_0,u_0,u_0,u_0,u_9,\ldots\}$. Clearly $\langle v_n\rangle$ is statistically bounded. But $\langle v_n\rangle$ is unbounded, since for every n (to be perfect square) $\varphi(v_n,u_0)=\varphi(u_0,v_n)=\varphi(u_0,u_n)>n+\varphi(u_0,u_0)$,
i.e., $\varphi(v_n,u_0)\nleq M+\varphi(u_0,u_0)$ for every n (to be perfect square) and for finite M, i.e., $\varphi(v_n,u_0)\nless M+\varphi(u_0,u_0)$ for $M>0$. Thus $\langle v_n\rangle$ is a statistically bounded sequence in *p.m.s.* (X,φ) which is unbounded, a contradiction.

Theorem 2.11. *Let (u_m) be a statistically bounded sequence in a p.m.s. (X,φ). Then there exists a bounded sequence (v_m) such that $v_m=u_m$ a.a. m.*

Proof. As (u_m) is statistically bounded, so there exist some $u\in X$ and $M>0$ such that $\delta(A)=0$, where $A=\{m\in\mathbb{N}:\varphi(u_m,u)-\varphi(u,u)>M\}$. Take

$$v_m=\begin{cases} u_m & \text{for } m\in\mathbb{N}-A \\ u & \text{for } m\in A.\end{cases}$$

Clearly $\{m\in\mathbb{N}:v_m\neq u_m\}\subseteq A$ and so $v_m=u_m$ a.a. m. It is easy to verify that $\varphi(v_m,u)-\varphi(u,u)\leq M$ for all $m\geq 1$.

Theorem 2.12. *Let $(Y,\varphi^{'})$ be a partial metric subspace of (X,φ).*

(i) If (v_m) is a sequence in Y and is statistical bounded w.r.t. partial metric $\varphi^{'}$ of Y, then (v_m) is statistical bounded w.r.t. partial metric φ of X.

(ii) If (u_m) is a sequence in Y and is statistical bounded w.r.t. partial metric φ of X, then (u_m) is statistical bounded w.r.t. partial metric $\varphi^{'}$ of Y.

3. Statistical convergence and statistical boundedness via dense subsequences

In this section, we explore the idea of statistical boundedness and statistical convergence of sequences in term of statistically bounded dense and statistically convergent dense subsequences respectively. Besides this, it is observed that in a *p.m.s.*, a statistically bounded sequence has a statistically dense, bounded subsequence.

Definition 3.1. *A subset T of $\mathbb{N}$ is statistically dense if $\delta(T) = 1$.*

Definition 3.2. *A subsequence $\{u_{m_n}\}$ of (u_m) is statistically dense if the set of all indices $(m_n)_{n\in\mathbb{N}}$ is statistically dense.*

Remark 3.3. *In ordinary convergence, every subsequence of a convergent sequence is convergent. The similar is not true in case of statistical convergences, i.e., subsequence of a statistically convergent sequence need not be statistically convergent.*

For this, consider a partial metric φ on $X = [0, \infty)$ defined as $\varphi(u, v) = \max\{u, v\}$. Take $(u_m) = \{1, 0, 0, 4, 0, 0, 0, 0, 9, \ldots\}$. Then (u_m) is statistically convergent sequence having a subsequence $\langle 1, 4, 9, 16, \ldots\rangle$ which is not statistically convergent.

The following theorem is a characterization of the statistical convergence in term of its statistical dense subsequences.

Theorem 3.4. *A sequence (u_m) is statistically convergent iff every statistically dense subsequence of (u_m) is statistically convergent.*

Proof. Let (u_m) is statistical convergent to $u \in X$ and having a statistically dense subsequence $\{u_{m_n}\}$ which is statistically divergent. Then for any $u \in X$, there is some $\varepsilon > 0$ such that

$$\liminf_n \frac{1}{n} card(\{m_n \in \mathbb{N}, m_n \le n : \varphi(u_{m_n}, u) > \varphi(u, u) + \varepsilon\}) = \lambda, \text{ where } \lambda \in (0, 1).$$

Since $\{u_{m_n}\}$ is subsequence of (u_m), we have

$$\{m \in \mathbb{N}, m \le n : \varphi(u_m, u) > \varphi(u, u) + \varepsilon\} \supseteq \{m_n \in \mathbb{N}, m_n \le n : \varphi(u_{m_n}, u) > \varphi(u, u) + \varepsilon\}.$$

This in turn implies, $\lim_{n\to\infty} \frac{1}{n} card(\{m \in \mathbb{N}, m \le n : \varphi(u_m, u) > \varphi(u, u) + \varepsilon\}) \ne 0$, a contradiction to the statistical convergence of (u_m).

Converse part follows easily as every sequence is a statistically dense subsequence of itself.

Remark 3.3 can be viewed in the light of the following:

Corollary 3.5. *A statistical dense subsequence of a statistically convergent sequence is statistically convergent.*

Theorem 3.6. *A sequence (u_m) in p.m.s. (X, φ) is statistically convergent to some $u \in X$ iff (u_m) has a statistically dense subsequence $\{u_{m_n}\}$ converging to u.*

Proof. Let (u_m) is statistically convergent to u. Then for each $\varepsilon > 0$, we have $\delta(\{m \in \mathbb{N} : |v_m - 0| > \varepsilon\}) = 0$ where $v_m = \varphi(u_m, u) - \varphi(u, u)$. Thus, we get (v_m) as a statistically convergent sequence of reals **(in usual sense)** converging statistically to 0. Using Theorem 1.4, there exists statistically dense subset $B = \{m_1 < m_2 < ... < m_n ...\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} v_{m_n} = 0$, i.e., $\lim_{n\to\infty} \varphi(u_{m_n}, u) = \varphi(u, u)$.

Converse part follows, from the inclusion $\{m \in \mathbb{N} : \varphi(u_m, u) > \varphi(u, u) + \varepsilon\} \subseteq (\mathbb{N} - B) \cup T$, where T is a finite subset of $\mathbb{N}$.

Theorem 3.7. *A sequence (u_m) in p.m.s. (X, φ) is statistically bounded if and only if there exists a set $T = \{m_1 < m_2 < ... < m_n ...\} \subset \mathbb{N}$ such that $\delta(T) = 1$ and $\{u_{m_n}\}_{n\in\mathbb{N}}$ is a bounded sequence.*

Proof. Let (u_m) be a statistically bounded sequence. Then for some $u \in X$ and $M > 0$ we have $\delta(\{m \in \mathbb{N} : |\varphi(u_m, u) - \varphi(u, u)| > M\}) = 0$. Let $v_m = \varphi(u_m, u) - \varphi(u, u)$ $(\in \mathbb{R})$. Then $\delta(\{m \in \mathbb{N} : |v_m| > M\}) = 0$. Thus (v_m) is a statistically bounded sequence of reals **(in usual sense)**. Hence by Theorem 1.6, there exists $T = \{m_1 < m_2 < \ldots < m_n ...\} \subset \mathbb{N}$ such that $\delta(T) = 1$ and $\langle v_{m_n}\rangle_{n\in\mathbb{N}}$ is a bounded sequence of reals. Then $|v_{m_n}| \le L$, i.e., $\varphi(u_{m_n}, u) - \varphi(u, u) < L$ for all $n \ge 1$ and for some $L > 0$. Thus $\{u_{m_n}\}$ is bounded sequence in *p.m.s.* (X, φ).

Conversely, since $\{u_{m_n}\}_{n\in\mathbb{N}}$ is a bounded sequence in (X, φ) so $\varphi(u_{m_n}, u) \le \varphi(u, u) + M$ for all $n \ge 1$ where $u \in X$ and $M > 0$. The result now follows easily in view of the inclusion

$$\{m \in \mathbb{N} : \varphi(u_m, u) > \varphi(u, u) + M\} \subseteq (\mathbb{N} - T).$$

Corollary 3.8. *A monotone and statistically bounded sequence of reals is statistically convergent.*

Proof. The proof is an easy consequence of Theorem 3.7 and Theorem 3.6.

Finally, we state the following results having the proof on similar lines of Remark 3.3, Theorem 3.4 and Corollary 3.5.

Remark 3.9. *In ordinary boundedness, every subsequence of a bounded sequence is bounded. The similar is not true in case of statistical boundedness, i.e., subsequence of a statistically bounded sequence may not be statistically bounded.*

Theorem 3.10. *A sequence (u_m) is statistically bounded iff every statistically dense subsequence of (u_m) is statistically bounded.*

Corollary 3.11. *A statistical dense subsequence of a statistically bounded sequence is statistically bounded.*

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