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GENERALIZED THREE CLASSES OF MIXED TYPE DOUBLE BERNOULLI- GEGENBAUER-GOULD AND HOPPER POLYNOMIALS

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Abstract: In this article, we investigate a three classes of generalized mixed type double Bernoulli-Gegenbauer-Gould and Hopper (BGG-H) polynomials. Some special polynomials of the generalized mixed type Bernoulli-Gegenbauer polynomials are discussed to obtain certain results and relations of our double (BGG-H) polynomials in terms of known and unknown functions. Some inequalities and limiting cases of double (BGG-H) polynomials are presented and then on using them we construct a matrix representation and obtain integral estimates.

Keywords and Phrases: Bernoulli's polynomials, Gegenbauer polynomials, Gould and Hopper's polynomials, double Bernoulli-Gegenbauer-Gould and Hopper polynomials, matrix representations, integral estimates.

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1. Introduction, some special functions and their values

Throughout this investigation, we consider two variables analogue of Gould and Hopper polynomials containing seven parameters given by $m_1, m_2 \in \mathbb{Z}_+$, (a set of positive integers); $h_1, h_2, \gamma_1, \gamma_2, p \in \mathbb{R}\backslash\{0\}$, (a set of real numbers except set of zeros) or belong to $\mathbb{C}\backslash\{0\}$, (a set of complex numbers except set of zeros).

A generalized Hermite polynomial of two variables is defined by the following generating function

$$e^{(h_1t^{m_1}+h_2T^{m_2})}\left(1+\gamma_1xt+\gamma_2yT\right)^p=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}H_{n,k}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y)\frac{t^n}{n!}\frac{T^k}{k!}. \quad (1.1)$$

Here all the parameters are independent of the variables x and y such that $|x| \le 1, |y| \le 1$, and $|\ \gamma_1 t+\ \gamma_2 T\ | < 1$.

Also in the formula (1.1), $\forall n \ge 0, k \ge 0; n, k \in \mathbb{N}\cup\{0\}, \mathbb{N}$ (a set of natural numbers) and the double series representation is found as

$$H_{n,k}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y) = \sum_{r=0}^{\left[\frac{n}{m_1}\right]}\sum_{s=0}^{\left[\frac{k}{m_2}\right]}(-p)_{n-m_1r+k-m_2s}\frac{n!k!\,(h_1)^r\,(-\gamma_1x)^{n-m_1r}\,(h_2)^s\,(-\gamma_2y)^{k-m_2s}}{r!\,(n-m_1r)!s!\,(k-m_2s)!} \quad (1.2)$$

where $[x]$ be the step function defined as $\forall x \in \mathbb{R}$, and $[x]$ means the greatest integer $\leqq x$, that is, $n \leqq x < n+1 \forall n \in \mathbb{Z}, \mathbb{Z}$ (a set of integers).

It is remarked that on putting $\gamma_1 = \gamma_2 = \gamma, m_1 = m_2 = m$ and $h_1 = h_2 = h$ in (1.1) - (1.2), we get a formula of Chandel, Agrawal and Kumar [3], having four parameters, (γ, m, h, p), studied by them in 1992.

Again the generalization of Hermite polynomials in the form of Gould and Hopper polynomials [5] are defined by

$$g_n^m(x,h) = \sum_{r=0}^{\left[\frac{n}{m}\right]}\frac{n!(h)^r(x)^{n-mr}}{r!(n-mr)!} \quad (1.3)$$

with the generating function

$$\sum_{n=0}^{\infty}g_n^m(x,h)\frac{t^n}{n!} = e^{xt+ht^m} \quad (1.4)$$

(see also in [14, pp. 76 and 86], respectively).

Clearly, in formula (1.2) on replacing x by $\frac{x}{p}$ and y by $\frac{y}{p}$ and again then taking $p \to \infty$ and then making an appeal to the formulae (1.3) and (1.4) we get a bi-product of the Gould and Hopper polynomials in the form

$$\lim_{p\to\infty} H_{n,k}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}\left(\frac{x}{p},\frac{y}{p}\right) = \sum_{r=0}^{\left[\frac{n}{m_1}\right]} \frac{n!\,(h_1)^r\,(\gamma_1 x)^{n-m_1 r}}{r!\,(n-m_1 r)!} \sum_{s=0}^{\left[\frac{k}{m_2}\right]} \frac{k!\,(h_2)^s\,(\gamma_2 y)^{k-m_2 s}}{s!\,(k-m_2 s)!} \tag{1.5}$$

$$= g_n^{m_1}(\gamma_1 x, h_1)\, g_k^{m_2}(\gamma_2 y, h_2).$$

On the other hand, Pathan and Khan [11] introduced a generalized Hermite polynomials of two variables in the form of a generating function as

$$e^{p(x+y)t-(xy+1)t^m} = \sum_{n=0}^{\infty} H_{n,m,p}(x,y)\frac{t^n}{n!},\ |t|<1 \tag{1.6}$$

where,

$$H_{n,m,p}(x,y) = \sum_{k=0}^{\left[\frac{n}{m}\right]} n!\frac{(1+xy)^k(-1)^k}{k!}\frac{p^{n-mk}(x+y)^{n-mk}}{(n-mk)!}.$$

It is noted that after some manipulations, some of the relations for the polynomials in (1.6) are found in terms of the polynomials given in (1.3)-(1.4).

For example, if in the formula (1.6) put $x = \frac{X}{2} \mp \frac{\sqrt{X^2+4(h+1)}}{2}$ and $y = \frac{X}{2} \pm \frac{\sqrt{X^2+4(h+1)}}{2}$, we get an identical results to (1.3) and (1.4) as

$$H_{n,m,p}\left(\frac{X}{2} \mp \frac{\sqrt{X^2+4(h+1)}}{2}, \frac{X}{2} \pm \frac{\sqrt{X^2+4(h+1)}}{2}\right) = \sum_{k=0}^{\left[\frac{n}{m}\right]} n!\frac{(h)^k}{k!}\frac{(pX)^{n-mk}}{(n-mk)!}$$

$$= g_n^m(pX,h). \tag{1.7}$$

Further, in the formula (1.7) set $m=2, h=-1, p=2$, a relation with Hermite polynomials, $H_n(.), n \geq 0$, is found by [14, p.76]

$$H_{n,2,2}(0,X) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!}\frac{(2X)^{n-2k}}{(n-2k)!} = g_n^2(2X,-1) = H_n(X). \tag{1.8}$$

Recently, Quintana [12] presented and analyzed a generalized mixed type two classes Bernoulli-Gegenbauer polynomials by defining the following generating functions

$$\left[\frac{t}{(e^t-1)\left(1-\frac{xt}{\pi}+\frac{t^2}{4\pi^2}\right)}\right]^{\alpha} e^{xt} = \sum_{n=0}^{\infty} v_n^{(\alpha)}(x)\frac{t^n}{n!} \tag{1.9}$$

where $|t| < 2\pi, |x| \leq 1$ and $\alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0\}$.

Again to analyze more results in this field , from the formula (1.9) for $\alpha = 0$ and $\alpha = 1$ we present some relations and formulae by different methods to Quintana [12] in the following way:

Therefore on setting $\alpha = 0$ in the formula (1.9) and get

$$\sum_{n=0}^{\infty} \quad v_n^{(0)}(x)\frac{t^n}{n!} = e^{xt}. \tag{1.10}$$

Now in the result (1.10) use the formula (see, Rainville [13, p.168]) given by

$$e^{xt} J_0 \left(t \left(1 - x^2\right)^{1/2}\right) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} \tag{1.11}$$

where the η order Bessel function is defined by the series
$J_\eta(z) = \left(\frac{z}{2}\right)^\eta \sum_{k=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{k!\Gamma(\eta+k+1)}, \eta$ is not a negative integer, $z \in \mathbb{C}$.

Also in the generating function (1.11), the Legendre polynomials of degree $n, P_n(x) \forall n \geq 0$, is defined by $\sum_{n=0}^{\infty} P_n(x)t^n = (1 - 2xt + t^2)^{-1/2}$, provided that $|t| < 1, |x| \leq 1$.

By which some familiar polynomials are given by

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right).$$

Therefore by the formulae (1.10) and (1.11), we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\left(-\frac{\left(1-x^2\right)}{4}\right)^k}{\Gamma(k+1)} v_{n-2k}^{(0)}(x)\frac{n!}{k!(n-2k)!}\frac{t^n}{n!} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!},$$

thus we derive a result

$$P_n(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-2k)!} \frac{\left(-\frac{\left(1-x^2\right)}{4}\right)^k}{\Gamma(k+1)} v_{n-2k}^{(0)}(x) \tag{1.12}$$

$\forall n = 0, 1, 2, 3, \ldots$

Now introducing the values of $P_n(x)$ in the formula (1.12), we obtain some special polynomials

$$v_0^{(0)}(x) = 1, \quad v_1^{(0)}(x) = x, \quad v_2^{(0)}(x) = x^2, \quad v_3^{(0)}(x) = x^3, \ldots. \tag{1.13}$$

Again in the formula (1.9) setting $\alpha = 1$, we get

$$\sum_{n=0}^{\infty} v_n^{(1)}(x)\frac{t^n}{n!} = \frac{te^{xt}}{(e^t-1)}\left(1-\frac{xt}{\pi}+\frac{t^2}{4\pi^2}\right)^{-1} = \sum_{n=0}^{\infty} n!\sum_{k=0}^{n}\frac{B_{n-k}(x)}{(n-k)!}\frac{U_k(x)}{(2\pi)^k}\frac{t^n}{n!}. \tag{1.14}$$

Here in (1.14), we have the generating functions, (see for example in [14, p.83 Eqn.(10)])

$$\left(1-2xt+t^2\right)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n$$

for the second kind Tchebycheff polynomials, $U_n(x)\ \forall n \geq 0$; and in [14, p.85 Eqn.(21)]14

$$\frac{te^{xt}}{(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

for Bernoulli polynomials, $B_n(x)\ \forall n \geq 0$.

Therefore by Eqn. (1.14), we get

$$v_n^{(1)}(x) = n!\sum_{k=0}^{n}\frac{B_{n-k}(x)}{(n-k)!}\frac{U_k(x)}{(2\pi)^k}, \forall n \geq 0. \tag{1.15}$$

Now in the formula (1.15), setting the familiar polynomials of second kind Tchebycheff polynomials, U_n, given by
$U_0(x) = 0, U_1(x) = (1-x^2)^{1/2}, U_2(x) = 2x(1-x^2)^{1/2}, U_3(x) = (4x^2-1)(1-x^2)^{1/2}$,
we obtain various special functions

$$v_0^{(1)}(x) = 0, v_1^{(1)}(x) = \frac{(1-x^2)^{1/2}}{2\pi}B_0(x), v_2^{(1)}(x) = \frac{(1-x^2)^{1/2}}{\pi}\left\{B_1(x)+\frac{x}{\pi}B_0(x)\right\} \tag{1.16}$$

$$v_3^{(1)}(x) = \frac{3(1-x^2)^{1/2}}{2\pi}\left\{B_2(x)+\frac{2x}{\pi}B_1(x)+\frac{(4x^2-1)}{2\pi^2}B_0(x)\right\}.$$

So that by (1.16), we derive some values in terms of Bernoulli numbers

$$v_0^{(1)}(0) = 0, v_1^{(1)}(0) = \frac{1}{2\pi}B_0(0), v_2^{(1)}(0) = \frac{B_1(0)}{\pi}, v_3^{(1)}(0) = \frac{3}{2\pi}\left\{B_2(0)+\frac{1}{2\pi^2}B_0(0)\right\}. \tag{1.17}$$

Motivated by above investigations, in the present article to explore new ideas in the field of mixed type polynomials, we make an appeal to the formulae (1.1)-(1.9) and thus introduce a generalized mixed type double Bernoulli-Gegenbauer-Gould and Hopper polynomials in following form of a generating function

$$\left[\frac{t}{\left(e^{t}-1\right)\left(1-\frac{xpt}{\theta}+\frac{t^{2}}{4\theta^{2}}\right)}\right]^{\alpha_1}\left[\frac{T}{\left(e^{T}-1\right)\left(1-\frac{ypT}{\varphi}+\frac{T^{2}}{4\varphi^{2}}\right)}\right]^{\alpha_2}$$
$$\times\left(1+\gamma_1 xt+\gamma_2 yT\right)^{p} e^{h_1 t^{m_1}+h_2 T^{m_2}}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_{K}P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p,\theta,\varphi)\frac{t^{n}}{n!}\frac{T^{k}}{k!}, \quad (1.18)$$

provided that $0<\frac{|t|}{2\theta}+\frac{|T|}{2\varphi}<\frac{1}{\gamma}, \forall\theta,\varphi>0, h_1,\gamma_1,h_2,\gamma_2,p\in\mathbb{R}\backslash\{0\}, \gamma=\max\{|\gamma_1|,|\gamma_2|\}$, $m_1,m_2\in\mathbb{N}$, $|x|,|y|\le 1$ and $\alpha_1,\alpha_2\in\left(-\frac{1}{2},\infty\right)\backslash\{0,1\}$.

2. Certain results and relations of (1.18) in terms of known and unknown functions

In this section, to derive the results and relations due to generating function (1.18) in terms of the known and unknown functions, we consider $\theta=\pi,\varphi=\pi$, and thus for all $|t|<2\pi, |T|<2\pi$, there exists an interesting generating function in following form

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_{K}P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^{n}}{n!}\frac{T^{k}}{k!}=\left[\frac{t}{\left(e^{t}-1\right)\left(1-\frac{xpt}{\pi}+\frac{t^{2}}{4\pi^{2}}\right)}\right]^{\alpha_1}$$
$$\times\left[\frac{T}{\left(e^{T}-1\right)\left(1-\frac{ypT}{\pi}+\frac{T^{2}}{4\pi^{2}}\right)}\right]^{\alpha_2}\left(1+\gamma_1 xt+\gamma_2 yT\right)^{p} e^{h_1 t^{m_1}+h_2 T^{m_2}} \quad (2.1)$$

Theorem 2.1. *For* $0<\frac{|t|}{2\pi}+\frac{|T|}{2\pi}<\frac{1}{\gamma}, h_1,\gamma_1,h_2,\gamma_2,p\in\mathbb{R}\backslash\{0\}, m_1,m_2\in\mathbb{N}, |x|,|y|\le 1$ *and* $\alpha_1,\alpha_2\in\left(-\frac{1}{2},\infty\right)\backslash\{0,1\}, \gamma=\max\{|\gamma_1|,|\gamma_2|\}$, *if the double polynomials is defined by*

$$A_{N_1,N_2}^{h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p}(x,y)=\sum_{s_1=0}^{N_1}\sum_{s_2=0}^{N_2}\binom{N_1}{s_1}\binom{N_2}{s_2}$$
$$\times(-xp)^{s_1}(-yp)^{s_2}H_{N_1-s_1,N_2-s_2}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y). \quad (2.2)$$

Then $\forall n\ge 0, k\ge 0$ *there exists a relation*

$${}_{K}P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)$$
$$=\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\binom{n}{r_1}\binom{k}{r_2}v_{r_1}^{(\alpha_1)}(xp)v_{r_2}^{(\alpha_2)}(yp)A_{n-r_1,k-r_2}^{h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p}(x,y), \quad (2.3)$$

where we suppose that all $v_{-n}^{(\alpha)}(.)$ are zero $\forall n \in \mathbb{N}, \alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0,1\}$.

Proof. Consider the function (2.1) in the form

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!}$$
$$= \left[\frac{t}{(e^t-1)\left(1-\frac{xpt}{\pi}+\frac{t^2}{4\pi^2}\right)}\right]^{\alpha_1} e^{pxt}\left[\frac{T}{(e^T-1)\left(1-\frac{ypT}{\pi}+\frac{T^2}{4\pi^2}\right)}\right]^{\alpha_2} e^{pyT}$$
$$\times e^{-pxt-pyT}\left(1+\gamma_1 xt+\gamma_2 yT\right)^p e^{h_1 t^{m_1}+h_2 T^{m_2}}. \tag{2.4}$$

Now in the right hand side of the Eqn. (2.4) use the formulae (1.1) and (1.9) and then apply series rearrangement techniques (see in Rainville [13, pp.56-58] and Srivastava and Manocha [14, pp.100-103]) we derive

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\binom{n}{r_1}\binom{k}{r_2} v_{r_1}^{(\alpha_1)}(xp)v_{r_2}^{(\alpha_2)}(yp)$$
$$\times\sum_{s_1=0}^{n-r_1}\sum_{s_2=0}^{k-r_2}\binom{n-r_1}{s_1}\binom{k-r_2}{s_2}(-xp)^{s_1}(-yp)^{s_2} H_{n-r_1-s_1,k-r_2-s_2}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y)\frac{t^n}{n!}\frac{T^k}{k!}. \tag{2.5}$$

Then make an appeal to the results (2.2) and (2.5), at once we find the relation (2.3).

Theorem 2.2. *If all conditions of the Theorem 2.1 are satisfied and suppose double polynomials is defined by*

$$B_{r_1,r_2}(x,y;p) = \left\{\sum_{s_1=0}^{r_1}\binom{r_1}{s_1} v_{r_1-s_1}^{(\alpha_1)}(px)(-px)^{s_1}\sum_{s_2=0}^{r_2}\binom{r_2}{s_2} v_{r_2-s_2}^{(\alpha_2)}(py)(-py)^{s_2}\right\} \tag{2.6}$$

then $\forall n \geq 0, k \geq 0$ there exists another results in the form

$${}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)$$
$$=\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\binom{n}{r_1}\binom{k}{r_2} B_{r_1,r_2}(x,y;p) H_{n-r_1,k-r_2}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y). \tag{2.7}$$

Proof. By the result (2.5), we also have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} H_{n,k}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y)\sum_{r_1=0}^{\infty} v_{r_1}^{(\alpha_1)}(xp)\sum_{r_2=0}^{\infty} v_{r_2}^{(\alpha_2)}(yp)$$
$$\times\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}(-px)^{s_1}(-py)^{s_2}\frac{t^{r_1+s_1}}{r_1!s_1!}\frac{T^{r_2+s_2}}{r_2!s_2!}\frac{t^n}{n!}\frac{T^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\binom{n}{r_1}\binom{k}{r_2}H_{n-r_1,k-r_2}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y)$$
$$\times\sum_{s_1=0}^{r_1}\sum_{s_2=0}^{r_2}\binom{r_1}{s_1}v_{r_1-s_1}^{(\alpha_1)}(px)(-px)^{s_1}\binom{r_2}{s_2}v_{r_2-s_2}^{(\alpha_2)}(py)(-py)^{s_2}\frac{t^n}{n!}\frac{T^k}{k!}. \tag{2.8}$$

Now in the result (2.8) make an appeal to the formula (2.6), we immediately find the relation (2.7).

Theorem 2.3. *If all conditions of the Theorem 2.1 are satisfied and suppose the polynomials are defined by*

$$\psi_{r_1}^{(\alpha_1)}(xp)=\sum_{s_1=0}^{r_1}\binom{r_1}{s_1}v_{r_1-s_1}^{(\alpha_1)}(xp)(-px)^{s_1} \tag{2.9}$$

and

$$\phi_{r_2}^{(\alpha_2)}(yp)=\sum_{s_2=0}^{r_2}\binom{r_2}{s_2}v_{r_2-s_2}^{(\alpha_2)}(yp)(-py)^{s_2}. \tag{2.10}$$

Then $\forall n\geq 0, k\geq 0$ *there exist following equality*

$${}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)$$
$$=\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\begin{vmatrix}\binom{n}{r_1}\psi^{(\alpha_1)_{r_1}}(xp) & 0\\ 0 & \binom{k}{r_2}\phi^{(\alpha_2)_{r_2}}(yp)\end{vmatrix} H_{n-r_1,k-r_2}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y). \tag{2.11}$$

Proof. Make an appeal to the theory and methods of the Theorems 2.1 and 2.2

we may write the generating formula (2.4) as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_{K}P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\binom{n}{r_1}\psi_{r_1}^{(\alpha_1)}(xp)\binom{k}{r_2}\phi_{r_2}^{(\alpha_2)}(yp)H_{n-r_1,k-r_2}^{(h_1,h_2,m_1,m_2,\gamma_1,\gamma_2,p)}(x,y)\frac{t^n}{n!}\frac{T^k}{k!}. \tag{2.12}$$

In (2.12) the polynomials $\psi_{r_1}^{(\alpha_1)}(xp)$ and $\phi_{r_2}^{(\alpha_2)}(yp)$ are defined in (2.9) and (2.10) respectively.

The formula (2.12) immediately gives us the formula (2.11).

Theorem 2.4. *If all conditions of the Theorem 2.1 are satisfied and suppose double bilinear polynomials is defined by*

$$A_{r_1,r_2}(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2;x,y,p)=\begin{vmatrix}\Phi_{r_1}^{(\alpha_1,m_1,h_1,p)}(xp) & 0\\ 0 & \Psi_{r_2}^{(\alpha_2,m_2,h_2,p)}(yp)\end{vmatrix}, \tag{2.13}$$

where, we denote

$$\Phi_{r_1}^{(\alpha_1,m_1,h_1,p)}(xp)=\sum_{s_1=0}^{r_1}\binom{r_1}{s_1}v_{r_1-s_1}^{(\alpha_1)}(xp)g_{s_1}^{m_1}(-xp;h_1)$$

and

$$\Psi_{r_2}^{(\alpha_2,m_2,h_2,p)}(yp)=\sum_{s_2=0}^{r_2}\binom{r_2}{s_2}v_{r_2-s_2}^{(\alpha_2)}(yp)g_{s_2}^{m_2}(-yp;h_2).$$

Then $\forall n\geq 0, k\geq 0$ *there exist following equality*

$${}_{K}P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)=\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\binom{n}{r_1}\binom{k}{r_2}(-p)_{n-r_1+k-r_2}$$
$$\times\begin{vmatrix}\Phi_{r_1}^{(\alpha_1,m_1,h_1,p)}(xp) & 0\\ 0 & \Psi_{r_2}^{(\alpha_2,m_2,h_2,p)}(yp)\end{vmatrix}(-\gamma_1 x)^{n-r_1}(-\gamma_2 y)^{k-r_2}. \tag{2.14}$$

Proof. Consider the formula (2.4) and write it in the form

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_{K}P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!}$$

$$=\left[\frac{t}{(e^t-1)\left(1-\frac{xpt}{\pi}+\frac{t^2}{4\pi^2}\right)}\right]^{\alpha_1} e^{pxt}\left[\frac{T}{(e^T-1)\left(1-\frac{ypT}{\pi}+\frac{T^2}{4\pi^2}\right)}\right]^{\alpha_2} e^{pyT}$$
$$\times e^{-pxt+h_1t^{m_1}}e^{-pyT+h_2T^{m_2}}\left(1+\gamma_1xt+\gamma_2yT\right)^p. \tag{2.15}$$

Then in right hand side of (2.15) use the generating formulae (1.4), (1.9) and the binomial formula, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{k}\binom{n}{r_1}\binom{k}{r_2}(-p)_{n-r_1+k-r_2}\left(-\gamma_1x\right)^{n-r_1}\left(-\gamma_2y\right)^{k-r_2}$$
$$\times A_{r_1,r_2}\left(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2;x,y,p\right)\frac{t^n}{n!}\frac{T^k}{k!} \tag{2.16}$$

Here in (2.16) the double bilinear sequence is defined by

$$A_{r_1,r_2}\left(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2;x,y,p\right)$$
$$=\sum_{s_1=0}^{r_1}\binom{r_1}{s_1}v_{r_1-s_1}^{(\alpha_1)}(xp)g_{s_1}^{m_1}\left(-xp,h_1\right)\sum_{s_2=0}^{r_2}\binom{r_2}{r_2}v_{r_2-s_2}^{(\alpha_2)}(yp)g_{s_2}^{m_2}\left(-yp,h_2\right) \tag{2.17}$$

Now in (2.16) make an appeal to the formulae, given in (2.13) and (2.17), we obtain the result (2.14).

3. Inequalities of ${}_K\boldsymbol{P}_{\boldsymbol{n,k}}^{(\boldsymbol{\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2})}(.)$

In this section, in order to find the inequalities involving the function ${}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(.)$, we first prove a lemma related to the exponential and binomial functions given by:

Lemma 3.1. *If* $0\leqq\gamma_1xt+\gamma_2yT<p,\forall p\in\mathbb{Z}_+$, *(a set of positive integers), then there exists following inequalities*

$$\left(1+\frac{\gamma_1xt+\gamma_2yT}{p}\right)^p\leqq e^{\gamma_1xt}e^{\gamma_2yT}\leqq\left(1-\frac{(\gamma_1xt+\gamma_2yT)}{p}\right)^{-p} \tag{3.1}$$

Proof. Comparing the three series (see also in [13, Rainville, p. 15, Lemma 1])

$$1+\frac{\gamma_1xt+\gamma_2yT}{p}=1+\frac{\gamma_1xt+\gamma_2yT}{p},$$
$$e^{\frac{\gamma_1xt}{p}}e^{\frac{\gamma_2yT}{p}}=e^{\frac{\gamma_1xt+\gamma_2yT}{p}}=1+\frac{\gamma_1xt+\gamma_2yT}{p}+\sum_{n=2}^{\infty}\frac{\left(\frac{\gamma_1xt+\gamma_2yT}{p}\right)^n}{n!} \tag{3.2}$$

and

$$\left(1-\frac{(\gamma_1 xt+\gamma_2 yT)}{p}\right)^{-1}=1+\frac{(\gamma_1 xt+\gamma_2 yT)}{p}+\sum_{n=2}^{\infty}\left(\frac{\gamma_1 xt+\gamma_2 yT}{p}\right)^{n},$$

we get the inequalities

$$\left(1+\frac{\gamma_1 xt+\gamma_2 yT}{p}\right)^{1}\leqq e^{\frac{\gamma_1 xt+\gamma_2 yT}{p}}\leqq\left(1-\frac{(\gamma_1 xt+\gamma_2 yT)}{p}\right)^{-1}.$$

Then raising (3.2) by the power p $\forall p\in\mathbb{Z}_+$, we find the required inequalities of (3.1).

Theorem 3.1. *If $\gamma_1>1,\gamma_2>1$ and all conditions of the Theorem 2.1 are satisfied, then by the generating function (2.1), there exists an inequality*

$$\begin{aligned}&{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right)\\&\leqq\left\{\sum_{r=0}^{n}\binom{n}{r}\quad v_{n-r}^{(\alpha_1)}(x)g_r^{m_1}\left((\gamma_1-1)\,x,h_1\right)\right\}\left\{\sum_{s=0}^{k}\binom{k}{s}v_{k-s}^{(\alpha_2)}(y)g_s^{m_2}\left((\gamma_2-1)\,y,h_2\right)\right\}.\end{aligned}\tag{3.3}$$

Proof. Consider the generating function (2.15) in which replace x by $\frac{x}{p}$ and y by $\frac{y}{p}$ to get it in the form

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right)\frac{t^n}{n!}\frac{T^k}{k!}\\&=\left[\frac{t}{(e^t-1)\left(1-\frac{xt}{\pi}+\frac{t^2}{4\pi^2}\right)}\right]^{\alpha_1}e^{xt}\left[\frac{T}{(e^T-1)\left(1-\frac{yT}{\pi}+\frac{T^2}{4\pi^2}\right)}\right]^{\alpha_2}e^{yT}\\&\times e^{-xt-yT}e^{h_1t^{m_1}+h_2T^{m_2}}\left(1+\frac{\gamma_1 xt+\gamma_2 yT}{p}\right)^{p}.\end{aligned}\tag{3.4}$$

Now making an appeal to the inequalities (3.2) in the formula (3.4), then for $\gamma_1>1,\gamma_2>1$, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right)\frac{t^n}{n!}\frac{T^k}{k!}\\&\leqq\left[\frac{t}{(e^t-1)\left(1-\frac{xt}{\pi}+\frac{t^2}{4\pi^2}\right)}\right]^{\alpha_1}e^{xt}\left[\frac{T}{(e^T-1)\left(1-\frac{yT}{\pi}+\frac{T^2}{4\pi^2}\right)}\right]^{\alpha_2}e^{yT}\\&\times e^{(\gamma_1-1)xt+h_1t^{m_1}+(\gamma_2-1)yT+h_2T^{m_2}}.\end{aligned}\tag{3.5}$$

Again on defining the functions given in (1.4) and (1.9) and thus using series arrangement techniques we obtain

$$
\begin{aligned}
&{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right)\\
&\leqq \sum_{r=0}^{n}\sum_{s=0}^{k}\binom{n}{r}\binom{k}{s}v_{n-r}^{(\alpha_1)}(x)g_r^{m_1}\left(\left(\gamma_1-1\right)x,h_1\right)v_{k-s}^{(\alpha_2)}(y)g_s^{m_2}\left(\left(\gamma_2-1\right)y,h_2\right). \quad (3.6)
\end{aligned}
$$

The inequality (3.6) gives us the formula (3.3).

Theorem 3.2. *If $\gamma_1 > 0, \gamma_2 > 0$, and all conditions of the Theorem 2.1 are satisfied, then by the generating function (2.1), there exists a limiting formula*

$$
\begin{aligned}
&\lim_{p\to\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right)\frac{t^n}{n!}\frac{T^k}{k!}\\
&=\left[\frac{t}{\left(e^t-1\right)\left(1-\frac{xt}{\pi}+\frac{t^2}{4\pi^2}\right)}\right]^{\alpha_1}\left[\frac{T}{\left(e^T-1\right)\left(1-\frac{yT}{\pi}+\frac{T^2}{4\pi^2}\right)}\right]^{\alpha_2}e^{\gamma_1xt+h_1t^{m_1}}e^{\gamma_2yT+h_2T^{m_2}}. \quad (3.7)
\end{aligned}
$$

Hence then for $\gamma_1 > 1, \gamma_2 > 1$ there exists a double polynomials

$$
\begin{aligned}
&\lim_{p\to\infty}{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right)\\
&=\sum_{r=0}^{n}\binom{n}{r}v_{n-r}^{(\alpha_1)}(x)g_r^{m_1}\left(\left(\gamma_1-1\right)x,h_1\right)\sum_{s=0}^{k}\binom{k}{s}v_{k-s}^{(\alpha_2)}(y)g_s^{m_2}\left(\left(\gamma_2-1\right)y,h_2\right) \quad (3.8)
\end{aligned}
$$

Proof. Performing the techniques of (3.4)) and then for $\gamma_1 > 0, \gamma_2 > 0$, taking the limit $p \to \infty$ in both the sides to get it in the form

$$
\begin{aligned}
&\lim_{p\to\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right)\frac{t^n}{n!}\frac{T^k}{k!}\\
&=\left[\frac{t}{\left(e^t-1\right)\left(1-\frac{xt}{\pi}+\frac{t^2}{4\pi^2}\right)}\right]^{\alpha_1}\left[\frac{T}{\left(e^T-1\right)\left(1-\frac{yT}{\pi}+\frac{T^2}{4\pi^2}\right)}\right]^{\alpha_2}\\
&\times e^{h_1t^{m_1}+h_2T^{m_2}}\lim_{p\to\infty}\left(1+\frac{\gamma_1xt+\gamma_2yT}{p}\right)^p. \quad (3.9)
\end{aligned}
$$

In the formula (3.9) on defining exponential function, we get the formula (3.7).

Again then for all $\gamma_1 > 1, \gamma_2 > 1$, making an appeal to the formulae (1.4) and (1.9), we get the formula (3.8).

4. First few polynomials of ${}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(.) \forall n,k \in \mathbb{N}\cup\{0\}$

On applying the formulae (3.3) and (3.4), we obtain special polynomials of ${}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(.)\ \forall n,k \in \mathbb{N}\cup\{0\}$.

Therefore for an example and for all $\gamma_1 > 1, \gamma_2 > 1$, and for $0 < \frac{|t|}{2\pi} + \frac{|T|}{2\pi} < \frac{1}{\gamma}, h_1, \gamma_1, h_2, \gamma_2, p \in \mathbb{R}\backslash\{0\}$, $m_1 = 3 = m_2, |x|, |y| \leq 1$ and $\alpha_1, \alpha_2 \in \left(-\frac{1}{2}, \infty\right)\backslash\{0,1\}$, $\gamma = \max\{|\gamma_1|, |\gamma_2|\}$, due to Quintana [12], we give first few polynomials of $v_n^{(\alpha_1)}(x)$ $\forall n \in \mathbb{N}\cup\{0\}$ as

$$v_0^{(\alpha_1)}(x) = 1, \quad v_1^{(\alpha_1)}(x) = \left(1+\frac{\alpha_1}{\pi}\right)x - \frac{\alpha_1}{2}$$

$$v_2^{(\alpha_1)}(x) = \left(1+\frac{2\alpha_1}{\pi}+\frac{\alpha_1(\alpha_1+1)}{\pi^2}\right)x^2 - \left(\alpha_1+\frac{(\alpha_1)^2}{\pi}\right)x + \frac{\alpha_1}{2}\left(\frac{(3\alpha_1-1)}{6}-\frac{1}{\pi^2}\right); \tag{4.1}$$

and due to (1.3) we derive following first few polynomials of $g_n^3((\gamma_1-1)x, h_1)\ \forall n \in \mathbb{N}\cup\{0\}$ $g_0^3((\gamma_1-1)x,h_1) = 1, g_1^3((\gamma_1-1)x,h_1) = (\gamma_1-1)x, g_2^3((\gamma_1-1)x,h_1) = (\gamma_1-1)^2x^2$, $g_3^3((\gamma_1-1)x,h_1) = (\gamma_1-1)^3x^3+6h_1$.

Thus making an appeal to the Theorems 3.2 and 3.3 and the formulae (4.1) we obtain first few polynomials of ${}_K P_{n,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}(.)\forall n,k \in \mathbb{N}\cup\{0\}$ as

$${}_K P_{0,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) \leqq \lim_{p\to\infty} {}_K P_{0,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) \tag{4.2}$$

$$= \left\{\sum_{r=0}^{0}\binom{0}{r} v_{0-r}^{(\alpha_1)}(x) g_r^3((\gamma_1-1)x,h_1)\right\}\left\{\sum_{s=0}^{0}\binom{0}{s} v_{0-s}^{(\alpha_2)}(y) g_s^3((\gamma_2-1)y,h_2)\right\}$$
$$= \left\{\binom{0}{r} v_{0-r}^{(\alpha_1)}(x) g_r^3((\gamma_1-1)x,h_1)\Big|_{r=0}\right\}\left\{\binom{0}{s}\quad v_{0-s}^{(\alpha_1)}(x) g_s^3((\gamma_1-1)x,h_1)\Big|_{s=0}\right\}$$
$$= 1$$
$$\Longrightarrow {}_K P_{0,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\tfrac{x}{p},\tfrac{y}{p};p\right)$$
$$\leqq \lim_{p\to\infty} {}_K P_{0,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p},\tfrac{y}{p};p\right) = 1.$$

Similarly we obtain

$${}_K P_{1,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p},\tfrac{y}{p};p\right) \leqq \lim_{p\to\infty} {}_K P_{1,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p},\tfrac{y}{p};p\right)$$
$$= \left(1+\tfrac{\alpha_1}{\pi}+(\gamma_1-1)\right)x - \tfrac{\alpha_1}{2},$$
$${}_K P_{2,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p},\tfrac{y}{p};p\right) \leqq \lim_{p\to\infty} {}_K P_{2,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p},\tfrac{y}{p};p\right)$$

$$= \left(1 + \tfrac{2\alpha_1}{\pi} + \tfrac{\alpha_1(\alpha_1+1)}{\pi^2} + \left(2 + \tfrac{2\alpha_1}{\pi}\right)(\gamma_1 - 1) + (\gamma_1 - 1)^2\right) x^2$$
$$- \left(\alpha_1 \left(1 + (\gamma_1 - 1)\right) + \tfrac{(\alpha_1)^2}{\pi}\right) x + \tfrac{\alpha_1}{2} \left(\tfrac{(3\alpha_1 - 1)}{6} - \tfrac{1}{\pi^2}\right),$$
$${}_K P_{n,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{n,0}^{(\alpha_1,h_1 3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \textstyle\sum_{r=0}^{n} \binom{n}{r} v_{n-r}^{(\alpha_1)}(x) g_r^3\left((\gamma_1 - 1)x, h_1\right),$$
$${}_K P_{0,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{0,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \left(1 + \tfrac{\alpha_2}{\pi} + (\gamma_2 - 1)\right) y - \tfrac{\alpha_2}{2}$$
$${}_K P_{1,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{1,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \left(\gamma_1\gamma_2 + \tfrac{\alpha_1}{\pi}(\gamma_2 - 1) + \tfrac{\alpha_2}{\pi}(\gamma_1 - 1)\right) xy - \tfrac{\alpha_1}{2}(\gamma_2 - 1) y - \tfrac{\alpha_2}{2}(\gamma_1 - 1) x$$
$${}_K P_{2,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{2,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \left\{\left(1 + \tfrac{\alpha_2}{\pi} + (\gamma_2 - 1)\right) y - \tfrac{\alpha_2}{2}\right\}$$
$$\times \left\{\left(1 + \tfrac{2\alpha_1}{\pi} + \tfrac{\alpha_1(\alpha_1+1)}{\pi^2} + \left(2 + \tfrac{2\alpha_2}{\pi}\right)(\gamma_1 - 1) + (\gamma_1 - 1)^2\right) x^2\right.$$
$$\left. - \left(\alpha_1\gamma_1 + \tfrac{(\alpha_1)^2}{\pi}\right) x + \tfrac{\alpha_1}{2}\left(\tfrac{(3\alpha_1 - 1)}{6} - \tfrac{1}{\pi^2}\right)\right\},$$
$${}_K P_{n,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{n,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \left\{\textstyle\sum_{r=0}^{n} \binom{n}{r} v_{n-r}^{(\alpha_1)}(x) g_r^3\left((\gamma_1 - 1)x, h_1\right)\right\} \left\{\left(1 + \tfrac{\alpha_2}{\pi}\right) y - \tfrac{\alpha_2}{2} + (\gamma_2 - 1) y\right\},$$
$${}_K P_{0,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{0,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \left\{\textstyle\sum_{s=0}^{k} \binom{k}{s} v_{k-s}^{(\alpha_2)}(y) g_s^3\left((\gamma_2 - 1)y, h_2\right)\right\},$$
$${}_K P_{1,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{1,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \left\{\left(1 + \tfrac{\alpha_1}{\pi}\right) x - \tfrac{\alpha_1}{2} + (\gamma_1 - 1) x\right\} \left\{\textstyle\sum_{s=0}^{k} \binom{k}{s} v_{k-s}^{(\alpha_2)}(y) g_s^3\left((\gamma_2 - 1)y, h_2\right)\right\}$$
$${}_K P_{2,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{2,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right)$$
$$= \left\{\left(1 + \tfrac{2\alpha_1}{\pi} + \tfrac{\alpha_1(\alpha_1+1)}{\pi^2}\right) x^2 - \left(\alpha_1 + \tfrac{(\alpha_1)^2}{\pi}\right) x + \tfrac{\alpha_1}{2}\left(\tfrac{(3\alpha_1 - 1)}{6} - \tfrac{1}{\pi^2}\right)\right.$$
$$\left. + \left(\left(2 + \tfrac{2\alpha_1}{\pi}\right)(\gamma_1 - 1) x^2 - \alpha_1 (\gamma_1 - 1) x\right) + (\gamma_1 - 1)^2 x^2\right\}$$
$$\times \left\{\textstyle\sum_{s=0}^{k} \binom{k}{s} v_{k-s}^{(\alpha_2)}(y) g_s^3\left((\gamma_2 - 1)y, h_2\right)\right\}$$
$${}_K P_{n,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right) \leqq \lim_{p\to\infty} {}_K P_{n,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\tfrac{x}{p}, \tfrac{y}{p}; p\right).$$
$$= \left\{\textstyle\sum_{r=0}^{n} \binom{n}{r} v_{n-r}^{(\alpha_1)}(x) g_r^3\left((\gamma_1 - 1)x, h_1\right)\right\} \left\{\textstyle\sum_{s=0}^{k} \binom{k}{s} v_{k-s}^{(\alpha_2)}(y) g_s^3\left((\gamma_2 - 1)y, h_2\right)\right\}$$

Hence due to (4.2), there exists a non-singular matrix given by (4.3)

$$\mathcal{A}_{n+1\times k+1}(x,y)=\lim_{p\to\infty}\begin{bmatrix} {}_KP_{0,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{0,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & \cdots & {}_KP_{0,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) \\ \vdots & \vdots & \vdots & \vdots \\ {}_KP_{n,0}^{(\alpha_1,h_13,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{n,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & \cdots & {}_KP_{n,k}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) \end{bmatrix} \tag{4.3}$$

From (4.3), particularly, we write

$$\mathcal{A}_{3\times 3}(x,y)=\lim_{p\to\infty}\begin{bmatrix} {}_KP_{0,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{0,1}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{0,2}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) \\ {}_KP_{1,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{1,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{1,2}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) \\ {}_KP_{2,0}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{2,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) & {}_KP_{2,2}^{(\alpha_1,h_1,3,\gamma_1;\alpha_2,h_2,3,\gamma_2)}\left(\frac{x}{p},\frac{y}{p};p\right) \end{bmatrix} \tag{4.4}$$

5. Application of ${}_K\boldsymbol{P}_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(.)$ to obtain integral estimates

To find out the estimates for any function $f(x,y)\forall x,y$ such that $a\le x\le b; a\le y\le b; |f(x,y)|\le M_1$ and $f(a,y)=0=f(b,y)\forall a\le y\le b$ and $f(x,a)=0=f(x,b)\forall a\le x\le b.$ we state the following theorem:

Theorem 5.1. *If in any square domain $a\le x\le b; a\le y\le b; \left|\left(\frac{\partial}{\partial x}\right)f(x,y)\right|\le M_1, \left|\left(\frac{\partial}{\partial y}\right)f(x,y)\right|\le M_2$. Also initial and end values are $f(a,y)=0=f(b,y)\forall a\le y\le b$ and $f(x,a)=0=f(x,b)$ $\forall a\le x\le b$, and $\alpha_1=\alpha_2, h_1=h_2, m_1=m_2, \gamma_1=\gamma_2$.*

Then for all conditions given in the Theorems 3.2 and 3.3 there exists following estimates

$$\left|\int_a^b\int_a^b f(x,y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right){}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)}\left(\frac{x}{p},\frac{y}{p};p\right)dxdy\right| < (M_1+M_2)\,F(a,b). \tag{5.1}$$

Here,

$$F(a,b)=\lim_{p\to\infty}\int_a^b\int_a^b\left|{}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)}\left(\frac{x}{p},\frac{y}{p};p\right)\right|dxdy.$$

Proof. Consider the double integral as

$$\int_a^b\int_a^b f(x,y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right){}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)}\left(\frac{x}{p},\frac{y}{p};p\right)dxdy \tag{5.2}$$

Now in the double integral (5.2), under the conditions given in the Theorems 3.2 and 3.3, using the theory of the Lemma 3.1 we find that

$$\begin{aligned}&\int_a^b\int_a^b f(x,y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right){}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)}\left(\frac{x}{p},\frac{y}{p};p\right)dxdy\\
&\leqq\int_a^b\left[\int_a^b\left\{\sum_{s=0}^{k}\binom{k}{s}v_{k-s}^{(\alpha_1)}(y)g_s^{m_1}\left((\gamma_1-1)y,h_1\right)\right\}\right.\\
&\times\left(\frac{\partial}{\partial x}\right)\left\{\sum_{r=0}^{n}\binom{n}{r}\quad v_{n-r}^{(\alpha_1)}(x)g_r^{m_1}\left((\gamma_1-1)x,h_1\right)\right\}f(x,y)dx\left.\right]dy\\
&+\int_a^b\left[\int_a^b\left\{\sum_{s=0}^{k}\binom{k}{s}v_{k-s}^{(\alpha_1)}(y)g_s^{m_1}\left((\gamma_1-1)y,h_1\right)\right\}\right.\\
&\times\left(\frac{\partial}{\partial y}\right)\left\{\sum_{r=0}^{n}\binom{n}{r}\quad v_{n-r}^{(\alpha_1)}(x)g_r^{m_1}\left((\gamma_1-1)x,h_1\right)\right\}f(x,y)dy\left.\right]dx\end{aligned} \tag{5.3}$$

Since here in this theorem $\alpha_1=\alpha_2, h_1=h_2, m_1=m_2, \gamma_1=\gamma_2$, therefore in (5.3), the change of order of integration techniques is followed.

Then under the conditions $f(a,y)=0=f(b,y)\forall a\leq y\leq b$ and $f(x,a)=0=f(x,b)\forall a\leq x\leq b$, and on applying the method of integration by parts in the inner integrals of (5.3), it gives as

$$\begin{aligned}&\left|\int_a^b\int_a^b\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right){}_KP_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)}\left(\frac{x}{p},\frac{y}{p};p\right)f(x,y)dxdy\right|\\
&\leqq\left|\int_a^b\int_a^b\left\{\sum_{s=0}^{k}\binom{k}{s}v_{k-s}^{(\alpha_1)}(y)g_s^{m_1}\left((\gamma_1-1)y,h_1\right)\right\}\right.\\
&\times\left\{\sum_{r=0}^{n}\binom{n}{r}v_{n-r}^{(\alpha_1)}(x)g_r^{m_1}\left((\gamma_1-1)x,h_1\right)\right\}\left(\frac{\partial}{\partial x}\right)f(x,y)dxdy\left.\right|\end{aligned}$$

$$+ \left| \int_a^b \int_a^b \left\{ \sum_{s=0}^{k} \binom{k}{s} v_{k-s}^{(\alpha_1)}(y) g_s^{m_1}\left((\gamma_1 - 1)\, y, h_1\right) \right\} \right.$$

$$\left. \times \left\{ \sum_{r=0}^{n} \binom{n}{r} \quad v_{n-r}^{(\alpha_1)}(x) g_r^{m_1}\left((\gamma_1 - 1)\, x, h_1\right) \right\} \left(\frac{\partial}{\partial y}\right) f(x,y) dy dx \right|$$

$$< \int_a^b \int_a^b \left| \left\{ \sum_{s=0}^{k} \binom{k}{s} v_{k-s}^{(\alpha_1)}(y) g_s^{m_1}\left((\gamma_1 - 1)\, y, h_1\right) \right\} \right.$$

$$\left. \left\{ \sum_{r=0}^{n} \binom{n}{r} v_{n-r}^{(\alpha_1)}(x) g_r^{m_1}\left((\gamma_1 - 1)\, x, h_1\right) \right\} \right| \times \left| \left(\frac{\partial}{\partial x}\right) f(x,y) \right| |dx||dy|$$

$$+ \int_a^b \int_a^b \left| \left\{ \sum_{s=0}^{k} \binom{k}{s} v_{k-s}^{(\alpha_1)}(y) g_s^{m_1}\left((\gamma_1 - 1)\, y, h_1\right) \right\} \right.$$

$$\left. \left\{ \sum_{r=0}^{n} \binom{n}{r} v_{n-r}^{(\alpha_1)}(x) g_r^{m_1}\left((\gamma_1 - 1)\, x, h_1\right) \right\} \right| \times \left| \left(\frac{\partial}{\partial y}\right) f(x,y) \right| |dx||dy|$$

$$< (M_1 + M_2) \lim_{p\to\infty} \int_a^b \int_a^b ||_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)} \left(\frac{x}{p}, \frac{y}{p}; p\right) |\, dxdy. \qquad (5.4)$$

Finally here on putting $(a,b) = \lim_{p\to\infty} \int_a^b \int_a^b \left| {}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)} \left(\frac{x}{p}, \frac{y}{p}; p\right) \right| dxdy$, we get the inequality (5.1).

Example 5.1. From (4.5) consider that $\lim_{p\to\infty} {}_K P_{1,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)} \left(\frac{x}{p}, \frac{y}{p}; p\right)$

$= \left((\gamma_1)^2 + \frac{\alpha_1}{\pi}(\gamma_1 - 1) + \frac{\alpha_1}{\pi}(\gamma_1 - 1)\right) xy - \frac{\alpha_1}{2}(\gamma_1 - 1)\, y - \frac{\alpha_1}{2}(\gamma_1 - 1)\, x.$

Also for any double function $f(x,y)$; $\forall a \le x \le b; a \le y \le b$; is such that $\left|\left(\frac{\partial}{\partial x}\right) f(x,y)\right| \le M_1, \left|\left(\frac{\partial}{\partial y}\right) f(x,y)\right| \le M_2.$

Then

$\left| \int_a^b \int_a^b f(x,y) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) {}_K P_{1,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)} \left(\frac{x}{p}, \frac{y}{p}; p\right) dxdy \right| < (M_1 + M_2)\, F(a,b)$

where,

$F(a,b) = \left((\gamma_1)^2 + \frac{2}{\pi}(\alpha_1\gamma_1 - \alpha_1)\right) \left(\frac{b^2 - a^2}{2}\right)^2 - \left(\frac{b^2 - a^2}{2}\right)(b-a)(\alpha_1\gamma_1 - \alpha_1).$

Solution. Making an appeal to the result (4.5) as

$$\lim_{p\to\infty} {}_K P_{1,1}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_1,h_1,m_1,\gamma_1)} \left(\frac{x}{p}, \frac{y}{p}; p\right)$$

$$= \int_a^b \int_a^b \left\{ \left((\gamma_1)^2 + \frac{2}{\pi}(\alpha_1\gamma_1 - \alpha_1)\right) xy - \left(\frac{\alpha_1\gamma_1}{2} - \frac{\alpha_1}{2}\right) y - \left(\frac{\alpha_1\gamma_1}{2} - \frac{\alpha_1}{2}\right) x \right\} dxdy$$

which on applying the double integration techniques give us the result (5.6).

6. Conclusions

By generalizing the method given in the present paper, some more interesting consequences of the generating functions for systems depending upon one and more variables and represented by multilinear extensions of classical results involving classical special functions may be obtained. The importance of Hermite polynomials is now well known; therefore a brief discussion of analogous linear and multiple generating functions will be very interesting.

Further, it may be remarked that, for the future directions of the study in the field, the discussion on the reducible cases of the linear and multi-linear generating functions may be very interesting. For example a ratio of the generating functions involving in numerator product of the generating function (1.4) of Gould and Hopper polynomials (1.3) with the generating function (1.9) as mixed type two classes Bernoulli-Gegenbauer polynomials due to Quintana [12] and in its denominator there is our generating function (2.1), then there exists a reduced double series given by

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{(\gamma_1,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!}$$
$$= \frac{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left\{\sum_{r=0}^{n}\binom{n}{r} g_{n-r}^{m_1}(xp,h_1)\, v_r^{(\alpha_1)}(xp)\right\}\left\{\sum_{s=0}^{k}\binom{k}{s} g_{n-s}^{m_2}(yp,h_2)\, v_k^{(\alpha_2)}(yp)\right\}\frac{t^nT^k}{n!k!}}{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_K P_{n,k}^{(\alpha_1,h_1,m_1,\gamma_1;\alpha_2,h_2,m_2,\gamma_2)}(x,y;p)\frac{t^nT^k}{n!k!}} \tag{6.1}$$

Then the double series in (6.1) is represented by the reducible generating function

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n.k}^{(\gamma_1,\gamma_2)}(x,y;p)\frac{t^n}{n!}\frac{T^k}{k!} = e^{2xpt+2ypT}\left(1+\gamma_1 xt+\gamma_2 yT\right)^{-p}. \tag{6.2}$$

It is noted that on multiplying both the sides of the formula (2.1) by $e^{2xpt+2ypT}$ and then using the definitions of the generating functions (1.4) and (1.9), we easily find the results (6.1) and (6.2). Also $\forall n \geq 0, k \geq 0$, the result (6.2) gives us the double polynomials in the form

$$C_{n.k}^{(\gamma_1,\gamma_2)}(x,y;p) = x^n y^k \sum_{r=0}^{n}\sum_{s=0}^{k}\binom{n}{r}\binom{k}{s}(p)_{n+k-r-s}(2p)^{r+s}\left(-\gamma_1\right)^{n-r}\left(-\gamma_2\right)^{k-s} \tag{6.3}$$

By (6.3), it is also noted that the double polynomials in (6.3) are independent of the parameters $\alpha_1, h_1, m_1, \alpha_2, h_2$ and m_2.

The first few special polynomials are derived due to generalized mixed type BernoulliGegenbauer polynomials. Then we obtained certain results and relations of our double (BGG-H) polynomials in terms of known and unknown functions. We also derived some of the inequalities and limiting cases of double (BGG-H) polynomials and then by using them we have constructed a matrix representation and obtained the integral estimates. Another result of interest is the formula (1.6), which in view of the work available in [11] may yield well-known results of polynomials associated with Humberts polynomials [14, p.86].

The results obtained in the Sections 1 and 5 are very applicable to derive various formulae concerning to the integrals involving the generalized special functions (for example see in [1]). Also on making an appeal to the results given in the Sections, 3, 4 and 5, we may analyse some properties of generalized Bernoulli polynomials and numbers found in the literature (for example see in [2], [4], [6], [7], [8], [9], [10], [15] and others).

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