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$\mathcal{N}$-th ORDER DIFFERENTIAL INEQUALITIES IN THE COMPLEX PLANE

Rajesh Kumar Maurya

Department of Mathematics,
Govt. Post Graduate College,
Gopeshwar, Uttarakhand, INDIA

E-mail : rajeshkrmaurya@gmail.com

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Abstract: There are numerous articles dealing with first and second order differential inequalities and differential subordinations and only three articles which are related to third order differential inequalities and subordinations. In this paper we generalise these inequalities for $\mathcal{N}$-th order differential inequalities for functions belonging to class of analytic functions f such that $f(0) = 0$.

Keywords and Phrases: Subordination, Differential subordination, Differential inequality, Maximum modulus.

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1. Introduction

Let U be the open unit disk in the complex plane $\mathbb{C}$, centered at origin, and let Ω and Δ be sets in $\mathbb{C}$. Let P be an analytic function defined on U, and D be a differential operator such that $D[P]$ is defined on U. Under what conditions on D, Ω and Δ that are needed so that

$$D[P] \subset \Omega \Rightarrow P(U) \subset \Delta \tag{1.1}$$

previous work has been done on first, second and third order differential inequalities of this type see ([2], [5]).

In this paper we intend to obtain concrete results on differential inequalities for the n-th order derivative for class of analytic functions.

Let $H = H(U)$ denote the class of functions analytic in the open unit disk U.For $n \in \mathbb{N}$ the set of natural number and $a \in \mathbb{C}$ define the class of functions denoted by $H[a, n]$ contains functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Denote the class $H[0, n] = A_n$. A function $f \in A_n(f(z) \neq 0$ for any $z \in U\backslash\{0\})$ We say that an analytic function f is subordinate to the analytic function F, and write $f \prec F$ in U iff there exists a Schwarz class function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$f(z) = g(w(z))$, for all $z \in U$. In particular if F is univalent in U, we have the following equivalence:

$$f \prec F \ in \ U \quad \Longleftrightarrow \quad f(0) = F(0) \text{ and } \ f(U) \subseteq F(U) \tag{1.2}$$

If either Ω or Δ in (1.1) is simply connected domain. Then it may be possible to write (1.1) in terms of subordination. If Δ is a simply connected domain containing the point a and $\Delta \neq \mathbb{C}$, then there is a conformal mapping q of U onto Δ such that $q(0) = a$, In this case (1.1) can be written as follows

$$\{\Psi(p(z)), zp^{'}(z), z^2p^{''}(z), ..., z^np^{(n)}(z); z)|z \in U\} \subset \Omega \tag{1.3}$$
$$\Rightarrow p(z) \prec q(z)$$

If Ω is also a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω such that $h(0) = \Psi(a, 0, 0, 0..0; 0)$.If in addition $\Psi(p(z)), zp^{'}(z), z^2p^{''}(z), ..., z^np^{(n)}(z); z)$ is analytic in U, then (1.1) can be written as follows:

$$\Psi(p(z)), zp^{'}(z), z^2p^{''}(z), ..., z^np^{(n)}(z); z) \prec h(z) \tag{1.4}$$
$$\Rightarrow p(z) \prec q(z)$$

Here we have three key constituents in this differential implication of the form (1.3), the differential operator Ψ, the set Ω and the 'dominating' function q. Given any two of these one would hope to find the third so that (1.3) is satisfied. In this article, we start with a given set Ω and a given function q, and determine a set of 'admissible' operators Ψ so that (1.3) holds.

Definition 1. *Let $\Psi : \mathbb{C}^n \times U \longrightarrow \mathbb{C}$ and h be univalent in U. If p is analytic in U and satisfies the n-th order differential subordination relation*

$$\Psi(p(z)), zp^{'}(z), z^2p^{''}(z), ..., z^np^{(n)}(z); z) \prec h(z) \tag{1.5}$$

p is called a solution of the differential subordination (1.5). A univalent function q is called a dominant of the solution of the differential subordination or more simply a dominant if $p \prec q$ *for p satisfying (1.5). A dominant* $\overline{q}$ *that satisfies* $\overline{q} \prec q$ *for all dominant of (1.5) is called the best dominant of (1.5). The best dominant is unique upto a rotation of U.*

Definition 2. *Let Q denote the set of functions q that are analytic and univalent on the set* $\overline{U}\backslash E(q)$*, where*

$$E(q) = \{\xi \in \partial U : \lim_{z \longrightarrow \xi} q(z) = \infty\} \tag{1.6}$$

and are such that $Min.|q^{'}(\xi)| = \rho > 0$ *for* $\xi \in \partial U\backslash\ E(q)$*. The subset of Q for which* $q(0) = a$ *is denoted by* $Q(a)$*.*

As a simple example , consider the function $q(z) = \frac{1+z}{1-z}$*. For this function, we have* $E(q) = \{1\}$ *and* $min.|q^{'}(\xi)|{=}\frac{1}{2} > 0$ *for* $\xi \in \partial U\backslash\ \{1\}$*.*

The Fundamental Lemmas

Definition 3. *In 1981, Sanford Miller and Petru Mocanu laid the foundation for the theory of differential subordinations in a paper published in Michigan Mathematics Journal (*[5]*),entitled Differential Subordinations and Univalent Functions. The following two lemmas, which appeared in that article, played a key role in the development of the theory that evolved for 2nd-order differential subordination , these are referred as the Miller/Mocanu lemmas.*

Lemma 1. A ([5], Miller/Mocanu lemma) *Let* $z_0 \in U$*, with* $r_{0=|z_0|}$*, and let* $f(z) = a_n z^n + a_{n+1} z^{n+1} + ...$ *be continuous on* $\overline{U_{r_0}}$ *and analytic on* $U_{r_0} \cup \{z_0\}$ *with f(z)not identical to zero function. and* $n \geq 2$*.If*

$$|f(z_0)| = Max\{|f(z)| : z_0 \in \overline{U_{r_0}}\} \tag{1.7}$$

Then there exists an $m \geq n$ *such that*

$$\frac{z_0 f'(z_0)}{f(z_0)} = m \tag{1.8}$$

and

$$Re\frac{z_0 f^{''}(z_0)}{f^{'}(z_0)} + 1 \geq m \tag{1.9}$$

Lemma 2. B ([5, Miller/Mocanu lemma]) *Let* $p(z) = a_n z^n + a_{n+1} z^{n+1} + ...$*be analytic in U with* $p(z)$ *not identical to a and* $n \geq 1$*, and let* $q \in Q(a)$*. If there* *exist point* $z_0 = r_0e^{i\theta_0} \in U$ *and* $w_0 \in \partial U\setminus\ E(q)$ *such that* $p(z_0) = q(w_0)$ *and* $p(\overline{U_{r_0}}) \subset q(U)$, *then exists an* $m \geq n \geq 1$ *such that*

$$z_0p^{'}(z_0) = mw_0q^{'}(w_0) \tag{1.10}$$

and

$$Re\frac{z_0p^{''}(z_0)}{p^{'}(z_0)} + 1 \geq m\left[Re\frac{w_0q^{''}(w_0)}{q^{'}(w_0)} + 1\right] \tag{1.11}$$

Lemma B is a generalization of lemma A and reduces to it for $p(z) = f(z)$ *and* $q(z) = z$.

Proof. of Lemma A: If we let $f(z) = R(r_0, \theta)e^{i\Phi(r_0,\theta)}$ for $z = r_0e^{i\theta}$ then

$$\frac{zf'(z)}{f(z)} = \frac{\partial\Phi}{\partial\theta} - \frac{i\partial R}{R\partial\theta}$$

Since R attains $\max imum$ at $z_0 = re^{i\theta_0}$ we must have $\frac{\partial R(z_0)}{\partial\theta} = 0$ at z_0 . Therefore we obtain $\frac{z_0f'(z_0)}{f(z_0)} = m$, where m is real we need to show $m \geq n$. If we let $g(z) = \frac{f(z_0z)}{[f(z_0)z^{n-1}]}$ for $z \in \overline{U}$, then g is continuous on $\overline{U}$

and analytic on $U \cup \{1\}$. Hence from the maximum modulus principle we have

$$|g(z)| \leq \max_{|z|=1}|g(z)| = \frac{1}{|f(z_0)|}\max_{|z|=1}|f(z_0z)| = 1, for\ z \in \overline{U}$$

Since $g(0) = 0$, by Schwarz lemma

$$|g(z)| \leq |z|\ and\ \leq \left|\frac{f(z_0z)}{f(z_0)}\right| \leq |z|^n.$$

In particular at the point $z = r, 0 \leq r < 1$ we have

$$Re\left[\frac{f(z_0z)}{f(z_0)}\right] \leq r^n.$$

Since $m = \frac{z_0f'(z_0)}{f(z_0)}$ we have

$$\begin{aligned} m &= \frac{d}{dr}\left[\frac{f(z_0r)}{f(z_0)}\right]|_{r=1} = \lim_{r\longrightarrow1}\frac{f(z_0r) - f(z_0)}{(r-1)f(z_0)} \\ &= \lim_{r\longrightarrow1}\left(1 - Re\frac{f(z_0r)}{f(z_0)}\right)\frac{1}{1-r}, \end{aligned}$$

taking real part we have

$$m = \lim_{r \longrightarrow 1} (1 - Re\frac{f(z_0 r)}{f(z_0)})\frac{1}{1-r} \geq \lim_{r \longrightarrow 1} \frac{1-r^n}{1-r} = n.$$

2. Main Results

Theorem 1. *Let $f \in H[0, n]$ then $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, for $\forall\ z \in U$. In Euler form f can also be written as*

$$f(z) = R(r, \theta) e^{i\psi(r.\theta)} \tag{2.1}$$

where $z = re^{i\theta}$, $|f(z)| = R(r, \theta)$ and $\psi(r.\theta)$ is amplitude of $f(z)$. Then

$$R(r, \theta) = \sum_{\lambda=0}^{\infty} C_\lambda X_\lambda(r) Y_\lambda(\theta)\ ; C_\lambda = A_\lambda B_\lambda$$

$$X(r) = A_\lambda e^{\frac{\lambda(\ln r)^2}{2} + a_\lambda \ln r}\ ; Y(r) = B_\lambda e^{\frac{-\lambda\theta^2}{2} + b_\lambda \ln r}$$

where $C_\lambda, A_\lambda, B_\lambda, a_\lambda, b_\lambda$ are real constants

Proof. Taking log on both side of (2.1) we get

$$\ln f(z) = \ln R(r, \theta) + i\psi(r,\ \theta),$$

since f is analytic, therefore $\ln f(z)$ is also analytic hence Cauchy Riemann equations for this are

$$\frac{\partial \ln R}{\partial r} = \frac{\partial \psi}{r \partial \theta} \Rightarrow \frac{r}{R}\frac{\partial R}{\partial r} = \frac{\partial \psi}{\partial \theta}, \tag{2.2}$$

and

$$\frac{1}{r}\frac{\partial \ln R}{\partial \theta} = -\frac{\partial \psi}{\partial r} \Rightarrow \frac{1}{rR}\frac{\partial R}{\partial \theta} = -\frac{\partial \psi}{\partial r}. \tag{2.3}$$

Differentiation partially (2.2) w.r.t. r and (2.3) w.r.t. θ respectively. we get

$$\frac{\partial^2 \psi}{\partial r \partial \theta} = \frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{R^2}\left(\frac{\partial R}{\partial r}\right)^2 + \frac{r}{R}\frac{\partial^2 R}{\partial r^2}, \tag{2.4}$$

$$\frac{\partial^2 \psi}{\partial \theta \partial r} = \frac{1}{rR^2}\left(\frac{\partial R}{\partial \theta}\right)^2 - \frac{1}{rR}\frac{\partial^2 R}{\partial \theta^2}, \tag{2.5}$$

being amplitude of analytic function $\psi(r.\theta)$ have second continuous partial derivative w.r.t. r and θ hence from (2.4) and (2.5) we have

$$\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{R^2}\left(\frac{\partial R}{\partial r}\right)^2 + \frac{r}{R}\frac{\partial^2 R}{\partial r^2} = \frac{1}{rR^2}\left(\frac{\partial R}{\partial \theta}\right)^2 - \frac{1}{rR}\frac{\partial^2 R}{\partial \theta^2}, \tag{2.6}$$

suppose $R(r,\theta) = X(r)Y(\theta)$ is a solution of partial differential equation in (2.6), where X is function of r only and Y is function of θ only. Then

$$\frac{X'Y}{XY} - \frac{r(X'Y)^2}{X^2Y^2} + \frac{rX''Y}{XY} = \frac{(XY')^2}{rX^2Y^2} - \frac{XY''}{rXY}$$

$$\Rightarrow r\left(\frac{X'}{X} - \frac{r(X')^2}{X^2} + \frac{rX''}{X}\right) = \frac{(Y')^2}{Y^2} - \frac{Y''}{Y}.$$

Since L.H.S is a function of r only and R.H.S. is function of θ only hence either side must be equal to a constant say λ. Then we have

$$r(\frac{X'}{X} - \frac{r(X')^2}{X^2} + \frac{rX''}{X}) = \lambda$$

and

$$\frac{Y''}{Y} - \frac{(Y')^2}{Y^2} = -\lambda$$

solving above equations we get

$$X_\lambda(r) = A_\lambda e^{\frac{\lambda(\ln r)^2}{2} + a_\lambda \ln r} \text{ and } Y_\lambda(r) = B_\lambda e^{\frac{-\lambda\theta^2}{2} + b_\lambda\theta}$$

hence we have the result

$$R(r,\theta) = \sum_{\lambda=0}^{\infty} C_\lambda X_\lambda(r) Y_\lambda(\theta) \ ; C_\lambda = A_\lambda B_\lambda$$

Since if $\lambda < 0$ then $lim_{r\to 0} X_\lambda(r) = 0$ any $\lambda < 0$ hence it implies $R(0,\theta) = 0\ \forall$ $f \in H[a,n]$ which is wrong if $a \neq 0$. So we must take $\lambda \geqslant 0$.

Theorem 2. *If $f \in H[0,n]$ i.e. $f(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ . \ . \ .$ such that*

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{U_r}, r < 1\}$$

and

$$|f'(z_1)| = \max\{|f'(z)| : z \in \overline{U_r}, r < 1\}$$

Then $z_1 = z_0$, i.e. maximum of modulus of $f'(z)$ is attained at the same point on which $|f(z)|$ attains its maximum.

Proof. We have $f(z) = Re^{i\psi}$ taking log on both side we get

$$\ln f = R + i\psi$$

differentiating with respect to z we have

$$\frac{1}{f}\frac{df}{dz} = \left(\frac{1}{R}\frac{\partial R}{\partial \theta} + i\frac{\partial \psi}{\partial \theta}\right)\frac{d\theta}{dz}$$

where

$$z = re^{i\theta} \Rightarrow \frac{d\theta}{dz} = \frac{-i}{z}$$

this gives

$$\frac{z}{f}\frac{df}{dz} = \frac{-i}{R}\frac{\partial R}{\partial \theta} + \frac{\partial \psi}{\partial \theta} \tag{2.7}$$

for a fixed r, at maximum of R ,$\frac{\partial R}{\partial \theta}|_{\theta_0} = 0$ and $\frac{\partial^2 R}{\partial \theta^2}|_{\theta_0} \leq 0$ from (2.7)

$$f^{'}(z) = \left(\frac{\partial \psi}{\partial \theta} - \frac{i}{R}\frac{\partial R}{\partial \theta}\right)\frac{f(z)}{z} = \left(\frac{\partial \psi}{\partial \theta} - \frac{i}{R}\frac{\partial R}{\partial \theta}\right)\frac{Re^{i\psi}}{re^{i\theta}},$$

so

$$|f^{'}(z)| = \frac{R}{r}\left|\frac{\partial \psi}{\partial \theta} - \frac{i}{R}\frac{\partial R}{\partial \theta}\right|,$$

now put

$$L = |f^{'}(z)|^2 = \left(\frac{R}{r}\right)^2\left(\left(\frac{\partial \psi}{\partial \theta}\right)^2 + \frac{1}{R^2}\left(\frac{\partial R}{\partial \theta}\right)^2\right), \tag{2.8}$$

when r is fixed, for maximum of L we must have $\frac{\partial L}{\partial \theta} = 0$ from (2.8) we have

$$\frac{\partial L}{\partial \theta} = \frac{2}{r^2}\left[\frac{\partial R}{\partial \theta}\left(\frac{\partial^2 R}{\partial \theta^2} + R\left(\frac{\partial \psi}{\partial \theta}\right)^2\right) + R^2\frac{\partial \psi}{\partial \theta}\frac{\partial^2 \psi}{\partial \theta^2}\right], \tag{2.9}$$

but from Cauchy Riemann equations in equation (2.2) and (2.3) in Theorem 1 we have

$$\frac{\partial^2 \psi}{\partial \theta^2} = -\frac{r}{R^2}\frac{\partial R}{\partial \theta}\frac{\partial R}{\partial r} + \frac{r}{R}\frac{\partial^2 R}{\partial \theta \partial r} \tag{2.10}$$

From Theorem 2.

$$\frac{\partial R}{\partial \theta} = \sum_{\lambda=0}^{\infty} C_\lambda X_\lambda(r)\frac{\partial Y_\lambda(\theta)}{\partial \theta} = \sum_{\lambda=0}^{\infty} C_\lambda X_\lambda(r)Y_\lambda(\theta)(-\lambda\theta + b_\lambda)$$

Since from expression of $X_\lambda(r), Y_\lambda(\theta)$ it is obvious that $A_\lambda, B_\lambda, C_\lambda, X_\lambda(r), Y_\lambda(\theta)$ all are greater than or equal to zero $\forall\ \lambda \in \mathbb{R}$ hence

$$\frac{\partial R}{\partial \theta} = 0 \Rightarrow -\lambda\theta + b_\lambda = 0 \ when\ \theta = \theta_0$$

therefore at $\theta = \theta_0$

$$\frac{\partial^2 R}{\partial r \partial \theta}|_{\theta_0} = \sum_{\lambda=0}^{\infty} C_\lambda \frac{\partial X_\lambda(r)}{\partial r} Y_\lambda(\theta)(-\lambda\theta + b_\lambda)|_{\theta_0} = 0 \tag{2.11}$$

from equations (2.10) and (2.11) we have

$$\frac{\partial R}{\partial \theta} = 0 \ ; \frac{\partial^2 \psi}{\partial \theta^2} = 0 \ at \ \theta = \theta_0$$

hence

$$\frac{\partial L}{\partial \theta} = 0 \ at \ \theta = \theta_0$$

We have to show that $\frac{\partial^2 L}{\partial \theta^2} \leq 0$ at $\theta = \theta_0$.

$$\frac{\partial^2 L}{\partial \theta^2} = \frac{2}{r^2} \left[\frac{\partial^2 R}{\partial \theta^2} \left(\frac{\partial^2 R}{\partial \theta^2} + R \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right) \right. \tag{2.12}$$

$$+ \frac{\partial R}{\partial \theta} \left(\frac{\partial^3 R}{\partial \theta^3} + \frac{\partial R}{\partial \theta} \left(\frac{\partial \psi}{\partial \theta} \right)^2 + 2R \frac{\partial \psi}{\partial \theta} \frac{\partial^2 \psi}{\partial \theta^2} \right) \tag{2.13}$$

$$\left. +2R \frac{\partial R}{\partial \theta} \frac{\partial \psi}{\partial \theta} \frac{\partial^2 \psi}{\partial \theta^2} + R^2 \left(\frac{\partial^2 \psi}{\partial \theta^2} \right)^2 + R^2 \frac{\partial \psi}{\partial \theta} \frac{\partial^3 \psi}{\partial \theta^3} \right].$$

Since at $\theta = \theta_0$, $\frac{\partial R}{\partial \theta} = 0$ and $\frac{\partial^2 \psi}{\partial \theta^2} = 0$ we have at $\theta = \theta_0$

$$\frac{\partial^2 L}{\partial \theta^2} = \frac{2}{r^2} \left[\frac{\partial^2 R}{\partial \theta^2} \left(\frac{\partial^2 R}{\partial \theta^2} + R \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right) + R^2 \frac{\partial \psi}{\partial \theta} \frac{\partial^3 \psi}{\partial \theta^3} \right] \tag{2.14}$$

again from CR equations we have

$$\frac{\partial^3 \psi}{\partial \theta^3} = \frac{\partial}{\partial \theta^2} \left(\frac{r}{R} \frac{\partial R}{\partial r} \right) \tag{2.15}$$

$$= r \left(\frac{2}{R^3} \left(\frac{\partial R}{\partial \theta} \right)^2 \frac{\partial R}{\partial r} - \frac{1}{R^2} \frac{\partial^2 R}{\partial \theta^2} \frac{\partial R}{\partial r} - \frac{2}{R^2} \frac{\partial R}{\partial \theta} \frac{\partial^2 R}{\partial r \partial \theta} + \frac{1}{R} \frac{\partial^3 R}{\partial \theta^2 \partial r} \right),$$

at $\theta = \theta_0$

$$\frac{\partial^3 \psi}{\partial \theta^3} = r \left(-\frac{1}{R^2} \frac{\partial^2 R}{\partial \theta^2} \frac{\partial R}{\partial r} + \frac{1}{R} \frac{\partial^3 R}{\partial \theta^2 \partial r} \right) \tag{2.16}$$

$$\text{so at } \theta = \theta_0 \ , \frac{\partial \psi}{\partial \theta} \frac{\partial^3 \psi}{\partial \theta^3} = -\frac{r^2}{R^3} \frac{\partial^2 R}{\partial \theta^2} \left(\frac{\partial R}{\partial r} \right)^2 + \frac{r^2}{R^2} \frac{\partial R}{\partial r} \frac{\partial^3 R}{\partial \theta^2 \partial r},$$

from equations (2.14), (2.16) and CR equations at $\theta = \theta_0$ we get

$$\frac{\partial^2 L}{\partial \theta^2} = \frac{2}{r^2}\left[\left(\frac{\partial^2 R}{\partial \theta^2}\right)^2 + r^2 \frac{\partial R}{\partial r}\frac{\partial^3 R}{\partial \theta^2 \partial r}\right], \tag{2.17}$$

again from CR equations

$$\begin{aligned} \frac{\partial^2 R}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(-rR\frac{\partial \psi}{\partial r}\right) \\ &= -r\frac{\partial R}{\partial \theta}\frac{\partial \psi}{\partial r} - rR\frac{\partial^2 \psi}{\partial \theta \partial r} \\ at\ \theta = \theta_0, \frac{\partial^2 R}{\partial \theta^2} &= -rR\frac{\partial^2 \psi}{\partial \theta \partial r} \\ \Rightarrow \left(\frac{\partial^2 R}{\partial \theta^2}\right)^2 &= r^2 R^2\left(\frac{\partial^2 \psi}{\partial \theta \partial r}\right)^2, \end{aligned} \tag{2.18}$$

similarly $\frac{\partial^3 R}{\partial \theta^2 \partial r}$

$$= -\frac{\partial \psi}{\partial r}\frac{\partial R}{\partial \theta} - r\frac{\partial^2 R}{\partial \theta \partial r}\frac{\partial \psi}{\partial r} - r\frac{\partial R}{\partial \theta}\frac{\partial^2 \psi}{\partial r^2} - R\frac{\partial^2 \psi}{\partial \theta \partial r} - r\frac{\partial R}{\partial \theta}\frac{\partial^2 \psi}{\partial \theta \partial r} - rR\frac{\partial^3 \psi}{\partial r \partial \theta^2} \tag{2.19}$$

at $\theta = \theta_0$,

$$\frac{\partial^3 R}{\partial \theta^2 \partial r} = -R\frac{\partial^2 \psi}{\partial \theta \partial r} - rR\frac{\partial^3 \psi}{\partial r \partial \theta^2},$$

from equations (2.17, 2.18, 2.19) and CR equations we have at $\theta = \theta_0$

$$\frac{\partial^2 L}{\partial \theta^2} = 2R^2\left[\frac{\partial^2 \psi}{\partial \theta \partial r} - \frac{R}{r^2}(\frac{\partial \psi}{\partial \theta})^2\right]\frac{\partial^2 \psi}{\partial \theta \partial r}, \tag{2.20}$$

from (2.10) it can be proven that

$$\frac{\partial^3 \psi}{\partial r \partial \theta^2} = 0\ at\ \ \theta = \theta_0$$

since we have

$$f(z) = R(r,\theta)e^{i\psi(r.\theta)}.$$

$$\Rightarrow z\frac{f'(z)}{f(z)} = \frac{\partial \psi}{\partial \theta} - i\frac{1}{R}\frac{\partial R}{\partial \theta}$$

$$\Rightarrow \frac{(f'(z) + zf''(z))f(z) - z(f'(z))^2}{(f(z))^2}$$

$$= \frac{r}{z}\left(\frac{\partial^2\psi}{\partial\theta\partial r} + i\frac{1}{R^2}\frac{\partial R}{\partial r}\frac{\partial R}{\partial\theta} - i\frac{1}{R}\frac{\partial^2 R}{\partial r\partial\theta}\right).$$

at $\theta = \theta_0$ we have

$$z_0\frac{f'(z_0)}{f(z_0)} + \frac{z_0^2 f''(z_0)}{f(z_0)} - \left(z_0\frac{f'(z_0)}{f(z_0)}\right)^2 = r\frac{\partial^2\psi}{\partial\theta\partial r},$$

sin be we have

$$\frac{z_0^2 f''(z_0)}{f(z_0)} - \left(z_0\frac{f'(z_0)}{f(z_0)}\right)^2 = \frac{z_0 f''(z_0)}{f'(z_0)}\frac{z_0 f'(z_0)}{f(z_0)} - \left(z_0\frac{f'(z_0)}{f(z_0)}\right)^2.$$

Corollary 1. *From Lemma* A ([2, Miller/Mocanu lemma]) *equations (1.8) and (1.9) we have*

$$\begin{aligned}\frac{z_0 f''(z_0)}{f'(z_0)}\frac{z_0 f'(z_0)}{f(z_0)} - \left(z_0\frac{f'(z_0)}{f(z_0)}\right)^2 &\geq (m-1)m - m^2 \\ &= -m \leq 0,\end{aligned}$$

therefore

$$\begin{aligned}\frac{z_0 f''(z_0)}{f'(z_0)} - m &\geq r\frac{\partial^2\psi}{\partial\theta\partial r} \\ &\Rightarrow \frac{z_0 f''(z_0)}{f'(z_0)} \geq r\frac{\partial^2\psi}{\partial\theta\partial r} \\ \textit{when } r &\to 1 \textit{ we have} \\ \frac{z_0 f''(z_0)}{f'(z_0)} &= \frac{\partial\psi}{\partial\theta} \geq \frac{\partial^2\psi}{\partial\theta\partial r},\end{aligned}$$

so for $r < 1$ we must have

$$\frac{\partial^2\psi}{\partial\theta\partial r} - \frac{R}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 \leq 0 \textit{ when } \theta = \theta_0 \textit{ and } f \in H[0,n]\ n \geq 1. \tag{2.21}$$

Therefore from equations (2.20) and (2.21) we have

$$\frac{\partial^2 L}{\partial\theta^2} \leq 0, \textit{when } \theta = \theta_0,$$

hence $z_1 = z_0$ i.e. f and f' both attains maximum of their modulus at the same point . From principle of regularity and mathematical induction we can prove that *an analytic function and all its derivative attains maximum of their modulus at the same point and this point need not be unique.*

Theorem 3. *Let $z_0 \in U$ and $r_0 = |z_0|$ and let*

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + ...$$

be continuous on $\overline{U}_{r_0}$ and analytic on U_{r_0} with $f(z)$ not identically zero, and $n \geq 1$. If

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{U_r}, r < 1\}$$

then there exists m_r for $n \geq k \geq 1$ such that

$$\frac{z_0 f^{(k)}(z_0)}{f^{(k-1)}(z_0)} \leq \frac{z_0 f^{(k-1)}(z_0)}{f^{(k-2)}(z_0)}. \tag{2.22}$$

$$\frac{z_0 f^{(k)}(z_0)}{f^{(k-1)}(z_0)} + k - 1 = m_k.$$

such that

$$m_k \geq m_{k-1} \geq .... \geq m_1 \geq n.$$

Proof. As we have seen in the proof of the lemma A(Miller and Mocanu lemma) when $f(z) = a_n z^n + a_{n+1} z^{n+1} + ...$ and z_0 is the point at which $|f(z)|$ is maximum on $\overline{U}_{r_0}$ then taking

$$g(z) = \frac{f(z_0 z)}{f(z_0) z^{n-1}},$$

then $|g(z)| \leq |z| \Rightarrow |\frac{f(z_0 z)}{f(z_0) z^{n-1}}| \leq |z|$ moreover $g(0) = 0$ and z^n is analytic univalent therefore $\frac{f(z_0 z)}{f(z_0)} \prec z^n$. Since from Theorem 3, $|f^{(k)}(z)|$ is maximum at z_0 for $1 \leq k \leq n$, similarly we can prove that

$$\begin{aligned} f'(z) &= n a_n z^n + (n+1) a_{n+1} z^{n+1} + ... \\ &= b_{n-1} z^{n-1} + b_n z^n + ... \end{aligned}$$

implies $\frac{f'(z_0 z)}{f(z_0)} \prec z^{n-1}$... continuing in this way we show that $\frac{f^{(n)}(z_0 z)}{f^{(n)}(z_0)} \prec z$. Since

$$z^n \prec z^{n-1} \prec z^{n-2} \prec z^{n-3} ... \prec z^2 \prec z \ for \ z \in U. \tag{2.23}$$

equation (2.23) implies that

$$\frac{f(z_0 z)}{f(z_0)} \prec \frac{f'(z_0 z)}{f'(z_0)} \prec \frac{f''(z_0 z)}{f''(z_0)} \prec \frac{f^{(k)}(z_0 z)}{f^{(k)}(z_0)} ... \prec \frac{f^{(n)}(z_0 z)}{f^{(n)}(z_0)} \prec z. \tag{2.24}$$

Since we have

$$\frac{d}{dr}\left[\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}\right]|_{r=1} = \lim_{r\to 1}\frac{f^{(k-1)}(z_0r)-f^{(k-1)}(z_0)}{(r-1)f^{(k-1)}(z_0)}. \tag{2.25}$$

$$= \lim_{r\to 1}\left[1-\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}\right]\frac{1}{1-r} = \lim_{r\to 1}\left[1-Re\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}\right]\frac{1}{1-r}.$$

From implications in equations (2.23) and (2.24) we have

$$\text{for } 1\le k\le n \tag{2.26}$$

$$\text{and } \lim_{r\to 1}\left[1-Re\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}\right]\frac{1}{1-r} \ge \lim_{r\to 1}\frac{1-r^{n-k+1}}{1-r} = n-(k-1),$$

$$\Rightarrow \frac{z_0f^{(k)}(z_0)}{f^{(k-1)}(z_0)}+k-1 = m_k \ge n,$$

$$\text{and } \lim_{r\to 1}\left[1-Re\frac{f^{(k)}(z_0r)}{f^{(k)}(z_0)}\right]\frac{1}{1-r} \le \lim_{r\to 1}\left[1-Re\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}\right]\frac{1}{1-r}.$$

Moreover from equation (2.24)

$$Re\frac{f^{(k)}(z_0r)}{f^{(k)}(z_0)} - Re\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)} \le r^{n-k}-r^{n+1-k}$$

$$\Rightarrow Re\frac{f^{(k)}(z_0r)}{f^{(k)}(z_0)} + r^{n+1-k} \le r^{n-k} + Re\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}$$

$$\Rightarrow -Re\frac{f^{(k)}(z_0r)}{f^{(k)}(z_0)} - r^{n+1-k} \ge -r^{n-k} - Re\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}$$

$$\lim_{r\to 1}\left\{\left(1-Re\frac{f^{(k)}(z_0r)}{f^{(k)}(z_0)}\right)\frac{1}{1-r}+\frac{r^{n-k}-r^{n+1-k}}{1-r}\right\} \ge \lim_{r\to 1}\left[1-Re\frac{f^{(k-1)}(z_0r)}{f^{(k-1)}(z_0)}\right]\frac{1}{1-r}$$

$$\frac{z_0f^{(k+1)}(z_0)}{f^{(k)}(z_0)}+1 \ge \frac{z_0f^{(k)}(z_0)}{f^{(k-1)}(z_0)} \ge n-(k-1)$$

$$\Rightarrow m_k \ge m_{k-1}.$$

Therefore from equations (2.25) and (2.26) we conclude that $m_k \ge m_{k-1} \ge .... \ge m_1 \ge n$ for all $1 \le k \le n$.

Remark 1. *Result in Theorem 3 generalizes Miller and mocanu lemma for n-th order derivative of an analytic function.*

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