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MAXIMAL AND MINIMAL PSEUDO SYMMETRIC IDEALS IN PARTIALLY ORDERED TERNARY SEMIGROUPS

Machchhindra T. Gophane and Dattatray N. Shinde*

Department of Mathematics,
Shivaji University,
Kolhapur - 416004, Maharashtra, INDIA

E-mail : mtg_maths@unishivaji.ac.in

*Department of Mathematics,
PES's College of Engineering,
Phaltan - 415523, Maharashtra, INDIA

E-mail : shindednmaths@gmail.com

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Abstract: We have introduced the notions of maximal and minimal pseudo symmetric ideals of a partially ordered ternary semigroup T and studied their properties. We show that every maximal pseudo symmetric ideal of a commutative partially ordered ternary semigroup with identity is a prime pseudo symmetric ideal. We gave an example to show that the converse of this statement is not true.

Keywords and Phrases: Partially ordered ternary semigroup, pseudo symmetric ideal, maximal pseudo symmetric ideal, minimal pseudo symmetric ideal.

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1. Introduction

The concept of a ternary semigroup was first introduced by Lehmer [9], who also established the theory of a ternary algebraic system in 1932. In 1965, F. M. Sioson introduced the ideal theory of ternary semigroups. Iampan [4] has defined the concept of partially ordered ternary semigroups and he also developed the theory of partially ordered ternary semigroups. In [11, 12], Siva Rami Reddy et al. have developed the ideal theory of a partially ordered ternary semigroup. In [6], Jyothi et al. explored the concepts of semipseudo symmetric ideals and pseudo symmetric ideals in partially ordered ternary semigroups. The minimal bi-ideals and maximal bi-ideals in ordered ternary semigroups was studied by Chinram et al. in [2]. Jailoka and Iampan [5] conducted a study on the properties of minimality and maximality of ordered quasi-ideals in ordered ternary semigroups.The notion of maximal ideals in ordered semigroups was studied by Kehayopulu et al. in [7, 8]. Changphas [1] studied the some important properties of maximal ideals in ternary semigroups.

Our aim of this article is to introduce the concepts of maximal and minimal pseudo symmetric ideals of a partially ordered ternary semigroup T and to study their properties.

2. Preliminaries

Definition 2.1. [9] *A non-empty set T with a ternary operation $[\ ] : T \times T \times T \longrightarrow T$ is said to be a ternary semigroup if $[\ ]$ satisfies the condition, $[p\ q\ r\ s\ t] = [[p\ q\ r]\ s\ t] = [p\ [q\ r\ s]\ t] = [p\ q\ [r\ s\ t]]$, for all $p, q, r, s, t \in T$.*

Definition 2.2. [4] *A ternary semigroup T is said to be a partially ordered ternary semigroup if there exist a partially ordered relation $\leq$ on T such that, $s \leq t \Rightarrow pqs \leq pqt, psq \leq ptq, spq \leq tpq$ for all $s, t, p, q \in T$.*

Definition 2.3. [11] *An element $0 \in T$ is called a zero of T if $0pq = p0q = pq0 = 0$ and $0 \leq t$ for all $p, q, t \in T$.*

Definition 2.4. [11] *An element $e \in T$ is called an identity element of T if $epp = ppe = pep = p$ and $p \leq e$ for all $p \in T$.*

Let $\emptyset \neq X \subseteq T$. Then the set $\{p \in T : p \leq x$, for some elements $x \in X\}$ is denoted by $(X]$. The set $(X]$ is also called as downward closure.

Definition 2.5. [6] *A partially ordered ternary semigroup T is called an commutative if $xyz = zxy = yzx = yxz = zyx = xzy\ \forall x, y, z \in T$.*

Definition 2.6. [3] *A non-empty subset X of T is said to be an ideal of T if*
(1) $TTX \subseteq X$, $XTT \subseteq X$, $TXT \subseteq X$.
(2) $(X] = X$.

Definition 2.7. [11] *Let X be a non-empty subset of T. The ideal of T generated by X, denoted by $\langle X\rangle$, is defined as the intersection of all ideals of T containing X.*

Definition 2.8. [6] *An ideal X of T is called a pseudo symmetric ideal of T if*

$a, b, c \in T,\ abc \in X \Rightarrow asbtc \in X\ \forall s, t \in T.$

Definition 2.9. [10] *Let X be a non-empty subset of T. The pseudo symmetric ideal of T generated by X, denoted by $(X)_{ps}$, is defined as the intersection of all pseudo symmetric ideals of T containing X. If $X = \{x\}$ for some $x \in T$, then the principal pseudo-symmetric ideal generated by x, denoted by $(x)_{ps}$.*

Definition 2.10. [10] *Let X be a pseudo symmetric ideal of T. Then X is said to be a maximal pseudo symmetric ideal of T if $X \neq T$ and there does not exist any proper pseudo symmetric ideal Y of T such that $X \subset Y \subset T$.*

3. Main Results

Let $\{(T_\alpha, [\ ]_\alpha, \leq_\alpha) : \alpha \in \Delta$ is any indexing set$\}$ be a non-empty collection of partially ordered ternary semigroups T_α .

Theorem 3.1. *The Cartesian product $\prod_{\alpha\in\Delta} T_\alpha$ with a ternary operation $[\ ] : \prod_{\alpha\in\Delta} T_\alpha \times \prod_{\alpha\in\Delta} T_\alpha \times \prod_{\alpha\in\Delta} T_\alpha \to \prod_{\alpha\in\Delta} T_\alpha$ defined by,*

$$((x_\alpha)_{\alpha\in\Delta}, (y_\alpha)_{\alpha\in\Delta}, (z_\alpha)_{\alpha\in\Delta}) \to [(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}]$$

where $[(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}] = ([x_\alpha y_\alpha z_\alpha]_\alpha)_{\alpha\in\Delta}$ and a partially ordered relation $\leq$ on $\prod_{\alpha\in\Delta} T_\alpha$ defined by,

$$\leq := \{((x_\alpha)_{\alpha\in\Delta}, (y_\alpha)_{\alpha\in\Delta}) \in \textstyle\prod_{\alpha\in\Delta} T_\alpha \times \prod_{\alpha\in\Delta} T_\alpha : x_\alpha \leq_\alpha y_\alpha \text{ for all } \alpha \in \Delta\}$$

is a partially ordered ternary semigroup.

Proof. Since $T_\alpha \neq \emptyset\ \forall \alpha \in \Delta$, we have $\prod_{\alpha\in\Delta} T_\alpha \neq \emptyset$. Firstly we will prove that $\prod_{\alpha\in\Delta} T_\alpha$ is a ternary semigroup.

Let $(x_\alpha)_{\alpha\in\Delta}, (y_\alpha)_{\alpha\in\Delta}, (z_\alpha)_{\alpha\in\Delta}, (a_\alpha)_{\alpha\in\Delta}, (b_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} T_\alpha$. Consider,

$$\begin{aligned}
[[(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}](a_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}] &= [([x_\alpha y_\alpha z_\alpha]_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}] \\
&= ([[x_\alpha y_\alpha z_\alpha]_\alpha (a_\alpha)(b_\alpha)]_\alpha)_{\alpha\in\Delta} \\
&= ([(x_\alpha)[y_\alpha z_\alpha a_\alpha]_\alpha (b_\alpha)]_\alpha)_{\alpha\in\Delta} \\
&= [(x_\alpha)_{\alpha\in\Delta}([y_\alpha z_\alpha a_\alpha]_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}] \\
&= [(x_\alpha)_{\alpha\in\Delta}[(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}](b_\alpha)_{\alpha\in\Delta}]
\end{aligned}$$

Similarly, we can prove that

$$[(x_\alpha)_{\alpha\in\Delta}[(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}](b_\alpha)_{\alpha\in\Delta}] = [(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}[(z_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}]].$$

Hence,

$$\begin{aligned}
&[[(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}](a_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}] \\
&\quad = [(x_\alpha)_{\alpha\in\Delta}[(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}](b_\alpha)_{\alpha\in\Delta}] \\
&\quad = [(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}[(z_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}]].
\end{aligned}$$

Therefore $\prod_{\alpha\in\Delta} T_\alpha$ is a ternary semigroup.

Now, let $(x_\alpha)_{\alpha\in\Delta}, (y_\alpha)_{\alpha\in\Delta}, (a_\alpha)_{\alpha\in\Delta}, (b_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} T_\alpha$ such that $(a_\alpha)_{\alpha\in\Delta} \leq (b_\alpha)_{\alpha\in\Delta} \Rightarrow a_\alpha \leq_\alpha b_\alpha \Rightarrow x_\alpha y_\alpha a_\alpha \leq_\alpha x_\alpha y_\alpha b_\alpha$ (since T_α is a partially ordered ternary semigroup) $\Rightarrow ([x_\alpha y_\alpha a_\alpha]_\alpha)_{\alpha\in\Delta} \leq ([x_\alpha y_\alpha b_\alpha]_\alpha)_{\alpha\in\Delta} \Rightarrow [(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}] \leq [(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}]$. Similarly, we can prove that $[(x_\alpha)_{\alpha\in\Delta}(a_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}] \leq [(x_\alpha)_{\alpha\in\Delta}(b_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}]$ and $[(a_\alpha)_{\alpha\in\Delta}(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}] \leq [(b_\alpha)_{\alpha\in\Delta}(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}]$. Thus $\prod_{\alpha\in\Delta} T_\alpha$ is a partially ordered ternary semigroup.

Lemma 3.1. *Let $\{(T_\alpha, [\ ]_\alpha, \leq_\alpha) : \alpha \in \Delta$ is any indexing set$\}$ be a non-empty collection of partially ordered ternary semigroup. If I_α is a pseudo symmetric ideal of T_α for each $\alpha \in \Delta$, then $\prod_{\alpha\in\Delta} I_\alpha$ is a pseudo symmetric ideal of $\prod_{\alpha\in\Delta} T_\alpha$.*

Proof. (1) We have $I_\alpha \neq \emptyset\ \forall \alpha \in \Delta$. Then there exists $x_\alpha \in I_\alpha$ for each $\alpha \in \Delta$. Therefore $(x_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} I_\alpha \subseteq \prod_{\alpha\in\Delta} T_\alpha$, $\prod_{\alpha\in\Delta} I_\alpha \neq \emptyset$.

(2) Let $(x_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} I_\alpha$ and $(y_\alpha)_{\alpha\in\Delta}, (z_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} T_\alpha$. Since $[x_\alpha y_\alpha z_\alpha]_\alpha \in [I_\alpha T_\alpha T_\alpha]_\alpha \subseteq I_\alpha$ for every $\alpha \in \Delta$, it gives that

$$[(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}] = ([x_\alpha y_\alpha z_\alpha]_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} I_\alpha.$$

Then $[(\prod_{\alpha\in\Delta} I_\alpha)(\prod_{\alpha\in\Delta} T_\alpha)(\prod_{\alpha\in\Delta} T_\alpha)] \subseteq (\prod_{\alpha\in\Delta} I_\alpha)$. Similarly, we can prove that

$$[(\prod_{\alpha\in\Delta} T_\alpha)(\prod_{\alpha\in\Delta} I_\alpha)(\prod_{\alpha\in\Delta} T_\alpha)] \subseteq (\prod_{\alpha\in\Delta} I_\alpha)$$

and

$$[(\prod_{\alpha\in\Delta} T_\alpha)(\prod_{\alpha\in\Delta} T_\alpha)(\prod_{\alpha\in\Delta} I_\alpha)] \subseteq (\prod_{\alpha\in\Delta} I_\alpha).$$

(3) Let $(y_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} I_\alpha$ and $(x_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} T_\alpha$ such that $(x_\alpha)_{\alpha\in\Delta} \leq (y_\alpha)_{\alpha\in\Delta}$. Since $y_\alpha \in I_\alpha, x_\alpha \in T_\alpha$ such that $x_\alpha \leq_\alpha y_\alpha$ and I_α is a pseudo symmetric ideal of T_α for every $\alpha \in \Delta$, we have $x_\alpha \in I_\alpha$ for every $\alpha \in \Delta$. Then $(x_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} I_\alpha$. Hence $\prod_{\alpha\in\Delta} I_\alpha$ is an ideal of $\prod_{\alpha\in\Delta} T_\alpha$.

(4) Let $(x_\alpha)_{\alpha\in\Delta}, (y_\alpha)_{\alpha\in\Delta}, (z_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} T_\alpha$ such that,

$$[(x_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}] \in \prod_{\alpha\in\Delta} I_\alpha \Rightarrow ([x_\alpha y_\alpha z_\alpha]_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} I_\alpha \Rightarrow [x_\alpha y_\alpha z_\alpha]_\alpha \in I_\alpha.$$

For all $(s_\alpha)_{\alpha\in\Delta}, (t_\alpha)_{\alpha\in\Delta} \in \prod_{\alpha\in\Delta} T_\alpha$, since I_α is a pseudo symmetric ideal of T_α for each $\alpha \in \Delta$ then for all $s_\alpha, t_\alpha \in T_\alpha$, we get $[x_\alpha s_\alpha y_\alpha t_\alpha z_\alpha]_\alpha \in I_\alpha$. This implies that

$$[(x_\alpha)_{\alpha\in\Delta}(s_\alpha)_{\alpha\in\Delta}(y_\alpha)_{\alpha\in\Delta}(t_\alpha)_{\alpha\in\Delta}(z_\alpha)_{\alpha\in\Delta}] \in \prod_{\alpha\in\Delta} I_\alpha.$$

Therefore, the set $\prod_{\alpha\in\Delta} I_\alpha$ is a pseudo symmetric ideal of $\prod_{\alpha\in\Delta} T_\alpha$.

The closed interval $T = [0, 1]$ is a partially ordered ternary semigroup with respect to usual multiplication of numbers and a usual partially ordered relation

$\leq$.

Lemma 3.2. *If $\alpha \in T$, then the set $I_\alpha = [0, \alpha]$ is a pseudo symmetric ideal of T.*
Proof. (1) Since $\alpha \in [0, \alpha] = I_\alpha$. Then $I_\alpha \neq \emptyset$.

(2) Let $x \in I_\alpha$ and $y, z \in T$. Since $0 \leq x \leq \alpha, 0 \leq y, z \leq 1$, we have $0 \leq xyz, yxz, yzx \leq \alpha$. Then $xyz, yxz, yzx \in I_\alpha$

(3) Let $y \in I_\alpha$ and $x \in T$ such that $x \leq y$. Therefore $0 \leq x, y \leq \alpha$ and $x \leq y$ implies $0 \leq x \leq \alpha$. Then $x \in I_\alpha$.

(4) Let $x, y, z \in T$ such that $xyz \in I_\alpha$. Since $0 \leq x, y, z \leq 1$ and $0 \leq xyz \leq \alpha$. For all $s, t \in T = [0, 1]$, $0 \leq xsytz \leq \alpha$ implies $xsytz \in I_\alpha$. Hence $I_\alpha = [0, \alpha]$ is a pseudo symmetric ideal of T.

Definition 3.1. [10] *A proper pseudo symmetric ideal X of T is called a prime pseudo symmetric ideal of T if $PQR \subseteq X \Rightarrow P \subseteq X$ or $Q \subseteq X$ or $R \subseteq X$ where P, Q, R are the pseudo symmetric ideals of T.*

Theorem 3.2. [12] *An ideal X of a commutative partially ordered ternary semigroup T is a prime ideal if and only if $xyz \in X$ implies either $x \in X$ or $y \in X$ or $z \in X$ for all $x, y, z \in T$.*

Proposition 3.1. *Every ideal of commutative partially ordered ternary semigroup is a pseudo symmetric ideal.*

Theorem 3.3. [11] *Let X be the non-empty subset of T,then $\langle X \rangle = (X \cup TTX \cup TXT \cup XTT \cup TTXTT]$.*

Proposition 3.2. *If T is a commutative partially ordered ternary semigroup with identity and X is a nonempty subset of T, then $(X)_{ps} = (TTX] = (XTT] = (TXT]$.*
Proof. Since T is commutative. By Proposition 3.1 and Theorem 3.3, we have $(X)_{ps} = (X \cup TTX \cup TXT \cup XTT \cup TTXTT] = (X \cup TTX \cup TTTTX] = (X \cup TTX]$. Let e be the identity element of T, we have $X = eeX \subseteq TTX$. Thus $(X)_{ps} = (TTX]$. Similarly, we can prove that $(X)_{ps} = (XTT]$ and $(X)_{ps} = (TXT]$.

Theorem 3.4. *If T is a commutative partially ordered ternary semigroup with identity and X is a maximal pseudo symmetric ideal of T, then X is a prime pseudo symmetric ideal of T.*
Proof. Let e be the identity element of T and X is a maximal pseudo symmetric ideal of T. To show that X is a prime pseudo symmetric ideal of T, let $x, y, z \in T$ such that $xyz \in X$ and $x, y, z \notin X$. Since T is commutative, we have $(X \cup \{x\})_{ps} = ((X \cup \{x\}) \cup TT(X \cup \{x\}) \cup T(X \cup \{x\})T \cup (X \cup \{x\})TT \cup TT(X \cup \{x\})TT] =$

$((X\cup\{x\})\cup TT(X\cup\{x\})]$. Since $X\cup\{x\}=ee(X\cup\{x\})\subseteq TT(X\cup\{x\})$, we have

$$(X\cup\{x\})_{ps}=(TT(X\cup\{x\})] \tag{3.1}$$

Since $x\notin X, X\subset X\cup\{x\}\subseteq(X\cup\{x\})_{ps}$. Since X is a maximal pseudo symmetric ideal and $(X\cup\{x\})_{ps}$ is a pseudo symmetric ideal of T, we have $(X\cup\{x\})_{ps}=T$. By (3.1),

$$T=(TT(X\cup\{x\})] \tag{3.2}$$

Similarly, we obtain

$$T=(TT(X\cup\{y\})] \tag{3.3}$$

and

$$T=(TT(X\cup\{z\})] \tag{3.4}$$

Since $y\in T$ and from equation (3.4), there exist $a_1,b_1\in T$ and $u\in X\cup\{z\}$ such that $y\leq a_1b_1u$. If $u\in X$ then $a_1b_1u\in TTX\subseteq X$ and $y\in X$. This is contradiction. If $u=z$ then

$$y\leq a_1b_1z \tag{3.5}$$

Again since $b_1\in T$ and from equation (3.3), there exist $a_2,b_2\in T$ and $v\in X\cup\{y\}$ such that $b_1\leq a_2b_2v$. If $v\in X$ then $y\in X$. This is contradiction. If $v=y$ then $b_1\leq a_2b_2y$ and so, from equation (3.5)

$$y\leq a_1(a_2b_2y)z \tag{3.6}$$

Finally, since $b_2\in T$ and from equation (3.2), there exist $a_3,b_3\in T$ and $w\in X\cup\{x\}$ such that $b_2\leq a_3b_3w$. If $w\in X$ then $y\in X$. This is contradiction. If $w=x$ then $b_2\leq a_3b_3x$ and so, from equation (3.6)

$$y\leq a_1(a_2(a_3b_3x)y)z\in T(T(TTx)y)z=TTTT(xyz)\subseteq TTX\subseteq X.$$

$y\in X$. This is impossible. Thus X is a prime pseudo symmetric ideal of T.

The converse statement of above Theorem 3.4 is not true. We illustrate this by the following example,

Example 3.1. Consider the commutative partially ordered ternary semigroup $S=T\times T=[0,1]\times[0,1]$. Clearly, $e=(1,1)$ is the identity element of S. By Lemma 3.2, $I_0=\{0\}$ is a pseudo symmetric ideal of T. By Lemma 3.1, $I=T\times\{0\}=[0,1]\times\{0\}$ is a pseudo symmetric ideal of S. By using Theorem 3.2, we can show that I is a prime pseudo symmetric ideal of S. In fact: Let $(a_1,b_1),(a_2,b_2),(a_3,b_3)\in S,[(a_1,b_1)(a_2,b_2)(a_3,b_3)]\in I$. Thus $(a_1a_2a_3,b_1b_2b_3)\in I=[0,1]\times\{0\}$, we have $b_1b_2b_3=0$, then $b_1=0$ or $b_2=0$ or $b_3=0$. This implies that $(a_1, b_1) \in I$ or $(a_2, b_2) \in I$ or $(a_3, b_3) \in I$. Therefore I is a prime pseudo symmetric ideal of S. By Lemma 3.2, $I_{1/2} = [0, 1/2]$ is a pseudo symmetric ideal of T. Since, $I \subset T \times I_{1/2} \subset T \times T = S$. Hence I is not maximal pseudo symmetric ideal.

Theorem 3.5. *Let X be a proper pseudo symmetric ideal of T. Then X is a maximal pseudo symmetric ideal of T if and only if $(X \cup \{x\})_{ps} = T$ for all $x \in T \setminus X$.*

Proof. Let $x \in T \setminus X$. Since $x \notin X, X \subset X \cup \{x\} \subseteq (X \cup \{x\})_{ps}$, X is a maximal pseudo symmetric ideal of T and $(X \cup \{x\})_{ps}$ is a pseudo symmetric ideal of T, we have $(X \cup \{x\})_{ps} = X$ or $(X \cup \{x\})_{ps} = T$. Since $x \in (X \cup \{x\})_{ps}$ and $x \notin X$, we have $(X \cup \{x\})_{ps} \neq X$. Therefore $(X \cup \{x\})_{ps} = T$.

Conversely, assume that $(X \cup \{x\})_{ps} = T \ \ \forall\ x \in T \setminus X$. Let I be a pseudo symmetric ideal of T such that $X \subseteq I$ and $X \neq I$. Then there exist an element $x \in I$ such that $x \notin X$, so $x \in T \setminus X$. Since $X \subseteq I$ and $x \in I$, we have $X \cup \{x\} \subseteq I$. By Definition 2.9, $(X \cup \{x\})_{ps} \subseteq I$ implies that $T = (X \cup \{x\})_{ps} \subseteq I$. Therefore $T = I$. Hence X is a maximal pseudo symmetric ideal of T.

Corollary 3.1. *If X is a maximal pseudo symmetric ideal of T then $(X \cup \{x\})_{ps} = T$ for all $x \in T \setminus X$.*

Theorem 3.6. *Let T be a partially ordered ternary semigroup for which there exists an element $x \in T$ such that $T \subseteq (x)_{ps}$. Then each proper pseudo symmetric ideal of T is contained in a maximal pseudo symmetric ideal of T.*

Proof. Let X be a proper pseudo symmetric ideal of T. We consider the set, $\mathcal{A} = \{Y : Y$ pseudo symmetric ideal of T, $X \subseteq Y \subset T\}$. Since $X \in \mathcal{A}$, we have $\mathcal{A} \neq \emptyset$, then the set $M = \bigcup\{Y : Y \in \mathcal{A}\}$ is a pseudo symmetric ideal of T and $X \subseteq M$. Now, we show that, the set M is a maximal pseudo symmetric ideal of T. If $M = T$ then $x \in M = \bigcup\{Y : Y \in \mathcal{A}\}$. Then there exists $Y \in \mathcal{A}$ such that $x \in Y$. Since Y is a pseudo symmetric ideal of T containing x. By hypothesis, $T \subseteq (x)_{ps} \subseteq Y$. Which is contradiction to choice of Y. Thus M is a proper pseudo symmetric ideal of T. Let L be a pseudo symmetric ideal of T such that $M \subseteq L$ and $L \neq T$. Then we have $X \subseteq M \subseteq L \subset T$, $L \in \mathcal{A}$, and $L \subseteq M$. Then $L = M$. Hence, the set M is a maximal pseudo symmetric ideal of T.

Corollary 3.2. *Let T be a partially ordered ternary semigroup for which there exists an element $x \in T$ such that $T \subseteq (x)_{ps}$. Let M_1 and M_2 be the two maximal pseudo symmetric ideals of T. Then $M_1 = M_2$.*

Lemma 3.3. *Let T be a partially ordered ternary semigroup for which there exists an element $x \in T$ such that $T \subseteq (x)_{ps}$. Let $\mathcal{A}$ be the set of all proper pseudo* *symmetric ideals of* T*. If* $\mathcal{A} \neq \emptyset$*, then the set* $\bigcup\{X : X \in \mathcal{A}\}$ *is the unique maximal pseudo symmetric ideal of* T*.*

proof. Since $\mathcal{A} \neq \emptyset$, then the set $\bigcup\{X : X \in \mathcal{A}\}$ is a pseudo symmetric ideal of T. Let $\bigcup\{X : X \in \mathcal{A}\} = T$ then $x \in \bigcup\{X : X \in \mathcal{A}\}$. Therefore there exists $X \in \mathcal{A}$ such that $x \in X$. Since X is a pseudo symmetric ideal of T containing x. By hypothesis, $T \subseteq (x)_{ps} \subseteq X$ implies $T = X$. This is impossible. Thus $\bigcup\{X : X \in \mathcal{A}\}$ is a proper pseudo symmetric ideal of T. Let L be a pseudo symmetric ideal of T such that $\bigcup\{X : X \in \mathcal{A}\} \subseteq L$ and $L \neq T$. Since $L \in \mathcal{A}$, we have $L \subseteq \bigcup\{X : X \in \mathcal{A}\}$. Then $L = \bigcup\{X : X \in \mathcal{A}\}$. Hence, the set $\bigcup\{X : X \in \mathcal{A}\}$ is the maximal pseudo symmetric ideal of T. If M is a maximal pseudo symmetric ideal of T. Then by Corollary 3.2, we have $M = \bigcup\{X : X \in \mathcal{A}\}$. So the set $\bigcup\{X : X \in \mathcal{A}\}$ is the unique maximal pseudo symmetric ideal of T.

Definition 3.2. *Let* T *be a partially ordered ternary semigroup without a zero element. Then* T *is called* P*-simple if* T *has no proper pseudo symmetric ideals.*

Example 3.2. Let $T = \{a, b\}$. A ternary operation [ ] on T defined by the following tables:

[ ]	a	b
aa	b	a
ab	a	b

[ ]	a	b
ba	a	b
bb	b	a

and the partial ordering relation $\leq := \{(a, a), (b, b)\}$. Then T is a partially ordered ternary semigroup. It is easy to see that $I_1 = \{a\}$ and $I_2 = \{b\}$ are not pseudo symmetric ideals of T (because, they are not ideals of T). However, $I = \{a, b\}$ is the only pseudo symmetric ideal of T and it is not a proper pseudo symmetric ideal of T. Thus, the partially ordered ternary semigroup $T = \{a, b\}$ is a P-simple.

Theorem 3.7. *For partially ordered ternary semigroup* T *without a zero element, the following statements are equivalent:*
(i) T *is* P*-simple.*
(ii) If for $x \in T$*,* $(TxTxT]$ *is a pseudo symmetric ideal of* T *then* $(TxTxT] = T$*.*
(iii) $(x)_{ps} = T \ \forall \ x \in T$*.*

Proof. (i) $\Rightarrow$ (ii): Suppose that, T is P-simple. For $x \in T$, $(TxTxT]$ is a pseudo symmetric ideal of T then by (i), $(TxTxT] = T$.

(ii) $\Rightarrow$ (i): Assume that, $(TxTxT] = T$ for all $x \in T$ and $(TxTxT]$ is a pseudo symmetric ideal of T. Therefore T is P-simple.

(i) $\Rightarrow$ (iii): Suppose that, T is P-simple. Then for $x \in T$, $(x)_{ps} \subseteq T$, by hypothesis we have $(x)_{ps} = T$ for all $x \in T$.

(iii) $\Rightarrow$ (i): Assume that, $(x)_{ps} = T$ for all $x \in T$. Let X be any pseudo symmetric ideal of T and $x \in X \Rightarrow T = (x)_{ps} \subseteq X \subseteq T$. Hence $T = X$. Therefore T is P-simple.

Definition 3.3. *A pseudo symmetric ideal X of a partially ordered ternary semigroup T without a zero element is called minimal pseudo symmetric ideal if X is a proper pseudo symmetric ideal of T and X does not properly contain any pseudo symmetric ideal of T.*

Theorem 3.8. *Let T be a partially ordered ternary semigroup without a zero element having proper pseudo symmetric ideals. Then every proper pseudo symmetric ideal of T is minimal if and only if the intersection of any two distinct proper pseudo symmetric ideals is empty.*

Proof. Let I_1 and I_2 be two distinct proper pseudo symmetric ideals of T. Firstly, assume that, I_1 and I_2 are minimal. Now, if $I_1 \cap I_2 \neq \emptyset$ then $I_1 \cap I_2$ is a pseudo symmetric ideal of T. Since, $I_1 \cap I_2 \subseteq I_1$ and I_1 is minimal, we have $I_1 \cap I_2 = I_1$. Since, $I_1 \cap I_2 \subseteq I_2$ and I_2 is minimal, we have $I_1 \cap I_2 = I_2$. So, $I_1 = I_1 \cap I_2 = I_2$. This is a contradiction. Hence $I_1 \cap I_2 = \emptyset$.

Conversely, let X be a proper pseudo symmetric ideal of T and Y be a pseudo symmetric ideal of T such that $Y \subseteq X$ then Y is a proper pseudo symmetric ideal of T. If $Y \neq X$ then by assumption, $\emptyset = Y \cap X = Y$. This is contradiction. Hence $Y = X$. Therefore X is a minimal pseudo symmetric ideal of T.

Definition 3.4. *A partially ordered ternary semigroup T with a zero element 0, $T^3 (= TTT) \neq \{0\}$ and $|T| > 1$ is called 0-P-simple if T has no nonzero proper pseudo symmetric ideals.*

Example 3.3. Let $T = \{0, a, b\}$. A ternary operation [ ] on T defined by the following tables:

[ ]	0	a	b
00	0	0	0
0a	0	0	0
0b	0	0	0

[ ]	0	a	b
a0	0	0	0
aa	0	b	a
ab	0	a	b

[ ]	0	a	b
b0	0	0	0
ba	0	a	b
bb	0	b	a

and the partial ordering relation $\leq := \{(0,0), (0,a), (0,b), (a,a), (b,b)\}$. Then T is a partially ordered ternary semigroup. It is easy to see that $I_1 = \{a\}, I_2 = \{b\}, I_3 = \{0, a\}, I_4 = \{0, b\}$ and $I_5 = \{a, b\}$ are not pseudo symmetric ideals of T (because, they are not ideals of T). However, $I = \{0, a, b\}$ is the only nonzero pseudo symmetric ideal of T and it is not a proper pseudo symmetric ideal of T. Thus, the partially ordered ternary semigroup $T = \{0, a, b\}$ is a 0-P-simple.

Theorem 3.9. *Let T be a partially ordered ternary semigroup with a zero element*

0 *such that* $T^3(= TTT) \neq \{0\}$ *and* $|T| > 1$. *Then* T *is 0-P-simple if and only if* $(x)_{ps} = T$ *for all* $x \in T \setminus \{0\}$.

Proof. Suppose that, T is 0-P-simple. Let $x \in T \setminus \{0\} \Rightarrow (x)_{ps} \neq \{0\}$. Since T is 0-P-simple, we have $(x)_{ps} = T$ for all $x \in T \setminus \{0\}$.

Conversely, Suppose that $(x)_{ps} = T \ \forall \ x \in T \setminus \{0\}$. Let I be a nonzero pseudo symmetric ideal of T, $x \in I \setminus \{0\} \Rightarrow (x)_{ps} = T$ and $(x)_{ps} \subseteq I \subseteq T$. This implies that $T = I$. Therefore T is 0-P-simple.

Definition 3.5. *A non-zero pseudo symmetric ideal* I *of a partially ordered ternary semigroup* T *with a zero element is called a 0-minimal pseudo symmetric ideal of* T *if* I *does not properly contain any nonzero pseudo symmetric ideal of* T.

Theorem 3.10. *Let* T *be a partially ordered ternary semigroup with a zero element having nonzero proper pseudo symmetric ideals. Then every nonzero proper pseudo symmetric ideal of* T *is 0-minimal if and only if the intersection of any two distinct nonzero proper pseudo symmetric ideals is* $\{0\}$.

Proof. Let I_1 and I_2 be two distinct nonzero proper pseudo symmetric ideals of T. Suppose that, I_1 and I_2 are 0-minimal. Now, if $I_1 \cap I_2 \neq \{0\}$ then $I_1 \cap I_2$ is a nonzero pseudo symmetric ideal of T. Since, $I_1 \cap I_2 \subseteq I_1$ and I_1 is 0-minimal, we have $I_1 \cap I_2 = I_1$. Since, $I_1 \cap I_2 \subseteq I_2$ and I_2 is 0-minimal, we have $I_1 \cap I_2 = I_2$. So, $I_1 = I_1 \cap I_2 = I_2$. This is a contradiction. Hence $I_1 \cap I_2 = \{0\}$.

Conversely, let X be a nonzero proper pseudo symmetric ideal of T and Y be a nonzero pseudo symmetric ideal of T such that $Y \subseteq X$ then Y is a nonzero proper pseudo symmetric ideal of T. If $Y \neq X$ then $\{0\} = Y \cap X = Y$. This is contradiction. Hence $Y = X$. Therefore X is a 0-minimal pseudo symmetric ideal of T.

Theorem 3.11. *Let* X *be a proper pseudo symmetric ideal of* T. *If either,*
(i) $T \setminus X = \{x\}$ *for some* $x \in T$
or
(ii) $T \setminus X \subseteq (TyTyT] \ \forall \ y \in T \setminus X$
Then X *is a maximal pseudo symmetric ideal of* T.

Proof. Let Y be a pseudo symmetric ideal of T such that X is a proper subset of Y.

Case (i). $T \setminus X = \{x\}$ for some $x \in T$. Since X is a proper subset of Y, we have $Y \setminus X \subseteq T \setminus X = \{x\}$. Thus $Y \setminus X = \{x\}$. Hence $Y = X \cup (Y \setminus X) = X \cup \{x\} = X \cup (T \setminus X) = T$. Therefore X is a maximal pseudo symmetric ideal of T.

Case (ii). $T \setminus X \subseteq (TyTyT]$ for all $y \in T \setminus X$. Let $y \in Y \setminus X \subseteq T \setminus X$, because, $Y \setminus X \neq \emptyset$. Thus $T \setminus X \subseteq (TyTyT] \subseteq (TYTYT] \subseteq Y$. Hence $T = X \cup (T \setminus X) \subseteq X \cup Y = Y \subseteq T$. Thus $Y = T$. Hence X is a maximal pseudo symmetric ideal of

T.

Theorem 3.12. *If X is a maximal pseudo symmetric ideal of T and $X \cup (x)_{ps}$ is a pseudo symmetric ideal of T for all $x \in T \setminus X$ then $T \setminus X \subseteq (x)_{ps}$ for all $x \in T \setminus X$.*
Proof. Suppose that X is a maximal pseudo symmetric ideal of T and $X \cup (x)_{ps}$ is a pseudo symmetric ideal of T for all $x \in T \setminus X$. Let $x \in T \setminus X$ then $X \subset X \cup (x)_{ps}$. Since $X \cup (x)_{ps}$ is a pseudo symmetric ideal of T and X is maximal, we have $X \cup (x)_{ps} = T$. Hence $T \setminus X \subseteq (x)_{ps}$.

Let $\mathfrak{A}$ be the union of all proper pseudo symmetric ideals in a partially ordered ternary semigroup without a zero element and let $\mathfrak{A}_\circ$ be the union of all nonzero proper pseudo symmetric ideals of a partially ordered ternary semigroup with a zero element.

Lemma 3.4. *Let T be a partially ordered ternary semigroup without a zero element then $T = \mathfrak{A}$ if and only if $(x)_{ps} \neq T \ \forall \ x \in T$.*
Proof. Suppose that, $T = \mathfrak{A}$. Let $x \in T$. Then $x \in \mathfrak{A}$. Therefore $x \in I$ for some proper pseudo symmetric ideal I of T. Hence, $(x)_{ps} \subseteq I \neq T$. This shows that $(x)_{ps} \neq T$.

Conversely, suppose that $(x)_{ps} \neq T \ \forall \ x \in T$ then $(x)_{ps} \subseteq \mathfrak{A}$. Let $x \in T$. Then $x \in \mathfrak{A}$. Thus $T = \mathfrak{A}$.

Lemma 3.5. *Let T be a partially ordered ternary semigroup with a zero element then $T = \mathfrak{A}_\circ$ if and only if $(x)_{ps} \neq T \ \forall \ x \in T$.*
Proof. Analogous to the proof of the Lemma 3.4.

Definition 3.6. *A partially ordered ternary semigroup T is called p-Noetherian partially ordered ternary semigroup if it satisfies the ascending chain condition for pseudo symmetric ideals of T, for any sequence $X_1 \subseteq X_2 \subseteq \ldots$ of pseudo symmetric ideals of T, then there exists a positive integer m such that $X_m = X_{m+1} = \ldots$.*

Lemma 3.6. *If T is a p-Noetherian partially ordered ternary semigroup containing proper pseudo symmetric ideals then T has a maximal pseudo symmetric ideal.*
Proof. Let X_1 be a pseudo symmetric ideal of T. If X_1 is not a maximal pseudo symmetric ideal, then there exists a proper pseudo symmetric ideal X_2 of T such that $X_1 \subseteq X_2$. If X_2 is not a maximal pseudo symmetric ideal, then there exists a proper pseudo symmetric ideal X_3 of T such that $X_1 \subseteq X_2 \subseteq X_3$. By continuing this way, we get an ascending chain $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ of pseudo symmetric ideals of T. Since T is p-Noetherian, then there exists a positive integer m such that $X_m = X_{m+1} = \ldots$. Therefore X_m is maximal pseudo symmetric ideal of T. Hence T has a maximal pseudo symmetric ideal.

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