South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 2 (2024), pp. 155-168

DOI: 10.56827/SEAJMMS.2024.2002.12 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

COEFFICIENT ESTIMATES FOR ρ-FOLD SYMMETRIC BI-UNIVALENT MA-MINDA TYPE FUNCTIONS

Hamid Shamsan, Renuka Devi K.* and Abdulhafid Modabish

Department of Mathematics,
Science College,
Sana'a University, Sana'a - 2124, YEMEN

E-mail : hmas19771@gmail.com, hafizmod2@gmail.com

*Department of Mathematics,
Government first grade College,
University of Mysore, Mysore - 570005, INDIA

E-mail : renukk84@gmail.com

(**Received:** *Jan. 16, 2024* **Accepted:** *Jul. 29, 2024* **Published:** *Aug. 30, 2024)*

Abstract: In the present investigation, we consider new subclasses of Σ_ρ consisting of regular and ρ-fold symmetric bi-univalent functions in the open unit disk. We obtain coefficient bounds for $|a_{\rho+1}|$ and $|a_{2\rho+3}|$ of the functions from these new subclasses.

Keywords and Phrases: Bi-univalent functions, ρ-Fold symmetric function, ρ-Fold symmetric bi-univalent function.

2020 Mathematics Subject Classification: 30C45.

1. Introduction

We start with following notations. The class of maps that are holomorphic on the unit open disk $\Delta = \{\zeta : \zeta \in \mathbb{C} \text{ with } |\zeta| < 1\}$ and of form

$$h(\zeta) = \zeta + \sum_{l=2}^{\infty} a_l \zeta^l \tag{1.1}$$

is denoted by $\mathfrak{A}$. The subclass of all functions of $\mathfrak{A}$ that are univalent in Δ is denoted by $\mathcal{S}$. The Köebe $\frac{1}{4}$ theorem [9] confirms that the image of Δ under each univalent function $h \in \mathfrak{A}$ contains a disk of radius $\frac{1}{4}$. Therefore, any univalent function h has an inverse h^{-1} satisfying $h^{-1}(h(\zeta)) = \zeta, \quad \zeta \in \Delta$ and

$$h^{-1}(h(\lambda)) = \lambda, \qquad \left(|\lambda| < r_0(h), \quad r_0(h) \geq \frac{1}{4}\right),$$

where

$$\gamma(\lambda) = h^{-1}(\lambda) = \lambda - a_2\lambda^2 + (2a_2^2 - a_3)\lambda^3 - (5a_2^3 - 5a_2a_3 + a_4)\lambda^4 + ... \tag{1.2}$$

A function $h \in \mathfrak{A}$ is said to be bi-univalent in Δ if h and h^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions defined in the unit disk Δ. The class of holomorphic bi-univalent functions was first presented and studied by Lewin [13] who proved that $|a_2| < 1.51$. Later, Brannan and Clunie [5] improved Lewin's result to $|a_2| \leq \sqrt{2}$.

A function is called a ρ-fold symmetric if it has the form

$$h(\zeta) = \zeta + \sum_{l=1}^{\infty} a_{l\rho+1}\zeta^{l\rho+1}, \quad \zeta \in \Delta, \rho \in \mathbb{N}. \tag{1.3}$$

We denote by $\mathcal{S}_\rho$ the class of ρ-fold symmetric univalent functions in Δ. Each bi-univalent function generates an ρ-fold symmetric bi-univalent function for any integer $\rho \in \mathbb{N}$. The normalized form of h is given as in (1.3) and the series expansion for h^{-1}, which was recently proven by Srivastava et al. [20], is given as follows:

$$\gamma(\lambda) = h^{-1}(\lambda) = \lambda - a_{\rho+1}\lambda^{\rho+1} + \left[(\rho+1)a_{\rho+1}^2 - a_{2\rho+1}\right]\lambda^{2\rho+1} - \left[\frac{1}{2}(\rho+1)(3\rho+2)a_{\rho+1}^3 - (3\rho+2)a_{\rho+1}a_{2\rho+1} + a_{3\rho+1}\right]\lambda^{3\rho+1} + ... \tag{1.4}$$

where $h^{-1} = \gamma$, we denote by Σ_ρ the class of ρ-fold symmetric bi-univalent functions in Δ.

Examples of ρ-fold symmetric bi-univalent functions are

$$\left(\frac{\zeta^\rho}{1-\zeta^\rho}\right)^{\frac{1}{\rho}}, \quad \left[\frac{1}{2}\log\left(\frac{1+\zeta^\rho}{1-\zeta^\rho}\right)^{\frac{1}{\rho}}\right], \quad [-\log(1-\zeta^\rho)]^{\frac{1}{\rho}}, ...$$

and the corresponding inverse functions are

$$\left(\frac{\lambda^\rho}{1+\lambda^\rho}\right)^{\frac{1}{\rho}}, \quad \left(\frac{e^{2\lambda^\rho}-1}{e^{2\lambda^\rho}+1}\right)^{\frac{1}{\rho}}, \quad \left(\frac{e^{\lambda^\rho}-1}{e^{\lambda^\rho}}\right)^{\frac{1}{\rho}}, ...$$

Brannan and Taha [6] and Taha [21] considered certain subclasses of bi-univalent functions formed by strongly starlike, convex, and starlike functions. They presented bi-convex and bi-starlike functions, as well as found non-sharp estimates for the coefficients $|a_2|$ and $|a_3|$. Nowadays, many authors introduced and studied bounds for various subclasses of bi-univalent functions ([4, 2, 3, 8, 15, 17, 10, 12, 1, 11]). For two regular functions h and γ in Δ, the subordination between them is written as $h \prec \gamma$. The function $h(\zeta)$ is subordinate to $\gamma(\zeta)$ if there is a Schwarz function G with $G(0)$ =0, $|G(z)| < 1$, for all $\zeta \in \Delta$, such that $h(\zeta) = \gamma(G(\zeta))$ for all $\zeta \in \Delta$.

Motivated by the previously published works and Rosihan et al. [1], in the next section we introduce new subclasses of bi-univalent functions $\mathcal{H}_{\Sigma_\rho}(\psi)$ and $\mathcal{ST}_{\Sigma_\rho}(\eta, \psi)$. Let ψ be a holomorphic function with positive real part in Δ such that $\psi(0) = 1$, $\psi(0) > 0$ and $\psi(\Delta)$ is symmetric with respect to real axis. Such a function has the form

$$\psi(\zeta) = 1 + \xi_1\zeta + \xi_2\zeta^2 + \xi_3\zeta^3 + ..., \quad (\xi_1 > 0). \tag{1.5}$$

Lemma 1.1. [16] *If the function $p \in \mathcal{P}$ is given by the series*

$$p(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + ..., \tag{1.6}$$

then

$$|c_n| \leq 2 \quad (n = 1, 2, ...).$$

2. Main Results

Definition 2.1. *Let $h \in \Sigma_\rho$. Then $h \in \mathcal{H}_{\Sigma_\rho}(\psi)$ if it satisfies the condition $h'(\zeta) \prec \psi(\zeta)$ and $\gamma'(\lambda) \prec \psi(\lambda)$, where $\gamma(\lambda) = h^{-1}(\lambda)$.*

Theorem 2.2. *Let $h \in \mathcal{H}_{\Sigma_\rho}(\psi)$ and given by (1.3). Then*

$$|a_{\rho+1}| \leq \frac{\sqrt{2}\xi_1^{\frac{3}{2}}}{\sqrt{|(\rho+1)(2\rho+1)\xi_1^2 + 2(\rho+1)^2\xi_1 - 2(\rho+1)^2\xi_2|}} \tag{2.1}$$

and

$$|a_{2\rho+1}| \leq \frac{\xi_1}{(2\rho+1)} + \frac{\xi_1^2}{2(\rho+1)}. \tag{2.2}$$

Proof. Let $h \in \mathcal{H}_{\Sigma_\rho}(\psi)$ and $\gamma = h^{-1}$. Hence there are regular functions $\Phi, \Psi : \Delta \to \Delta$, with $\Phi(0) = \Psi(0) = 0$, satisfying

$$h'(\zeta) = \psi(\Phi(z)) \quad \textit{and} \quad g'(\lambda) = \psi(\Psi(\lambda)). \tag{2.3}$$

Define the functions p_1 and p_2 by
$p_1(\zeta) = \frac{1+\Phi(\zeta)}{1-\Phi(\zeta)} = 1 + c_\rho \zeta^\rho + c_{2\rho}\zeta^{2\rho} + ...$ and $p_2(\zeta) = \frac{1+\Psi(\zeta)}{1-\Psi(\zeta)} = 1 + b_\rho \zeta^\rho + b_{2\rho}\zeta^{2\rho} + ...$,
or, equivalently,

$$\Phi(\zeta) = \frac{p_1(\zeta)-1}{p_1(\zeta)+1} = \frac{1}{2}\left(c_\rho\zeta^\rho + (c_{2\rho} - \frac{c_\rho^2}{2})\zeta^{2\rho} + ...\right) \tag{2.4}$$

and

$$\Psi(\zeta) = \frac{p_2(\zeta)-1}{p_2(\zeta)+1} = \frac{1}{2}\left(b_\rho\zeta^\rho + (b_{2\rho} - \frac{b_\rho^2}{2})\zeta^{2\rho} + ...\right). \tag{2.5}$$

Obviously that p_1 and p_2 are regular in Δ and $p_1(0) = p_2(0) = 1$. Since $p_1, p_2 \in \mathcal{P}$, Therefore $|b_i| \le 2$ and $|c_i| \le 2, \quad (i \in \mathbb{N})$.
Now, by substituting from (2.4) and (2.5) into (2.3), and using (1.5), we get

$$h'(\zeta) = \psi\left(\frac{p_1(\zeta)-1}{p_1(\zeta)+1}\right)$$

$$= \psi\left(\frac{c_\rho\zeta^\rho + c_{2\rho}\zeta^{2\rho} + c_{3\rho}\zeta^{3\rho} + ...}{2 + b_\rho\zeta^\rho + b_{2\rho}\zeta^{2\rho} + b_{3\rho}\zeta^{3\rho} + ...}\right)$$

$$= \psi\left[\frac{1}{2}c_\rho\zeta^\rho + \frac{1}{2}(c_{2\rho} - \frac{c_\rho^2}{2})\zeta^{2\rho} + \frac{1}{2}(c_{3\rho} - c_\rho c_{2\rho} + \frac{c_\rho^3}{4})\zeta^{3\rho} + ...\right]$$

$$= 1 + \frac{\xi_1 c_\rho}{2}\zeta^\rho + \left[\frac{\xi_1}{2}(c_{2\rho} - \frac{c_\rho^2}{2}) + \frac{\xi_2 c_\rho^2}{4}\right]\zeta^{2\rho}$$

$$+ \left[\frac{\xi_1}{2}(c_{3\rho} - c_\rho c_{2\rho} + \frac{c_\rho^3}{4}) + \frac{\xi_2 c_\rho}{2}(c_{2\rho} - \frac{c_\rho^2}{2}) + \frac{\xi_3 c_\rho^3}{8}\right]\zeta^{3\rho} + ... \tag{2.6}$$

and

$$\gamma'(\lambda) = \psi\left(\frac{p_2(\lambda)-1}{p_2(\lambda)+1}\right) = 1 + \frac{1}{2}\xi_1 b_\rho\lambda^\rho + \left(\frac{1}{2}\xi_1\left(b_{2\rho} - \frac{b_\rho^2}{2}\right) + \frac{1}{4}\xi_2 b_\rho^2\right)\lambda^{2\rho} + ... \tag{2.7}$$

Since $h \in \Sigma_k$ is given by (1.3), therefore its inverse $\gamma = h^{-1}$ has the expansion

$$\gamma(\lambda) = h^{-1}(\lambda) = \lambda - a_{\rho+1}\lambda^{\rho+1} + \left[(\rho+1)a_{\rho+1}^2 - a_{2\rho+1}\right]\lambda^{2\rho+1} - ....$$

Since
$h'(\zeta) = 1 + (\rho+1)a_{\rho+1}\zeta^\rho + (2\rho+1)a_{2\rho+1}\zeta^{2\rho} + ... \quad$ and

$\gamma'(\lambda) = 1 - (\rho+1)a_{\rho+1}\lambda^{\rho} + (2\rho+1)\left[(\rho+1)a_{\rho+1}^2 - a_{2\rho+1}\right]\lambda^{2\rho} + ...$, it follows from (2.6) and (2.7) that

$$(\rho+1)a_{\rho+1} = \frac{\xi_1 c_\rho}{2}. \tag{2.8}$$

$$(2\rho+1)a_{2\rho+1} = \frac{\xi_1}{2}\left(c_{2\rho} - \frac{c_\rho^2}{2}\right) + \frac{\xi_2 c_\rho^2}{4}. \tag{2.9}$$

$$-(\rho+1)a_{\rho+1} = \frac{\xi_1 b_\rho}{2}. \tag{2.10}$$

and

$$(2\rho+1)\left[(\rho+1)a_{\rho+1}^2 - a_{2\rho+1}\right] = \frac{\xi_1}{2}\left(b_{2\rho} - \frac{b_\rho^2}{2}\right) + \frac{\xi_2 b_\rho^2}{4}. \tag{2.11}$$

From (2.8) and (2.10), we obtain

$$c_\rho = -b_\rho. \tag{2.12}$$

and

$$2a_{\rho+1}^2 = \frac{\xi_1^2(c_\rho^2 + b_\rho^2)}{4(\rho+1)^2}. \tag{2.13}$$

By combining the equations (2.9) and (2.11) and using (2.13), we obtain

$$a_{\rho+1}^2 = \frac{\xi_1^3\left(c_{2\rho} + b_{2\rho}\right)}{2(\rho+1)\left[(2\rho+1)\xi_1^2 - 2(\rho+1)\xi_2 + 2(\rho+1)\xi_1\right]}.$$

Using Lemma 1.6 for the coefficients $b_{2\rho}$ and $c_{2\rho}$, we get

$$|a_{\rho+1}| \leq \frac{\sqrt{2}\xi_1^{\frac{3}{2}}}{\sqrt{\left|(\rho+1)\left[(2\rho+1)\xi_1^2 - 2(\rho+1)^2\xi_2 + 2(\rho+1)^2\xi_1\right]\right|}}.$$

Hence we get the inequality (2.1). Now, by subtracting (2.11) from (2.9) and from (2.12), we have $c_\rho^2 = b_\rho^2$, hence

$$a_{2\rho+1} = \frac{1}{4(2\rho+1)}\xi_1(c_{2\rho} - b_{2\rho}) + \frac{1}{8(\rho+1)}(\xi_1^2 c_\rho^2).$$

Using (2.13) and Lemma 1.6 for the coefficients $b_{2\rho}$ and $c_{2\rho}$, we obtain

$$|a_{2\rho+1}| \leq \frac{\xi_1}{(2\rho+1)} + \frac{\xi_1^2}{2(\rho+1)}.$$

Which completes the proof.

As $\rho = 1$, we get a result, presented by Rosihan et al. [1].

Corollary 2.3. *Let $h \in \mathcal{H}_\Sigma(\psi)$ and given by (1.1). Then*

$$|a_2| \leq \frac{\xi_1^{\frac{3}{2}}}{\sqrt{|3\xi_1^2 - 4\xi_2 + 4\xi_1|}} \quad and \quad |a_3| \leq \frac{\xi_1}{3} + \frac{\xi_1^2}{4}. \tag{2.14}$$

Definition 2.4. *A function $f \in \Sigma_\rho$ is belong to the class $\mathcal{ST}_{\Sigma_\rho}(\eta, \psi)$, $\eta \geq 0$, if the following subordinations hold*

$$\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\eta\zeta^2 h''(\zeta)}{h(\zeta)} \prec \psi(z), \qquad (\zeta \in \Delta,)$$

and

$$\frac{\lambda\gamma'(\lambda)}{\gamma(\lambda)} + \frac{\eta\lambda^2\gamma''(\lambda)}{\gamma(\lambda)} \prec \psi(\lambda), \qquad (\lambda \in \Delta,)$$

where $\gamma(\lambda) = h^{-1}(\lambda)$.

Theorem 2.5. *Let h given by (1.3) be in the class $\mathcal{ST}_{\Sigma_\rho}(\eta, \psi)$. Then*

$$|a_{\rho+1}| \leq \frac{\xi_1^{\frac{3}{2}}}{\sqrt{\left|\left[\rho + 2\rho(1+\rho)\alpha\right]\xi_1^2 + (\xi_1 - \xi_2)\left[1 + (1+\rho)\alpha\right]^2\right|}}. \tag{2.15}$$

and

$$|a_{2\rho+1}| \leq \frac{(\rho+1)\left[\xi_1 + |\xi_2 - \xi_1|\right]}{2\rho^2\left[1 + 2(\rho+1)\alpha\right]}. \tag{2.16}$$

Proof. Let $h \in \mathcal{ST}_{\Sigma_\rho}(\eta, \psi)$. Hence there are regular functions $\Phi, \Psi : \Delta \to \Delta$, with $\Phi(0) = \Psi(0) = 0$, satisfying

$$\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\eta\zeta^2 h''(\zeta)}{h(\zeta)} = \psi(\Phi(\zeta)), \qquad (\zeta \in \Delta,) \tag{2.17}$$

and

$$\frac{\lambda\gamma'(\lambda)}{\gamma(\lambda)} + \frac{\eta\lambda^2\gamma''(\lambda)}{\gamma(\lambda)} = \psi(\Psi(\lambda)), \qquad (\lambda \in \Delta,) \tag{2.18}$$

where $\gamma(\lambda) = h^{-1}(\lambda)$. By (2.17), we have

$$\zeta + (\rho+1)(1+\alpha\rho)a_{\rho+1}\zeta^{\rho+1} + (2\rho+1)(1+2\eta\rho)a_{2\rho+1}\zeta^{2\rho+1} + ... =$$
$$\left\{1 + \frac{1}{2}\xi_1 c_\rho \zeta^\rho + \left(\frac{1}{2}\xi_1\left(c_{2\rho} - \frac{c_\rho^2}{2}\right) + \frac{1}{4}\xi_2 c_\rho^2\right)\zeta^{2\rho} + ...\right\}$$
$$\left\{\zeta + a_{\rho+1}\zeta^{\rho+1} + a_{2\rho+1}\zeta^{2\rho+1} + ...\right\}.$$

By equating the coefficients on both sides we obtain

$$\left[\rho + \rho(1+\rho)\eta\right]a_{\rho+1} = \frac{\xi_1 c_\rho}{2}. \tag{2.19}$$

$$\left[2\rho + 2\rho(1+2\rho)\eta\right]a_{2\rho+1} - \left[\rho + \rho(1+\rho)\eta\right]a_{\rho+1}^2 = \frac{1}{2}\xi_1\left(c_{2\rho} - \frac{c_\rho^2}{2}\right) + \frac{1}{4}\xi_2 c_\rho^2. \tag{2.20}$$

Also, from (2.18), we have

$$\lambda - (\rho+1)(1+\eta\rho)a_{\rho+1}\lambda^{\rho+1} + (2\rho+1)(1+2\eta\rho)((\rho+1)a_{\rho+1}^2 - a_{2\rho+1})\lambda^{2\rho+1} + ... =$$
$$\left\{1 + \frac{1}{2}\xi_1 b_\rho \lambda + \left(\frac{1}{2}\xi_1\left(b_{2\rho} - \frac{b_\rho^2}{2}\right) + \frac{1}{4}\xi_2 b_\rho^2\right)\lambda^{2\rho} + ...\right\}$$
$$\left\{\lambda - a_{\rho+1}\lambda^{\rho+1} + \left[(\rho+1)a_(\rho+1)^2 - a_{2\rho+1}\right]\lambda^{2\rho+1} + ...\right\}.$$

By equating the coefficients on both sides we obtain

$$-\left[\rho + \rho(1+\rho)\eta\right]a_{\rho+1} = \frac{\xi_1 b_\rho}{2}. \tag{2.21}$$

$$\left[\rho(2\rho+1) + \rho(\rho+1)(4\rho+1)\right]a_{\rho+1}^2 - 2\left[\rho + \rho(2\rho+1)\eta\right]a_{2\rho+1} = \frac{1}{2}\xi_1\left(b_{2\rho} - \frac{b_\rho^2}{2}\right) + \frac{1}{4}\xi_2 b_\rho^2. \tag{2.22}$$

From (2.19) and (2.21), we obtain

$$c_\rho = -b_\rho. \tag{2.23}$$

and

$$2a_{\rho+1}^2 = \frac{\xi_1^2(c_\rho^2 + b_\rho^2)}{4\left[\rho + \rho(\rho+1)\eta\right]^2}. \tag{2.24}$$

By combining the equations (2.20) and (2.22) and using (2.24), we obtain

$$a_{\rho+1}^2 = \frac{\xi_1^3 \left(b_{2\rho} + c_{2\rho}\right)}{4\rho^2 \left[\left(\rho + 2\rho(\rho+1)\eta\right)\xi_1^2 + (\xi_1 - \xi_2)\left((1 + (\rho+1)\eta\right)^2\right]}.$$

Using Lemma 1.6 for $b_{2\rho}$ and $c_{2\rho}$, we have

$$|a_{\rho+1}^2| \leq \frac{\xi_1^3}{\rho^2 \left| \left[\left(\rho + 2\rho(\rho+1)\eta\right)\xi_1^2 + (\xi_1 - \xi_2)\left((1 + (\rho+1)\eta\right)^2\right] \right|}.$$

Since $\xi_1 > 0$, the inequality (2.15) obtained from the last inequality. Now, by subtracting (2.22) from (2.20) and from (2.23), we obtain $c_\rho^2 = b_\rho^2$, hence

$$a_{2\rho+1} = \frac{\frac{\xi_1}{2}\left[\left((1+2\rho) + (1+\rho)(1+4\rho)\eta\right)c_{2\rho} + \left(1 + (1+\rho)\eta\right)b_{2\rho}\right]}{4\rho^2\left[1 + (1+2\rho)\eta)\right]\left[1 + 2(1+\rho)\eta\right]}$$

$$+ \frac{(1+\rho)b_\rho^2(\xi_2 - \xi_1)}{8\rho^2\left[1 + 2(1+\rho)\eta\right]}.$$

Using (2.24) and Lemma 1.6 for the coefficients $b_{2\rho}$ and $c_{2\rho}$, we get

$$|a_{2\rho+1}| \leq \frac{(1+\rho)\left[\xi_1 + |\xi_2 - \xi_1|\right]}{2\rho^2\left[1 + 2(1+\rho)\eta\right]}.$$

This is the requirement inequality in (2.16).

As $\rho = 1$, we obtained a result, presented by Rosihan et al. [1].

Corollary 2.6. *Let h given by (1.1) be in the class $\mathcal{ST}_\Sigma(\eta, \psi)$. Then*

$$|a_2| \leq \frac{\xi_1^{\frac{3}{2}}}{\sqrt{\left|\left[(1+4\eta)\xi_1^2 + (\xi_1 - \xi_2)(1+2\eta)^2\right|\right.}}.$$

and

$$|a_3| \leq \frac{\xi_1 + |\xi_2 - \xi_1|}{(1+4\eta)}.$$

As $\rho = 1$ and for $\eta = 0$, we obtained the Ma-Minda's coefficient estimates for bi-starlike functions.

Corollary 2.7. *Let h given by (1.1) be in the class $\mathcal{ST}_{\Sigma}(\psi)$. Then*

$$|a_2| \leq \frac{\xi_1^{\frac{3}{2}}}{\sqrt{\left|\xi_1^2 + (\xi_1 - \xi_2\right|}}.$$

and

$$|a_3| \leq \xi_1 + |\xi_2 - \xi_1|.$$

Definition 2.8. *A function $h \in \Sigma_\rho$ is said to be in the class $\mathcal{M}_{\Sigma_\rho}(\eta, \psi)$, $\eta \geq 0$, if the following subordinations hold*

$$(1-\eta)\frac{\zeta h'(\zeta)}{h(\zeta)} + \eta\left(1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right) \prec \psi(\zeta), \qquad (\zeta \in \Delta,)$$

and

$$(1-\eta)\frac{\lambda \gamma'(\lambda)}{\gamma(\lambda)} + \eta\left(1 + \frac{\lambda \gamma''(\lambda)}{\gamma'(\lambda)}\right) \prec \psi(\lambda), \qquad (\lambda \in \Delta,)$$

where $\gamma(\lambda) = h^{-1}(\lambda)$.

Theorem 2.9. *Let h given by (1.3) be in the class $\mathcal{M}_{\Sigma_\rho}(\eta, \psi)$. Then*

$$|a_{\rho+1}| \leq \frac{\sqrt{2\xi_1^3}}{\sqrt{\left|\left((1+\rho)(1+\rho^2\eta))\xi_1^2 + 2\rho^2(\xi_1 - \xi_2)\left(1 + \eta\rho\right)^2\right|}}. \tag{2.25}$$

and

$$|a_{2\rho+1}| \leq \frac{\big((3\rho+1) + \rho^2(3\rho+5)\eta\big)\left[\xi_1 + |\xi_2 - \xi_1|\right]}{2\rho(\rho+1)(1+\rho^2\eta)(1+2\rho\eta)}. \tag{2.26}$$

Proof. Let $h \in \mathcal{M}_{\Sigma_k}(\eta, \psi)$. Hence there are regular functions $\Phi, \Psi : \Delta \to \Delta$, with $\Phi(0) = \Psi(0) = 0$, satisfying

$$(1-\eta)\frac{\zeta h'(\zeta)}{h(\zeta)} + \eta\left(1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right) = \psi(\Phi(\zeta)), \qquad (\zeta \in \Delta,) \tag{2.27}$$

and

$$(1-\eta)\frac{\lambda \gamma'(\lambda)}{\gamma(\lambda)} + \eta\left(1 + \frac{\lambda \gamma'(\lambda)}{\gamma'(\lambda)}\right) = \psi(\Psi(\lambda)), \qquad (\lambda \in \Delta,) \tag{2.28}$$

where $\gamma(\lambda) = h^{-1}(\lambda)$. By (2.27), we have

$$\zeta+\left[2(\rho+1)+\eta\rho^2\right]a_{\rho+1}\zeta^{\rho+1}+\left[\left(2(1+2\rho)+4\eta\rho^2\right)a_{2\rho+1}+\left(\rho^2+2\rho+1\right)a_{\rho+1}^2\right]\zeta^{2\rho+1}+...$$

$$=\left\{1+\frac{1}{2}\xi_1 c_\rho\zeta^\rho+\left(\frac{1}{2}\xi_1\left(c_{2\rho}-\frac{c_\rho^2}{2}\right)+\frac{1}{4}\xi_2 c_\rho^2\right)\zeta^{2\rho}+...\right\}$$

$$\left\{\zeta+(\rho+2)a_{\rho+1}\zeta^{\rho+1}+\left[(\rho+1)a_{\rho+1}^2+2(\rho+1)a_{2\rho+1}\right]\zeta^{2\rho+1}+...\right\}.$$

By equating the coefficients on both sides we get

$$\left(\rho+\eta\rho^2\right)a_{\rho+1}=\frac{\xi_1 c_\rho}{2}. \tag{2.29}$$

$$2\rho\left[1+2\eta\rho\right]a_{2\rho+1}-\rho\left[1+\rho(2+\rho)\eta\right]a_{\rho+1}^2=\frac{1}{2}\xi_1\left(c_{2\rho}-\frac{c_\rho^2}{2}\right)+\frac{1}{4}\xi_2 c_\rho^2. \tag{2.30}$$

Also, from (2.28), we have

$$\lambda-\left[2(1+\rho)+\eta\rho^2\right]a_{\rho+1}\lambda^{\rho+1}+\left\{(\rho+1)\left[(5\rho+3)+4\eta\rho^2\right]a_{\rho+1}^2-2\left[(2\rho+1)2\eta\rho^2\right]a_{2\rho+1}\right\}\lambda^{2\rho+1}+...$$

$$=\left\{1+\frac{1}{2}\xi_1 b_\rho\lambda^\rho+\left(\frac{1}{2}\xi_1\left(b_{2\rho}-\frac{b_\rho^2}{2}\right)+\frac{1}{4}\xi_2 b_\rho^2\right)\lambda^{2\rho}+...\right\}$$

$$\left\{\lambda-(\rho+2)a_{\rho+1}\lambda^{\rho+1}+\left[(\rho+1)(2\rho+3)a_{\rho+1}^2-2(\rho+1)a_{2\rho+1}\right]\lambda^{2\rho+1}+...\right\}.$$

By equating the coefficients on both sides we get

$$-\left(\rho+\eta\rho^2\right)a_{\rho+1}=\frac{\xi_1 b_\rho}{2}. \tag{2.31}$$

$$\left[(2\rho+1)+\rho^2(2\rho+3)\eta\right]a_{\rho+1}^2-2\rho\left[1+2\eta\rho\right]a_{2\rho+1}=\frac{1}{2}\xi_1\left(b_{2\rho}-\frac{b_\rho^2}{2}\right)+\frac{1}{4}\xi_2 b_\rho^2. \tag{2.32}$$

From (2.29) and (2.31), we obtain

$$c_\rho=-b_\rho. \tag{2.33}$$

and

$$2a_{\rho+1}^2=\frac{\xi_1^2(c_\rho^2+b_\rho^2)}{4\left[\rho+\eta\rho^2\right]^2}. \tag{2.34}$$

By combining the equations (2.30) and (2.32) and using (2.34), we obtain

$$a_{\rho+1}^2 = \frac{\xi_1^3\left(b_{2\rho}+c_{2\rho}\right)}{2(\rho+1)(1+\eta\rho^2)\xi_1^2+4\rho^2(\xi_1-\xi_2)(1+\eta\rho)^2}.$$

Using Lemma 1.6 for $b_{2\rho}$ and $c_{2\rho}$, we obtain

$$|a_{\rho+1}^2| \leq \frac{2\xi_1^3}{\left|(\rho+1)(1+\eta\rho^2)\xi_1^2+2\rho^2(\xi_1-\xi_2)(1+\eta\rho)^2\right|}.$$

Thus we get the result (2.25).
Now, by subtracting (2.32) from (2.30) and from (2.33), we obtain $c_\rho^2 = b_\rho^2$, therefore

$$a_{2\rho+1} = \frac{\frac{\xi_1}{2\rho}\left[\left((2\rho+1)+\rho^2(2\rho+3)\eta\right)c_{2\rho}+\rho\left(1+\rho(\rho+2)\eta\right)b_{2\rho}\right]}{2(\rho+1)(1+\rho^2\eta)(1+2\rho\eta)}$$

$$+\frac{b_\rho^2(\xi_2-\xi_1)\left[(3\rho+1)+\rho^2(3\rho+5)\eta\right]}{8\rho(\rho+1)(1+\rho^2\eta)(1+2\rho\eta)}.$$

Using (2.34) and Lemma 1.6 for the coefficients b_2 and c_2, we get

$$|a_{2\rho+1}| \leq \frac{\left((3\rho+1)+\rho^2(3\rho+5)\eta\right)\left[\xi_1+|\xi_2-\xi_1|\right]}{2\rho(\rho+1)(1+\rho^2\eta)(1+2\rho\eta)}.$$

which completes the proof.

As $\rho = 1$, we have a result, presented by Rosihan et al. [1].

Corollary 2.10. *Let f given by (1.1) and belongs to the class $\mathcal{M}_\Sigma(\eta,\psi)$. Then*

$$|a_2| \leq \frac{\xi_1\sqrt{\xi_1}}{\sqrt{\left|(1+\eta)\xi_1^2+(\xi_1-\xi_2)(1+\eta)^2\right|}}.$$

and

$$|a_3| \leq \frac{\xi_1+|\xi_2-\xi_1|}{(1+\eta)}.$$

As $\rho = 1$ and $\eta = 1$, we get the Ma-Minda's coefficient estimates for bi-convex functions, however if $\eta = 0$, we obtained the Ma-Minda's coefficient estimates for bi-starlike functions.

Corollary 2.11. *Let h given by (1.1) be in the class* $\mathcal{CV}_\Sigma(\psi)$. *Then*

$$|a_2| \leq \frac{\xi_1^{\frac{3}{2}}}{\sqrt{2\left|\xi_1^2 + 2\xi_1 - 2\xi_2\right|}}.$$

and

$$|a_3| \leq \frac{1}{2}(\xi_1 + |\xi_2 - \xi_1|).$$

References

[1] Ali R. M., Lee S. K., Ravichandran V. and Supramaniam S., Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Applied Mathematics Letters, 25(3) (2012), 344-351.

[2] Aldawish I., Swamy S. R., Frasin B. A., A Special Family of m-Fold Symmetric Bi-Univalent Functions Satisfying Subordination Condition., Fractal Fract, 6 (2022), 271.

[3] Al-shbeil I., Khan N., Tchier F., Xin Q., Malik S. N., Khan S., Coefficient Bounds for a Family of s-Fold Symmetric Bi-Univalent Functions, Axioms, 12 (2023), 317.

[4] Altinkaya Ş. and Yalçin S., Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points, Journal of Function Spaces, Article ID 145242 (2015), 5 pages.

[5] Brannan D. A. and Clunie J. G., Aspects of Contemporary Complex Analysis (Proc. of the NATO Advanced Study Institute Held at the University of Durham, Durham; July 1-20, 1979), Academic Press, New York, London (1980).

[6] Brannan D. A. and Taha T. S., On some classes of bi-univalent functions, Stud. Univ. Babe¸s-Bolyai Math., 31(2) (1986), 70-77.

[7] De-Branges L., A proof of the Bieberbach conjecture, Acta Math., 154 (1985), 137-152.

[8] Breaz D., Cot$\hat{i}$lă L. I., The study of Coefficient estimates and Fekete-Szegö inequalities for the new classes of m-Fold symmetric bi-univalent functions defined using an operator, J. inequalities Appl., 2023 (2023), 15.

[9] Duren, P. L. Univalent functions, Grundlchren der Math. Wissenschaften, Springer-Verlag, New York-Berlin, 1983.

[10] Frasin B. A. and Aouf M. K., New subclasses of bi-univalent functions, Appl. Math. Lett., 24 (2011), 1569-1573.

[11] Hamid S., Read Q. and Latha S., Coefficient estimates for bi-univalent ma-minda type functions associated with q-derivative, To appear in TWMS Journal of Applied and Engineering Mathematics.

[12] Hayami T. and Owa S., Coefficient bounds for bi-univalent functions, Pan Amer. Math. J., 22(4) (2012), 15-26.

[13] Lewin M., On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967), 63-68.

[14] Ma W. and Minda D., A unified treatment of some special classes of univalent functions, In Proceeding of the Conference on Complex Analysis, Z. Li, F. Ren, L. Yang and S. Zhang (Eds), Int. Press (1994), 157-169.

[15] Oros G.I., Cot$\hat{i}$l$\breve{a}$ L. I., Coefficient estimates and the Fekete-Szeg$\ddot{o}$ problem for new classes of m-Fold symmetric bi-univalent functions, Mathematics, 10 (2022), 129.

[16] Pommerenke C., Univalent functions, Vandenhoeck and Ruprecht, Gottingen, 1975.

[17] Sabir P. O., Srivastava H. M., Atshan W. G., Mohammed P. O., Chorfi N., Vivas-Cortez M., A Family of Holomorphic and m-Fold Symmetric Bi-Univalent Functions Endowed with Coefficient Estimate Problems, Mathematics, 11 (2023), 3970.

[18] Seoudy T. M. and Aouf M. K., Coefficient estimates of new classes of q-starlike and q-convex functions of complex order, J. Math. Inequal., 10 (2016), 135-145.

[19] Srivastava H. M., Mishra A. K. and Gochhayat P., Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (2010), 1188-1192.

[20] Srivastava H. M., Sivasubramanian S. and Sivakumar R., Initial coefficient bounds for a subclass of k-fold symmetric bi-univalent functions, Tbilisi Math. J., 7(2) (2014), 1-10.

[21] Taha T. S., Topics in Univalent Function Theory, University of London, PhD Thesis, 1981.