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ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH $(p-q)-$ WANAS OPERATOR

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Abstract: In this research article, we have defined new subclasses $GW^{\sigma,\delta}_{s,t,p,q}(\alpha,\beta)$ and $TGW^{\sigma,\delta}_{s,t,p,q}(\alpha,\beta)$ of analytic functions involving Wanas Operator. We have discussed coefficients conditions for these introduced subclasses. Further properties such as partial sum and integral means results are also investigated for these subclasses.

Keywords and Phrases: $(p-q)-$ Wanas Operator, coefficient bounds, partial sums, integral means.

2020 Mathematics Subject Classification: 30C45.

1. Introduction

Let A denote the class of all analytic functions in the open unit disk $U = \{z \in C : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} b_k z^k, z \in U. \tag{1.1}$$

Denote by S, the subclass of A consisting functions that are univalent, T as the subclass of A consisting functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} b_k z^k, b_k \geq 0, z \in U \tag{1.2}$$

introduced and studied by Silverman [3].
In 2019, Wanas [9] introduced the operator called Wanas Operator $W_{s,t}^{\sigma,\delta}: A \to A$ as

$$W_{s,t}^{\sigma,\delta} f(z) = z + \sum_{k=2}^{\infty} \left[\sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} \left(\frac{s^\tau + kt^\tau}{s^\tau + t^\tau} \right) \right]^{\delta} b_k z^k, \tag{1.3}$$

where $s \in \mathbb{R}, t \in \mathbb{R}_0^+$ with $s + t > 0, k - 1, \sigma \in \mathbb{N}$ and $\delta \in \mathbb{N}_0$.
For $f \in A$, we have the $(p - q)-$ Wanas Operator [10] as

$$W_{s,t,p,q}^{\sigma,\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{[\Psi_k(\sigma, s, t)]_{p,q}}{[\Psi_1(\sigma, s, t)]_{p,q}} \right)^{\delta} b_k z^k = z + \sum_{k=2}^{\infty} D_{k,\delta} b_k z^k \tag{1.4}$$

$$\text{where} \quad D_{k,\delta} = \left(\frac{[\Psi_k(\sigma, s, t)]_{p,q}}{[\Psi_1(\sigma, s, t)]_{p,q}} \right)^{\delta}, \tag{1.5}$$

$$\Psi_k(\sigma, s, t) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} \big(s^\tau + kt^\tau\big), \Psi_1(\sigma, s, t) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} \big(s^\tau + t^\tau\big)$$

$s \in \mathbb{R}, t \in \mathbb{R}_0^+$ with $s + t > 0, k - 1, \sigma \in \mathbb{N}$, $\delta \in \mathbb{N}_0, 0 < q < p \leq 1$ and $z \in U$.
In continuation of the study of Rosy et al [2], Sunil Verma et al [7] and T. Thulasiram et al [8], we introduce new subclasses of A involving $(p - q)-$ Wanas operator.

Definition 1.1. *Let $\alpha \geq 0, 0 \leq \beta < 1, 0 < q \leq p$ and $\delta > 0$. A function $f \in A$ is said to be in $GW_{s,t,p,q}^{\sigma,\delta}(\alpha, \beta)$ if it satisfies the following condition,*

$$\Re\left(\frac{W_{s,t,p,q}^{\sigma,\delta} f(z)}{z} \right) \geq \alpha \left| (W_{s,t,p,q}^{\sigma,\delta} f(z))' - \frac{W_{s,t,p,q}^{\sigma,\delta} f(z)}{z} \right| + \beta \tag{1.6}$$

where $W_{s,t,p,q}^{\sigma,\delta} f(z)$ is given by (1.4).
We further let $TGW_{s,t,p,q}^{\sigma,\delta}(\alpha, \beta) = GW_{s,t,p,q}^{\sigma,\delta}(\alpha, \beta) \cap T$.
For $\Psi_k(\sigma, s, t) = 1, \Psi_1(\sigma, s, t) = 1$, the class $GW_{s,t,p,q}^{\sigma,\delta}(\alpha, \beta)$ reduces to the class $SD(\alpha)$.

2. The Subclasses $GW_{s,t,p,q}^{\sigma,\delta}(\alpha, \beta)$ and $TGW_{s,t,p,q}^{\sigma,\delta}(\alpha, \beta)$

Theorem 2.1. *A function $f(z) \in GW_{s,t,p,q}^{\sigma,\delta}(\alpha, \beta)$ if*

$$\sum_{k=2}^{\infty} [1 + \alpha(k - 1)] D_{k,\delta} |b_k| \leq 1 - \beta \tag{2.1}$$

where $\alpha \geq 0, 0 \leq \beta < 1$ *and* $D_{k,\delta}$ *is given by (1.5).*

Proof. Since $\alpha \geq 0, 0 \leq \beta < 1$, it suffices to show that

$$\alpha\left|(W^{\sigma,\delta}_{s,t,p,q}f(z))' - \frac{(W^{\sigma,\delta}_{s,t,p,q}f(z))}{z}\right| - \Re\left\{\frac{(W^{\sigma,\delta}_{s,t,p,q}f(z))}{z} - 1\right\} \leq 1-\beta.$$

We have

$$\begin{aligned}
\alpha&\left|(W^{\sigma,\delta}_{s,t,p,q}f(z))' - \frac{(W^{\sigma,\delta}_{s,t,p,q}f(z))}{z}\right| - \Re\left\{\frac{(W^{\sigma,\delta}_{s,t,p,q}f(z))}{z} - 1\right\} \\
&\leq \alpha\left|(W^{\sigma,\delta}_{s,t,p,q}f(z))' - \frac{(W^{\sigma,\delta}_{s,t,p,q}f(z))}{z}\right| - \left|\frac{(W^{\sigma,\delta}_{s,t,p,q}f(z))}{z} - 1\right| \\
&\leq \alpha\sum_{k=2}^{\infty}(k-1)D_{k,\delta}|b_k| + \sum_{k=2}^{\infty}D_{k,\delta}|b_k| \\
&= \sum_{k=2}^{\infty}(1+\alpha(k-1))D_{k,\delta}|b_k|.
\end{aligned}$$

The last expression is bounded above by $(1-\beta)$ if

$$\sum_{k=2}^{\infty}(1+\alpha(k-1))D_{k,\delta}|b_k| \leq 1-\beta.$$

Theorem 2.2. *For* $\alpha \geq 0, 0 \leq \beta < 1$ *, a function* $f(z) \in TGW^{\sigma,\delta}_{s,t,p,q}(\alpha,\beta)$ *if and only if*

$$\sum_{k=2}^{\infty}[1+\alpha(k-1)]D_{k,\delta}|b_k| \leq 1-\beta. \tag{2.2}$$

Proof. Suppose $f(z)$ of the form (1.2) is in the class $TGW^{\sigma,\delta}_{s,t,p,q}(\alpha,\beta)$. Then

$$\Re\left(\frac{TW^{\sigma,\delta}_{s,t,p,q}f(z)}{z}\right) - \alpha\left|(TW^{\sigma,\delta}_{s,t,p,q}f(z))' - \frac{(TW^{\sigma,\delta}_{s,t,p,q}f(z))}{z}\right| \geq \beta$$

$$\Longrightarrow \Re(1 - \sum_{k=2}^{\infty}D_{k,\delta}|b_k|z^{k-1}) - \alpha|\sum_{k=2}^{\infty}(k-1)D_{k,\delta}\ b_k z^{k-1}| \geq \beta$$

Letting z to take real values and as $|z| \to 1$, we have

$$1 - \sum_{k=2}^{\infty}D_{k,\delta}|b_k| - \alpha\sum_{k=2}^{\infty}(k-1)D_{k,\delta}|b_k| \geq \beta$$

$$\Longrightarrow \qquad \sum_{k=2}^{\infty}(1+\alpha(k-1))D_{k,\delta}|b_k| \leq 1-\beta$$

where $\alpha \geq 0, 0 \leq \beta < 1$, $D_{k,\delta}$ is given by (1.5).

Corollary 2.1. *If $f \in TGW_{s,t,p,q}^{\sigma,\delta}(\alpha,\beta)$, then*

$$|b_k| \leq \frac{(1-\beta)}{(1+\alpha(k-1))D_{k,\delta}} \tag{2.3}$$

where $k \geq 2, \alpha \geq 0, 0 \leq \beta < 1$ and $D_{k,\delta}$ is given by (1.5).

Equality holds for the function

$$f_k(z) = z - \frac{(1-\beta)}{(1+\alpha(k-1))D_{k,\delta}} z^k, \tag{2.4}$$

$\alpha \geq 0, 0 \leq \beta < 1$, $D_{k,\delta}$ is given by (1.5).

3. Partial Sums of Functions in the subclass $GW_{s,t,p,q}^{\sigma,\delta}(\alpha,\beta)$

Silverman [6] investigated the partial sums of starlike and convex functions, we examine the ratio of the function of the form (1.4) to its sequence of partial sums where the coefficients of f are sufficiently small to satisfy the condition (2.1). We determine the sharp lower bounds for $\Re\left[\frac{W_{s,t,p,q}^{\sigma,\delta}f(z)}{W_{s,t,p,q}^{\sigma,\delta}f_m(z)}\right], \Re\left[\frac{W_{s,t,p,q}^{\sigma,\delta}f_m(z)}{W_{s,t,p,q}^{\sigma,\delta}f(z)}\right]$, $\Re\left[\frac{W_{s,t,p,q}^{\sigma,\delta}f'(z)}{W_{s,t,p,q}^{\sigma,\delta}f'_m(z)}\right]$ and $\Re\left[\frac{W_{s,t,p,q}^{\sigma,\delta}f'_m(z)}{W_{s,t,p,q}^{\sigma,\delta}f'(z)}\right]$.

In the sequel, we make frequent use of the well known result that $\Re\left[\frac{1+w(z)}{1-w(z)}\right] > 0$, $z \in U$ if and only if $w(z) = \sum_{k=1}^{\infty} d_k z^k$ satisfy the inequality $|w(z)| \leq |z|$.

Theorem 3.1. *If a function $f(z) \in GW_{s,t,p,q}^{\sigma,\delta}(\alpha,\beta)$ and satisfies (2.1), then*

$$\Re\left[\frac{W_{s,t,p,q}^{\sigma,\delta}f(z)}{W_{s,t,p,q}^{\sigma,\delta}f_m(z)}\right] \geq 1 - \frac{1}{e_{m+1}}, z \in U, m \in \mathbb{N} \tag{3.1}$$

and

$$\Re\left[\frac{W_{s,t,p,q}^{\sigma,\delta}f_m(z)}{W_{s,t,p,q}^{\sigma,\delta}f(z)}\right] \geq \frac{e_{m+1}}{1+e_{m+1}}, z \in U, m \in \mathbb{N} \tag{3.2}$$

where $e_k = \dfrac{1+\alpha(k-1)}{1-\beta}$.

The estimates in (3.1) and (3.2) are sharp for every m *with extremal function* $f(z) = z + \frac{1}{e_{m+1}} z^{m+1}$.

Proof. Clearly $e_{k+1} > e_k > 1, k = 2, 3, 4, \ldots$
Therefore, we have

$$\sum_{k=2}^{m} D_{k,\delta}|b_k| + e_{m+1} \sum_{k=2}^{m} D_{k,\delta}|b_k| \leq \sum_{k=2}^{m} e_k D_{k,\delta}|b_k| \leq 1.$$

Consider,

$$\begin{aligned} g(z) &= e_{m+1}\left[\frac{W^{\sigma,\delta}_{s,t,p,q} f(z)}{W^{\sigma,\delta}_{s,t,p,q} f_m(z)} - \left(1 - \frac{1}{e_{m+1}}\right)\right] \\ &= \frac{1 + \sum_{k=m+1}^{\infty} D_{k,\delta} b_k z^{k-1} + e_{m+1} \sum_{k=2}^{m} D_{k,\delta} b_k z^{k-1}}{1 + \sum_{k=2}^{m} D_{k,\delta} b_k z^{k-1}} \\ &= \frac{1 + A(z)}{1 + B(z)}. \end{aligned}$$

Set, $\dfrac{1+A(z)}{1+B(z)} = \dfrac{1+w(z)}{1-w(z)}$, so that

$$\begin{aligned} w(z) &= \frac{A(z) - B(z)}{2 + A(z) + B(z)} \\ &= \frac{e_{m+1} \sum_{k=m+1}^{\infty} D_{k,\delta} b_k z^{k-1}}{2 + 2\sum_{k=2}^{\infty} D_{k,\delta} b_k z^{k-1} + e_{m+1} \sum_{k=m+1}^{\infty} D_{k,\delta} b_k z^{k-1}} \end{aligned}$$

and $w(0) = 0$.

$$|w(z)| \leq \frac{e_{m+1} \sum_{k=m+1}^{\infty} D_{k,\delta}|b_k|}{2 - 2\sum_{k=2}^{\infty} D_{k,\delta}|b_k| - e_{m+1} \sum_{k=m+1}^{\infty} D_{k,\delta}|b_k|}$$

if

$$\sum_{k=2}^{\infty} D_{k,\delta}|b_k| + e_{m+1} \sum_{k=m+1}^{\infty} D_{k,\delta}|b_k| \leq 1. \tag{3.3}$$

The L.H.S of (3.3) is bounded above by $\sum_{k=2}^{\infty} e_k D_{k,\delta}|b_k| \geq 0$

$$\sum_{k=2}^{m}(e_k - 1)D_{k,\delta}|b_k| + \sum_{k=m+1}^{\infty}(e_k - e_{m+1})D_{k,\delta}|b_k| \geq 0.$$

The above inequality holds because e_k is a non- decreasing sequence.
To see that the function $f(z) = z + \frac{1}{e_{m+1}} z^{m+1}.$ gives a sharp result, we observe that for $z = re^{\frac{i\pi}{m}}$,

$$\frac{W^{\sigma,\delta}_{s,t,p,q} f(z)}{W^{\sigma,\delta}_{s,t,p,q} f_m(z)} = \frac{z + \frac{1}{e_{m+1}} z^{m+1}}{z}$$
$$= 1 + \frac{z^m}{e_{m+1}} \to 1 - \frac{1}{e_{m+1}} \text{when } r \to 1^-$$

By setting, $$h(z) = (1 + e_{m+1}) \left[\frac{W^{\sigma,\delta}_{s,t,p,q} f_m(z)}{W^{\sigma,\delta}_{s,t,p,q} f(z)} - \left(\frac{e_{m+1}}{1 + e_{m+1}} \right) \right].$$

The proof of (3.2) is akin (3.1).

Theorem 3.2. *If a function $f(z) \in GW^{\sigma,\delta}_{s,t,p,q}(\alpha, \beta)$ and satisfies (2.1), then*

$$\Re \left[\frac{W^{\sigma,\delta}_{s,t,p,q} f'(z)}{W^{\sigma,\delta}_{s,t,p,q} f'_m(z)} \right] \geq 1 - \frac{m+1}{e_{m+1}}, z \in U, m \in \mathbb{N} \tag{3.4}$$

and

$$\Re \left[\frac{W^{\sigma,\delta}_{s,t,p,q} f'_m(z)}{W^{\sigma,\delta}_{s,t,p,q} f'(z)} \right] \geq \frac{d_{m+1}}{m + 1 + e_{m+1}}, z \in U, m \in \mathbb{N} \tag{3.5}$$

where $e_k = \dfrac{1 + \alpha(k-1)}{1 - \beta}.$

The estimates in (3.4) and (3.5) are sharp for every m with extremal function $f(z) = z + \frac{1}{e_{m+1}} z^{m+1}.$

Proof. Clearly $e_{k+1} > e_k > 1, k = 2, 3, 4, \ldots.$ Therefore, we have

$$\sum_{k=2}^{m} D_{k,\delta}|b_k| + e_{m+1} \sum_{k=2}^{m} D_{k,\delta}|b_k| \leq \sum_{k=2}^{m} e_k D_{k,\delta}|b_k| \leq 1.$$

By setting, $$g(z)=\frac{e_{m+1}}{m+1}\left[\frac{W^{\sigma,\delta}_{s,t,p,q}f'(z)}{W^{\sigma,\delta}_{s,t,p,q}f'_m(z)}-\left(1-\frac{m+1}{e_{m+1}}\right)\right], z\in U$$

and $$h(z)=\frac{m+1+e_{m+1}}{m+1}\left[\frac{W^{\sigma,\delta}_{s,t,p,q}f'_m(z)}{W^{\sigma,\delta}_{s,t,p,q}f'(z)}-\left(\frac{e_{m+1}}{m+1+e_{m+1}}\right)\right], z\in U$$

the result follows and the details are omitted.

4. Integral means inequalities of $TGW^{\sigma,\delta}_{s,t,p,q}(\alpha,\beta)$

We obtain integral means inequalities for the functions in the subclass $TGW^{\sigma,\delta}_{s,t,p,q}(\alpha,\beta)$.

Lemma 4.1. [1] *If the functions f and g are analytic in $\mathbb{U}$ with $g\prec f$, then for $\kappa>0$, and $0<r<1$*

$$\int_0^{2\pi}|g(re^{i\theta})|^{\kappa}d\theta\le\int_0^{2\pi}|f(re^{i\theta})|^{\kappa}d\theta. \tag{4.1}$$

In [3], Silverman found the function $f(z)=z-\frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured [4] and settled [5], that

$$\int_0^{2\pi}|f(re^{i\theta})|^{\kappa}d\theta\le\int_0^{2\pi}|f_2(re^{i\theta})|^{\kappa}d\theta \tag{4.2}$$

for all $f\in T,\kappa>0$ and $0<r<1$. In [5], he also proves his conjecture for the subclasses of starlike functions of order α and convex functions of order α.
Using theorem 2.2 and lemma 4.1, we prove the following result.

Theorem 4.1. *Suppose $f(z)\in TGW^{\sigma,\delta}_{s,t,p,q}(\alpha,\beta)$ and $f_2(z)$ is defined by*

$$f_2(z)=z-\frac{(1-\beta)}{(1+\alpha(k-1))D_{k,\delta}}z^2,$$

then for $z=re^{i\theta}, 0<r<1$, we have,

$$\int_0^{2\pi}|f(z)|^{\kappa}d\theta\le\int_0^{2\pi}|f_2(z)|^{\kappa}d\theta. \tag{4.3}$$

Proof. For $f(z)=z-\sum_{k=2}^{\infty}b_kz^k$ and $f_2(z)=z-\frac{(1-\beta)}{(1+\alpha(k-1))D_{k,\delta}}z^2$ then we show that

$$\int_0^{2\pi}\left|1-\sum_{k=2}^{\infty} b_k z^k\right|^{\kappa} d\theta \le \int_0^{2\pi}\left|1-\frac{1+\beta}{(1+\alpha(k-1))D_{k,\delta}} z\right|^{\kappa} d\theta.$$

By Lemma 4.1, it suffices to show that

$$\Longrightarrow 1-\sum_{k=2}^{\infty} b_k|z^{k-1}| \le 1-\frac{1+\beta}{(1+\alpha(k-1))D_{k,\delta}}|z|$$

$$\Longrightarrow 1-\sum_{k=2}^{\infty} b_k|z^{k-1}| = 1-\frac{1+\beta}{(1+\alpha(k-1))D_{k,\delta}}|z|$$

$$\Longrightarrow \left|\sum_{k=2}^{\infty} b_k z^{k-1}\right| = \left|\frac{1+\beta}{(1+\alpha(k-1))D_{k,\delta}} z\right|$$

$$|w(z)| = \sum_{k=2}^{\infty}\frac{1+\beta}{(1+\alpha(k-1))D_{k,\delta}} b_k|z|^{k-1} \le |z|.$$

5. Conclusion

This article makes use of the $(p-q)$-Wanas operator and defines two new subclasses of analytic univalent functions. This defined classes analyzes coefficient estimates, partial sums and integral mean results for the functions belonging to them. Furthermore, we believe that this study can be extended to analyzing various other properties.

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