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NEW APPLICATIONS OF A q-CALCULUS OPERATOR REGARDING NEW SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract: In the present article, a new subclasses of analytic function is introduced by making use of linear multiplier fractional q-differentiable operator. For functions belonging to these classes we obtained coefficient estimates, extreme points, q- Bernardi integral operator and many more properties.

Keywords and Phrases: Analytic functions, Univalent functions, Bernardi Operator, Linear Multiplier Fractional q-Differentialble Operator.

2020 Mathematics Subject Classification: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$\phi(\tau) = \tau + \sum_{l=2}^{\infty} r_l \tau^l \tag{1.1}$$

which are analytic and univalent in the open unit disc $\mathcal{U}$.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second terms onwards are negative. That is, an analytic function $\phi \in \mathcal{T}$ if it has a Taylor series expansion of the form

$$\phi(\tau) = \tau - \sum_{l=2}^{\infty} |r_l|\tau^l \qquad (r_l \geq 0) \tag{1.2}$$

which are analytic in the open disc $\mathcal{U}$.

Now recall the following q-analogue definitions given by Gasper and Rahman [7]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)},$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function ϕ defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [7])

$$D_q\phi(\tau) = \frac{\phi(\tau) - \phi(\tau q)}{\tau(1-q)}, \ \ (\tau \neq 0, \, q \neq 0).$$

$$\int_0^{\tau} \phi(t)d_q(t) = \tau(1-q)\sum_{m=0}^{\infty} q^m\phi(\tau q^m).$$

A linear multiplier fractional q-differentiable operator [13] is defined as

$$\mathbf{D}_{q,\mu}^{\rho,0}\phi(\tau) = \phi(\tau)$$

$$\mathbf{D}_{q,\mu}^{\rho,1}\phi(\tau) = (1-\mu)\Omega_q^{\rho}\phi(\tau) + \mu\tau\mathbf{D}_q\left(\Omega_q^{\rho}\phi(\tau)\right),$$

$$\mathbf{D}_{q,\mu}^{\rho,2}\phi(\tau) = \mathbf{D}_{q,\mu}^{\rho,1}\left(\mathbf{D}_{q,\mu}^{\rho,1}\phi(\tau)\right)$$

$$\vdots$$

$$\mathbf{D}_{q,\mu}^{\rho,t}\phi(\tau) = \mathbf{D}_{q,\mu}^{\rho,1}\left(\mathbf{D}_{q,\mu}^{\rho,t-1}\phi(\tau)\right) \tag{1.3}$$

We note that if $\phi \in \mathcal{A}$ is given by (1.1), then by (1.3), we have

$$\mathbf{D}_{q,\mu}^{\rho,t}\phi(\tau) = \tau + \sum_{l=2}^{\infty} B(l,\rho,\mu,t,q)r_l\tau^l \tag{1.4}$$

where

$$B(l,\rho,\mu,t,q) = \left(\frac{\Gamma_q(2-\rho)\Gamma_q(l+1)}{\Gamma_q(l+1-\rho)}\left[([l]_q-1)\mu+1\right]\right)^t. \tag{1.5}$$

It can be seen that, for different parametric values, the operator $\mathbf{D}^{\rho,t}_{q,\mu}$ reduces to many known and new integral and differential operators. In particular, when $\rho = 0$, $q \to 1^-$ the operator $\mathbf{D}^{\rho,t}_{q,\mu}$ reduces to the operator introduced by Al-Oboudi [1] and if $\rho = 0$, $\mu = 1$ and $q \to 1^-$ it reduces to the operator introduced by Sălăgean [15].

Now using linear multiplier fractional q-differentiable operator $\mathbf{D}^{\rho,t}_{q,\mu}\phi(\tau)$, we define the following subclasses $\mathcal{S}_q(\epsilon,\zeta,\rho,\mu,b)$ and $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ of analytic function.

Definition 1.1. *For* $-1 \le \epsilon < 1$, $\zeta \ge 0$, $0 < \rho < 2$, $b \in \mathbb{C} - \{0\}$ *and* $0 < q < 1$, *let* $\mathcal{S}_q(\epsilon,\zeta,\rho,\mu,b)$ *be the subclass of* $\mathcal{A}$ *consisting of functions of the form* (1.1), *and satisfying the analytic criterion*

$$\Re\left\{1-\frac{2}{b}+\frac{2}{b}\left(\frac{\tau D_q[D^{\rho,t}_{q,\mu}\phi]}{D^{\rho,t}_{q,\mu}\phi}\right)\right\} > \zeta\left\{\frac{2}{b}\left(\frac{\tau D_q[D^{\rho,t}_{q,\mu}\phi]}{D^{\rho,t}_{q,\mu}\phi}\right)-\frac{2}{b}\right\}+\epsilon, \quad \tau\in\mathcal{U}. \tag{1.6}$$

Let $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b) = \mathcal{S}_q(\epsilon,\zeta,\rho,\mu,b) \cap \mathcal{T}$.

It can be seen that, the special cases of the class $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ as $q \to 1^-$ and for different choices of the parameters we get the results obtained by Altintat and Owa [2], Bharathi, Parvatham and Swaminathan [4], Caglar and Orhan [5], Caglar, Orhan and Srivastava [6], Gour, Joshi and Purohit [8], Padamanabhan and Jayamala [12], Owa and Srivastava [11], Kim and Ronning [9], Ravikumar N, S Latha and B A Frasin [14], Selvakumaran, Rajaguru, Purohit and Suthar [16].

2. Main Results

First we prove the necessary and sufficient conditions for the class $\mathcal{S}_q(\epsilon,\zeta,\rho,\mu,b)$ and $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$.

Theorem 2.1. *A sufficient condition for a function* ϕ *of the form* (1.1), *to be in* $\mathcal{S}_q(\epsilon,\zeta,\rho,\mu,b)$ *is that*

$$\sum_{l=2}^{\infty}\left[2(1+\zeta)([l]_q-1)+b(1-\epsilon)\right]B(l,\rho,\mu,t,q)|r_l| \le b(1-\epsilon). \tag{2.1}$$

Proof. Suppose $\phi \in \mathcal{S}_q(\epsilon,\zeta,\rho,\mu,b)$,

$$\zeta\left|\frac{2}{b}\left(\frac{\tau D_q[D^{\rho,t}_{q,\mu}\phi]}{D^{\rho,t}_{q,\mu}\phi}-1\right)\right| - \Re\left\{\frac{2}{b}\left(\frac{\tau D_q[D^{\rho,t}_{q,\mu}\phi]}{D^{\rho,t}_{q,\mu}\phi}-1\right)\right\}$$

$$\leq (1+\zeta)\left|\frac{2}{b}\left(\frac{\tau D_q[D^{\rho,t}_{q,\mu}\phi]}{D^{\rho,t}_{q,\mu}\phi}-1\right)\right|$$

$$\leq \frac{2(1+\zeta)}{b}\left[\frac{\sum_{l=2}^{\infty}([l]_q-1)B(l,\rho,\mu,t,q)|r_l||\tau|^{l-1}}{1-\sum_{l=2}^{\infty}B(l,\rho,\mu,t,q)|r_l||\tau|^{l-1}}\right].$$

Letting $\tau\to 1$ we have,

$$\leq \frac{2(1+\zeta)}{b}\left[\frac{\sum_{l=2}^{\infty}([l]_q-1)B(l,\rho,\mu,t,q)|r_l|}{1-\sum_{l=2}^{\infty}B(l,\rho,\mu,t,q)|r_l|}\right].$$

This is bounded above by $1-\epsilon$ if

$$\sum_{l=2}^{\infty}\left[2(1+\zeta)([l]_q-1)+b(1-\epsilon)\right]B(l,\rho,\mu,t,q)|r_l|\leq b(1-\epsilon).$$

Hence the proof.

Theorem 2.2. *A function* $\phi\in\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ *and* τ *if*

$$\sum_{l=2}^{\infty}\frac{\left[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)\right]B(l,\rho,\mu,t,q)}{b(1-\epsilon)}\leq 1.$$

Proof. In view of Theorem 2.1, we need to prove the necessity part of the theorem. If $\phi\in\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ and τ real then

$$\Re\left\{1-\frac{2}{b}+\frac{2}{b}\left(\frac{\tau D_q[D^{\rho,t}_{q,\mu}\phi]}{D^{\rho,t}_{q,\mu}\phi}\right)\right\}-\epsilon\geq\zeta\left\{\frac{2}{b}\left(\frac{\tau D_q[D^{\rho,t}_{q,\mu}\phi]}{D^{\rho,t}_{q,\mu}\phi}\right)-\frac{2}{b}\right\}$$

That is,

$$\frac{2}{b}\left[\sum_{l=2}^{\infty}(1-[l]_q)B(l,\rho,\mu,t,q)|r_l|\tau^{l-1}\right]$$

$$\leq(1-\epsilon)\left[1-\sum_{l=2}^{\infty}B(l,\rho,\mu,t,q)|r_l|\tau^{l-1}\right].$$

Letting $\tau \to 1,$ along the real axis we obtain the desired inequality,

$$\sum_{l=2}^{\infty} \frac{2(\zeta-1)(1-[l]_q)+b(1-\epsilon)}{b(1-\epsilon)} B(l,\rho,\mu,t,q) \leq 1.$$

Theorem 2.3. *Let* $\phi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ *define* $\phi_1(\tau)=\tau$ *and*

$$\phi_l(\tau)=\tau-\frac{b(1-\epsilon)}{[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)}\tau^l.$$

Then $\phi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ *if and only if* ϕ *can be expressed as*

$$\phi(\tau)=\sum_{l=1}^{\infty}\chi_l\phi_l(\tau), \quad \text{where} \quad \chi_l \geq 0 \quad \text{and} \quad \sum_{l=1}^{\infty}\chi_l=1. \tag{2.2}$$

Proof. If $\phi(\tau)=\sum_{l=1}^{\infty}\chi_l\phi_l(\tau)$ with $\sum_{l=1}^{\infty}\chi_l=1,\ \chi_l\geq 0$ then

$$\sum_{l=2}^{\infty}[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)\chi_l\left[\frac{b(1-\epsilon)}{[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)}\right]$$

$$=\sum_{l=2}^{\infty}\chi_l b(1-\epsilon)=(1-\chi_1)(1-\epsilon)b \leq b(1-\epsilon).$$

Hence $\phi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$. Conversely, let $\phi(\tau)=\tau-\sum_{l=2}^{\infty}|r_l|\tau^l \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$, define

$$\chi_l=\frac{[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)}{b(1-\epsilon)}|r_l|$$

and define $\chi_1=1-\sum_{l=2}^{\infty}\chi_l$. From Theorem 2.1,

$$\sum_{l=2}^{\infty}\chi_l \leq 1 \quad \text{and} \quad \chi_1 \geq 0.$$

Therefore, we can see that $\phi(\tau)$ can be expressed in the (2.2).

Corollary 2.4. *Let* $\phi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$, *then*

$$|r_l| < \frac{b(1-\epsilon)}{[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)}.$$

Theorem 2.5. *Let* $\mu_1 < \mu_2$ *then* $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu_2,b) \subset \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu_1,b)$.
Proof. Let $\phi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu_2,b)$, then we have

$$\frac{\sum_{l=2}^{\infty}[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu_2,t,q)}{b(1-\epsilon)} \leq 1,$$

but hence $B(l,\rho,\mu,t,q)$ is an increasing function of μ,
$B(l,\rho,\mu_1,t,q) < B(l,\rho,\mu_2,t,q)$, so we have

$$\frac{\sum_{l=2}^{\infty}[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu_1,t,q)}{b(1-\epsilon)}|r_l|$$

$$< \frac{\sum_{l=2}^{\infty}[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu_2,t,q)}{b(1-\epsilon)}|r_l| \leq 1,$$

then $\phi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu_1,b)$.

Theorem 2.6. *The class* $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ *is closed under convex linear combination.*
Proof. Let ϕ and ψ be the arbitrary elements of $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ then for every $\eta (0<\eta<1)$, we need to show that
$(1-\eta)\phi+\eta\psi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu_1,b)$. Thus we have

$$(1-\eta)\phi+\eta\psi = \tau - \sum_{l=2}^{\infty}[(1-\eta)|r_l|+\eta|b_l|]\tau^l$$

and

$$\frac{\sum_{l=2}^{\infty}[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)}{b(1-\epsilon)}[(1-\eta)|r_l|+\eta|b_l|] < 1.$$

Theorem 2.7. *Let* $\phi \in \mathcal{TS}_q(\epsilon, \zeta, \rho, \mu, b)$. *For the q- analogous Bernardi's integarl operator defined by*

$$L_{q,\gamma}\phi(\tau) = \frac{[\gamma+1]_q}{\tau^\gamma}\int_0^\tau s^{\gamma-1}\phi(s)d_qs,$$

then we have $L_{q,\gamma}\phi \in \mathcal{TS}_q(\epsilon, \zeta, \rho, \mu, b)$.

Proof. We have

$$L_{q,\gamma}\phi(\tau) = \frac{[\gamma+1]_q}{\tau^\gamma}\tau(1-q)\sum_{j=0}^{\infty} q^j(\tau q)^{\gamma-1}\phi(\tau q^j)$$

$$= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma}\phi(\tau q^j)$$

$$= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma}\sum_{l=1}^{\infty} q^{j\gamma}|r_l|\tau^l$$

$$= \tau + \sum_{l=2}^{\infty}\frac{[\gamma+1]_q}{[\gamma+l]_q}|r_l|\tau^l.$$

Since $\phi \in \mathcal{TS}_q(\epsilon, \zeta, \rho, \mu, b)$ and since $\dfrac{[\gamma+1]_q}{[\gamma+l]_q} < 1, \quad \forall \quad l \geq 2,$ we have

$$\sum_{l=2}^{\infty}[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)|r_l|\frac{[\gamma+1]_q}{[\gamma+l]_q} < b(1-\epsilon).$$

Theorem 2.8. *Let* $\phi \in \mathcal{TS}_q(\epsilon, \zeta, \rho, \mu, b)$ *then* $L_{q,\gamma}\phi(\tau)$ *is q- starlike of order* $0 \leq \epsilon_3 \leq 1$ *in* $|\tau| \leq R_1$ *where*

$$R_1 = inf\left\{\left[\frac{[\gamma+l]_q}{[\gamma+1]_q}\frac{(1-\epsilon_3)[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]b(l,\rho,\mu,t,q)}{[l]_q([l]_q-\epsilon_3)b(1-\epsilon)}\right]^{\frac{1}{l-1}} : m \in \mathbb{N}/\{1\}.\right\}$$

Proof. It is sufficient to prove

$$\left|\frac{\tau(D_qL_{q,\gamma}\phi(\tau))}{L_{q,\gamma}\phi(\tau)} - 1\right| < 1-\epsilon_3, \qquad \tau \in \mathcal{U}.$$

Now

$$\left|\frac{\tau(D_qL_{q,\gamma}\phi(\tau))}{L_{q,\gamma}\phi(\tau)} - 1\right|$$

$$= \left| \frac{\sum_{l=2}^{\infty}([l]_q - 1)|r_l|\tau^{l-1}\frac{[\gamma+1]_q}{[\gamma+l]_q}}{1+\sum_{l=2}^{\infty}|r_l||\tau^{l-1}|\frac{[\gamma+1]_q}{[\gamma+l]_q}} \right|.$$

This last expression is less than $1-\epsilon_3$, since

$$|\tau^{l-1}| \leq \left\{ \left[\frac{[\gamma+l]_q}{[\gamma+1]_q} \frac{(1-\epsilon_3)[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]b(l,\rho,\mu,t,q)}{([l]_q-\epsilon_3)b(1-\epsilon)} \right] \right\}.$$

Using the fact that ϕ is convex if and only if $\tau D_q\phi$ is starlike, we obtain the following.

Theorem 2.9. *Let* $\phi \in \mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ *then* $L_{q,\gamma}\phi(\tau)$ *is q- convex of order* $0 \leq \epsilon_3 \leq 1$ *in* $|\tau| \leq R_1$ *where*

$$R_1 = inf \left\{ \left[\frac{[\gamma+l]_q}{[\gamma+1]_q} \frac{(1-\epsilon_3)[2(\zeta-1)(1-[l]_q)+b(1-\epsilon)]B(l,\rho,\mu,t,q)}{[l]_q([l]_q-\epsilon_3)b(1-\epsilon)} \right]^{\frac{1}{l-1}} : m \in \mathbb{N}/\{1\}. \right\}$$

3. Conclusions

This article investigated two subclasses of analytic functions $\mathcal{S}_q(\epsilon,\zeta,\rho,\mu,b)$ and $\mathcal{TS}_q(\epsilon,\zeta,\rho,\mu,b)$ on unit disc U. For functions belonging to these classes, we calculated co-efficient estimate, extreme points, convex linear combination, q-Bernalli integral operator, q-Starlike and q-convex functions. Further research can be conducted to investigate more properties using these classes.

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