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STRONG (WEAK) NEIGHBOURHOOD COVERING SETS OF A GRAPH

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Abstract: The ve-degree of a vertex $u \in V(G)$, denoted by $d_{ve}(u)$, is the number of edges in the subgraph $\langle N[u] \rangle$. A vertex u is said to n-cover (neighbourhood-cover) an edge e if e is an edge of the subgraph $\langle N[u] \rangle$. A set $S \subseteq V(G)$ is called a n-covering set of a graph G if every edge in G is n-covered by some vertex in S. The n-covering number $\alpha_n(G)$ is the minimum cardinality of a n-covering set of G. In this paper, we introduce new parameters such as strong (weak) n-covering number and strong (weak) n-independence number using ve-degrees of vertices, and we establish a relationship between them. Further, we define and study n-cover balanced sets.

Keywords and Phrases: ve-degree, n-cover, strong n-covering number, n-cover balanced graph.

2020 Mathematics Subject Classification: 05C07, 05C69, 05C70.

1. Introduction

By a graph G, we refer to a finite, simple, undirected graph with a vertex set $V(G)$ and an edge set $E(G)$. Let $|V(G)| = p$ denote the order of G, and $|E(G)| = q$ denote the size of G. The terminologies and notations used here follow those in [3, 8]. For any $v \in V(G)$, the set $N[v] = \{u \in V(G) : uv \in E(G)\} \cup \{v\}$ represents the closed neighborhood of v. If $S \subseteq V(G)$, then the induced subgraph $\langle S \rangle$ of G has vertex set S and edge set $E(\langle S \rangle) = \{uv \in E(G) \mid u \in S \text{ and } v \in S\}$. A vertex v is said to cover an edge e if e is incident on v. A set $D \subseteq V(G)$ is called a vertex cover of G if every edge in G is covered by some vertex in D. The vertex covering number $\alpha(G)$ is the minimum cardinality of a vertex cover of G. The concepts of strong and weak vertex coverings were first introduced by S. S. Kamath and R. S. Bhat [4]. For an edge $e = uv$, vertex v strongly covers the edge e if $d(v) \geq d(u)$. In such a case, vertex u weakly covers e. A set $S \subseteq V(G)$ is a strong (weak) vertex cover of a graph G if every edge in G is strongly (weakly) covered by some vertex in S. The strong (weak) vertex covering number $s\alpha(G)$ ($w\alpha(G)$) is the minimum cardinality of a strong (weak) vertex cover of G. These two parameters satisfy the following inequality: for any graph G, $s\alpha(G) \leq w\alpha(G) \leq \alpha(G)$.

The concept of the ve-degree of a vertex was introduced by S. S. Kamath and R. S. Bhat [5]. The ve-degree of a vertex $u \in V(G)$, denoted by $d_{ve}(u)$, is the number of edges in the subgraph $\langle N[u] \rangle$. If G is a triangle-free graph, then $d_{ve}(u) = d(u)$ for every $u \in V(G)$. The maximum ve-degree of a graph G is denoted by $\Delta_{ve}(G)$, and the minimum ve-degree of G is denoted by $\delta_{ve}(G)$. A graph G is said to be ve-regular if $d_{ve}(u) = d_{ve}(w)$ for every $u, w \in V(G)$.

In 1985, E. Sampathkumar and P. S. Neeralagi [6] initiated the study of the neighborhood set of a graph. A set $S \subseteq V(G)$ is called a neighborhood set of G if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by $N[v]$. The neighborhood number $n_0(G)$ is the minimum cardinality of a neighborhood set of G. A vertex v is said to n-cover (neighborhood-cover) an edge e if e is an edge of the induced subgraph $\langle N[v] \rangle$. A set $S \subseteq V(G)$ is called an n-covering set of a graph G if every edge in G is n-covered by some vertex in S. The n-covering number, denoted as $\alpha_n(G)$, is the minimum cardinality of a n-covering set of G. Note that, for any graph G without isolated vertices, any n-covering set of G is also a neighborhood set of G, and vice versa. Therefore, $n_0(G) = \alpha_n(G)$ for any graph G without isolated vertices. Additionally, if a graph G has k isolated vertices, then $n_0(G) = \alpha_n(G) + k$. S. G. Bhat [1] introduced the concept of n-independent (neighborhood-independent) sets. A set $S \subseteq V(G)$ is said to be n-independent if every edge $e \in E(\langle S \rangle)$ is n-covered by a vertex in $V(G) - S$. The n-independence number $\beta_n(G)$ of a graph G is the maximum cardinality of an n-independent set of G. These two parameters satisfy the following relation: $\alpha_n(G) + \beta_n(G) = p$. The properties of n-covering sets and n-independent sets were further studied in [2].

2. Strong (weak) n-covering sets and strong (weak) n-independent sets of a graph

Definition 2.1. *A vertex $u \in V(G)$ strongly (weakly) n-covers an edge $e \in E(G)$ if u n-covers e and $d_{ve}(u) \geq d_{ve}(w)$ $(d_{ve}(u) \leq d_{ve}(w))$ for every w which n-covers e.*

Definition 2.2. *A set $S \subseteq V(G)$ is said to be a strong (weak) n-covering set of G if vertices in S strongly (weakly) n-covers all the edges of G. The strong (weak) n-covering number $s\alpha_n(G)$ $(w\alpha_n(G))$ of G is the minimum cardinality of a strong (weak) n-covering set of G. That is, $s\alpha_n(G) = \min\{|S| : S \text{ is a strong n-covering set}\}$.*

Definition 2.3. *A set $S \subseteq V(G)$ is said to be a strong (weak) n-independent set of G if for every edge e in $\langle S \rangle$, there exists a vertex $w \in V(G) - S$ such that w weakly (strongly) n-covers e. The strong (weak) n-independence number $s\beta_n(G)$ $(w\beta_n(G))$ of G is the maximum cardinality of a strong (weak) n-independent set of G.*

Remark 2.1.

(i) *For a null p-vertex graph $\overline{K_p}$, we assume that $s\alpha_n(\overline{K_p}) = w\alpha_n(\overline{K_p}) = 0$ and $s\beta_n(\overline{K_p}) = w\beta_n(\overline{K_p}) = p$.*

(ii) *Let G be a non-trivial and non-null graph and $u_{\Delta_{ve}}$ $(u_{\delta_{ve}})$ be a vertex of G of maximum (minimum) ve-degree. Then $V(G) - \{u_{\delta_{ve}}\}$ $(V(G) - \{u_{\Delta_{ve}}\})$ is a strong (weak) n-covering set of G. Further, $\{u_{\Delta_{ve}}\}$ $(\{u_{\delta_{ve}}\})$ is a strong (weak) n-independent set of G.*

Example 2.1. For the graph G_1 shown in Figure 1, the ve-degrees are as follows: $d_{ve}(v_1) = 3$, $d_{ve}(v_2) = d_{ve}(v_3) = 5$, $d_{ve}(v_4) = 4$, $d_{ve}(v_5) = 2$, and $d_{ve}(v_6) = 1$.

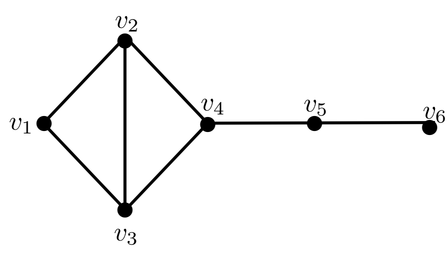

Figure 1: Graph G_1

Note that $\{v_3, v_5\}$ is an n-covering set of G_1, $\{v_3, v_4, v_5\}$ is a strong n-covering set of G_1, and $\{v_1, v_4, v_5, v_6\}$ is a weak n-covering set of G_1. Furthermore, $\{v_1, v_2, v_4, v_6\}$ is an n-independent set of G_1, $\{v_2, v_3\}$ is a strong n-independent set of G_1, and $\{v_1, v_3, v_6\}$ is a weak n-independent set of G_1. Hence, $\alpha_n(G_1) = 2$, $s\alpha_n(G_1) = 3$, $w\alpha_n(G_1) = 4$, $\beta_n(G_1) = 4$, $s\beta_n(G_1) = 2$, and $w\beta_n(G_1) = 3$.

2.1. Preliminary Results

We compute strong (weak) n-covering number and strong (weak) n-independence number of some standard graphs.

Proposition 2.1.

(i) *For a path P with $p \geq 3$ vertices, $\alpha_n(P) = s\alpha_n(P) = \left\lfloor \frac{p}{2} \right\rfloor$, $w\alpha_n(P) = \left\lceil \frac{p+1}{2} \right\rceil$, $s\beta_n(P) = \left\lfloor \frac{p-1}{2} \right\rfloor$ and $\beta_n(P) = w\beta_n(P) = \left\lceil \frac{p}{2} \right\rceil$.*

(ii) *For a cycle C with $p \geq 4$ vertices, $\alpha_n(C) = s\alpha_n(C) = w\alpha_n(C) = \left\lceil \frac{p}{2} \right\rceil$ and $\beta_n(C) = s\beta_n(C) = w\beta_n(C) = \left\lceil \frac{p}{2} \right\rceil$.*

(iii) *For a complete bipartite graph $K_{m,l}$, $\alpha_n(K_{m,l}) = s\alpha_n(K_{m,l}) = s\beta_n(K_{m,l}) = \min\{m, l\}$ and $w\alpha_n(K_{m,l}) = \beta_n(K_{m,l}) = w\beta_n(K_{m,l}) = \max\{m, l\}$.*

(iv) *For a complete graph K_p with p vertices, $\alpha_n(K_p) = s\alpha_n(K_p) = w\alpha_n(K_p) = 1$ and $\beta_n(K_p) = s\beta_n(K_p) = w\beta_n(K_p) = p - 1$.*

(v) *For a wheel graph W_p with $p \geq 5$ vertices, $\alpha_n(W_p) = s\alpha_n(W_p) = 1$, $w\alpha_n(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$, $\beta_n(W_p) = w\beta_n(W_p) = p - 1$ and $s\beta_n(W_p) = \left\lceil \frac{p}{2} \right\rceil$.*

(vi) *For a windmill graph $Wd(k, l)$ with $k \geq 2$ and $l \geq 2$, $\alpha_n(Wd(k, l)) = s\alpha_n(Wd(k, l)) = 1$, $w\alpha_n(Wd(k, l)) = l$, $\beta_n(Wd(k, l)) = w\beta_n(Wd(k, l)) = l(k - 1)$ and $s\beta_n(Wd(k, l)) = l(k - 2) + 1$.*

(vii) *For a Dutch windmill graph $D_k^{(m)}$ with $k > 4$ and $m \geq 2$, $\alpha_n(D_k^{(m)}) = s\alpha_n(D_k^{(m)}) = 1 + m \left\lceil \frac{k-2}{2} \right\rceil$, $w\alpha_n(D_k^{(m)}) = m \left\lceil \frac{k}{2} \right\rceil$, $\beta_n(D_k^{(m)}) = w\beta_n(D_k^{(m)}) = m \left\lfloor \frac{k}{2} \right\rfloor$ and $s\beta_n(D_k^{(m)}) = 1 + m \left\lceil \frac{k-3}{2} \right\rceil$.*

Proposition 2.2. *Let G be a connected graph of order $p > 1$. Then,*

(i) *$s\alpha_n(G) = 1$ if and only if there exists $v \in V(G)$ such that $d(v) = p - 1$.*

(ii) *$w\alpha_n(G) = 1$ if and only if $G = K_p$.*

Remark 2.2.

(i) *For any graph* G, $\alpha_n(G) \leq \min\{s\alpha_n(G), w\alpha_n(G)\}$ *and* $\max\{s\beta_n(G), w\beta_n(G)\} \leq \beta_n(G)$.

(ii) *If a graph* G *has no triangles, then* $s\alpha_n(G) = s\alpha(G)$ *and* $w\alpha_n(G) = w\alpha(G)$.

2.2. Gallai-type results

We first prove the following and obtain Gallai-type results for the new parameters defined.

Proposition 2.3. *Let* $G = (V, E)$ *be a graph. For any set* $S \subseteq V(G)$,

(i) S *is a strong n-covering set of* G *if and only if* $V(G) - S$ *is a weak n-independent set of* G.

(ii) S *is a weak n-covering set of* G *if and only if* $V(G) - S$ *is a strong n-independent set of* G.

Proof. Let S be a strong n-covering set of G and $W = V(G) - S$. Let e be an edge in the subgraph $\langle W \rangle$. Since S is a strong n-covering set, there exists $u \in S$ such that u strongly n-covers e. Thus, W is a weak n-independent set of G. Conversely, let W be a weak n-independent set and $S = V(G) - W$. Let $e \in E(G)$. Then, we consider the following two cases:
Case 1. If $e \in E(\langle W \rangle)$, then there exists $u \in V(G) - W = S$ such that u strongly n-covers e.
Case 2. If $e \notin E(\langle W \rangle)$, then u be a vertex in $V(G)$ which strongly n-covers e. Now, suppose $u \in W$, then $e \in E(\langle N[u] \rangle) \subseteq E(\langle W \rangle)$, which is a contradiction. This implies that, $u \in V(G) - W = S$.
Hence, S is a strong n-covering set of G. With the similar arguments, we can prove that the complement of a weak n-covering set of G is a strong n-independent set of G.

Theorem 2.1. *For any graph* G *of order* $p > 1$,

(i) $s\alpha_n(G) + w\beta_n(G) = p$

(ii) $w\alpha_n(G) + s\beta_n(G) = p$.

Proof. Let S be a strong n-covering set of G such that $|S| = s\alpha_n(G)$. Then by Proposition 2.3, $V(G) - S$ is a weak n-independent set of G. Hence, $w\beta_n(G) \geq |V(G) - S| = p - s\alpha_n(G)$. Therefore, $s\alpha_n(G) + w\beta_n(G) \geq p$. Again, if W is a weak n-independent set of G such that $|W| = w\beta_n(G)$. Then $V(G) - W$ is a strong n-covering set by Proposition 2.3. Hence, $s\alpha_n(G) \leq |V(G) - W|$. That is, $s\alpha_n(G) + w\beta_n(G) \leq p$. Then from the above inequalities (i) follows. Similarly, (ii) holds.

3. Strong and weak ve-degree of a vertex

Definition 3.1. *The strong (weak) ve-degree of a vertex $u \in V(G)$, denoted by $d_{sve}(u)$ ($d_{wve}(u)$), is the number of edges strongly (weakly) n-covered by u. Then $\Delta_{sve}(G)$ ($\Delta_{wve}(G)$) and $\delta_{sve}(G)$ ($\delta_{sve}(G)$) represent the maximum strong (weak) ve-degree and minimum strong (weak) ve-degree of G, respectively.*

Definition 3.2. *The regular ve-degree of a vertex $u \in V(G)$, denoted by $d_{rve}(u)$, is the number of edges which are both strongly and weakly n-covered by u. The balanced ve-degree of a vertex $u \in V(G)$, denoted by $d_{bve}(u)$, is the number of edges which are neither strongly nor weakly n-covered by u.*

Definition 3.3. *A vertex $u \in V(G)$ is called strong (weak) ve-silent if $d_{sve}(u) = 0$ ($d_{wve}(u) = 0$). A set $S \subseteq V(G)$ is said to be strong (weak) ve-silent set if for every vertex $u \in S$, $d_{sve}(u) = 0$ ($d_{wve}(u) = 0$). The strong (weak) ve-silent number $\Theta_{sve}(G)$ ($\Theta_{wve}(G)$) is the maximum cardinality of a strong (weak) ve-silent set of G.*

Remark 3.1. *For any graph G, $\Delta_{ve}(G) = \Delta_{sve}(G)$.*

Theorem 3.1. *Let G be a graph. Then for any vertex $u \in V(G)$, $d_{ve}(u) = d_{sve}(u) + d_{wve}(u) + d_{bve}(u) - d_{rve}(u)$.*

Proof. Consider a vertex $u \in V(G)$. Let D be the set of all edges n-covered by u, S be the set of edges strongly n-covered by u, W be the set of edges weakly n-covered by u, R be the set of edges both strongly and weakly n-covered by u, and B be the set of edges neither strongly nor weakly n-covered by u. By definition, $S \cap W = R$, $S \cap B = \emptyset$, $W \cap B = \emptyset$, and $R \cap B = \emptyset$. Hence, we have $d_{ve}(u) = |D| = |S \cup W \cup B| = |S| + |W| + |B| - |S \cap W| - |S \cap B| - |W \cap B| + |S \cap W \cap B|$. Since $S \cap B = \emptyset$, $W \cap B = \emptyset$, and $R \cap B = \emptyset$, this simplifies to $d_{ve}(u) = |S| + |W| + |B| - |R|$. Therefore, $d_{ve}(u) = d_{sve}(u) + d_{wve}(u) + d_{bve}(u) - d_{rve}(u)$.

4. Bounds on $s\alpha_n(G)$ and $w\alpha_n(G)$

Proposition 4.1. *If there exists a strong (weak) n-covering set of a graph G which is also a strong (weak) n-independent set of G, then*

(i) $s\alpha_n(G) + w\alpha_n(G) \leq p$

(ii) $s\beta_n(G) + w\beta_n(G) \geq p$.

Proof. Let S be a strong n-covering set of a graph G which is also a strong n-independent set of G. Then, $s\alpha_n(G) \leq |S|$. Also, by the Proposition 2.3, $V(G) - S$ is a weak n-covering set of G. That is, $w\alpha_n(G) \leq |V(G) - S| = p - |S|$. Thus, (i) holds. Using the Theorem 2.1 in (i), we get $s\beta_n(G) + w\beta_n(G) \geq p$. Similar argument holds for weak n-covering set of G which is also a weak n-independent set of G.

Proposition 4.2. *Let G be a graph with order p and size q. Then*

(i) $\left\lceil \frac{q}{\Delta_{ve}(G)} \right\rceil \leq s\alpha_n(G) \leq p - \Theta_{sve}(G)$

(ii) $\left\lceil \frac{q}{\Delta_{wve}(G)} \right\rceil \leq w\alpha_n(G) \leq p - \Theta_{wve}(G)$.

Proof. Since a vertex $u \in V(G)$ can strongly n-cover at most $\Delta_{ve}(G)$ edges and we have to strongly n-cover all the q edges, we need at least $\left\lceil \frac{q}{\Delta_{ve}(G)} \right\rceil$ vertices to strongly n-cover all the edges of G. This implies the lower bound in (i) holds. Let $S \subseteq V(G)$ be a strong ve-silent set of G with maximum cardinality. That is, $|S| = \Theta_{sve}(G)$. Since, every vertex in S is strong ve-silent, no vertex in S can strongly n-cover any edge in G. Therefore, $V(G) - S$ is a strong n-covering set of G. Hence, $s\alpha_n(G) \leq p - |S| = p - \Theta_{sve}(G)$. With the similar arguments, the bounds in (ii) holds.

Example 4.1. We observe that any complete graph K_p attain the lower bounds in (i) and (ii) of the Proposition 4.2. For any wheel graph W_p, note that $s\alpha_n(W_p) = 1 = p - (p-1) = p - \Theta_{sve}(W_p)$. Thus, W_p attains the upper bound in (i). Also, for the graph G_1 given in the Figure 1, we have $w\alpha_n(G_1) = 4 = 6 - 2 = p - \Theta_{wve}(G_1)$. Hence, G_1 attains the upper bound in (ii). Thus the above bounds in the Proposition 4.2 are sharp.

Remark 4.1.

(i) *For any ve-regular graph G, $\alpha_n(G) = s\alpha_n(G) = w\alpha_n(G)$ and $\beta_n(G) = s\beta_n(G) = w\beta_n(G)$. But, the converse need not be true. Note that, a wheel graph W_p with $p \geq 5$ is not ve-regular, but $\alpha_n(W_p) = s\alpha_n(W_p)$ and $\beta_n(W_p) = w\beta_n(W_p)$. The graph G_2 given in the Figure 2 is not ve-regular, but $s\alpha_n(G_2) = 3 = w\alpha_n(G_2)$ and $s\beta_n(G_2) = 2 = w\beta_n(G_2)$. Also, for the graph G_3 in the Figure 2, we have $\alpha_n(G_3) = w\alpha_n(G_3)$ and $\beta_n(G_3) = s\beta_n(G_3)$, but G_3 is not ve-regular.*

(ii) *The numbers $s\alpha_n(G)$ and $w\alpha_n(G)$ are incomparable in general. For example, in Figure 2, $\{v_1, v_4, v_5, v_6\}$ is a weak n-covering set of G_3 and $\{v_1, v_2, v_3, v_7, v_8\}$*

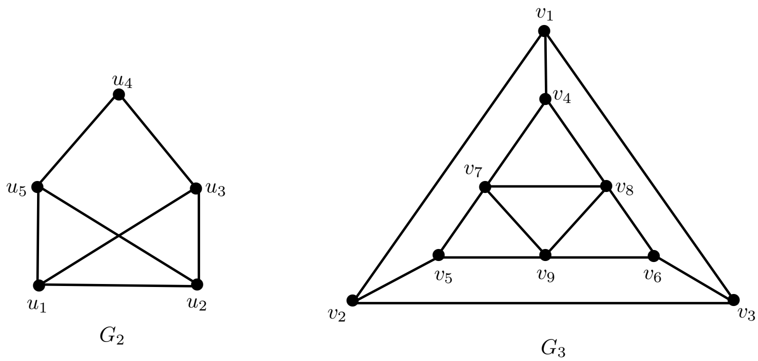

Figure 2: Graphs G_2 and G_3

is a strong n-covering set of G_3. *Hence,* $s\alpha_n(G_3) = 5 > 4 = w\alpha_n(G_3)$. *On the other hand, for a wheel graph* W_p *with* $p \geq 5$, *we have* $s\alpha_n(W_p) < w\alpha_n(W_p)$.

5. n-cover Balanced Graphs

E. Sampathkumar and L. Pushpa Latha [7] introduced the concept of domination balanced graphs. In a similar way, we define n-cover balanced graphs.

Definition 5.1. *A graph* G *is said to be n-cover balanced if there exists a strong n-covering set* S_1 *of* G *and a weak n-covering set* S_2 *of* G *such that* $S_1 \cap S_2 = \phi$.

Example 5.1. A wheel graph W_p is a n-cover balanced graph. Consider the graph G_1 given in the Figure 1. Note that, v_4 uniquely strongly n-covers the edge v_4v_5 and weakly n-covers the edges v_2v_4 and v_3v_4. This implies that, v_4 belongs to any strong (weak) n-covering set of G_1. Thus, G_1 is not a n-cover balanced graph.

Proposition 5.1. *For any graph* G, *the following statements are equivalent:*

(i) G *is n-cover balanced.*

(ii) There exists a strong n-covering set of G *which is a strong n-independent set of* G.

(iii) There exists a weak n-covering set of G *which is a weak n-independent set of* G.

Proof. Let G be a n-cover balanced graph. Then there exists a strong n-covering set S_1 of G and a weak n-covering set S_2 of G such that $S_1 \cap S_2 = \phi$. Let e be an edge of the subgraph $\langle S_1 \rangle$. Then there exists a vertex $u \in S_2 \subseteq V(G) - S_1$ such that u weakly n-covers e. Thus, S_1 is a strong n-independent set of G. Similarly, we can prove that S_2 is weak n-independent set of G. This implies that, $(i) \implies (ii)$

and $(i) \implies (iii)$ holds. To prove $(ii) \implies (i)$ and $(ii) \implies (iii)$: Let S be a strong n-covering set of G which is a strong n-independent set of G. Then by Proposition 2.3, $V(G) - S$ is a weak n-covering set of G and weak n-independent set of G. Thus, G is n-cover balanced. With similar arguments, $(iii) \implies (i)$ and $(iii) \implies (ii)$ holds.

Note 5.1. *We denote a strong (weak) n-covering set S of G with $|S| = s\alpha_n(G)$ ($|S| = w\alpha_n(G)$) as $s\alpha_n$-set ($w\alpha_n$-set) of G.*

Definition 5.2. *A n-cover balanced graph G is fully n-cover balanced if there exists a partition of vertex set $V(G) = S_1 \cup S_2$ such that S_1 is a $s\alpha_n$-set of G and S_2 is a $w\alpha_n$-set of G.*

Example 5.2. The graph G_4 in the Figure 3 is fully n-cover balanced, since $\{u_2, u_5\}$ is the $s\alpha_n$-set of G_4 and $\{u_1, u_3, u_4, u_6\}$ is the $w\alpha_n$-set of G_4.

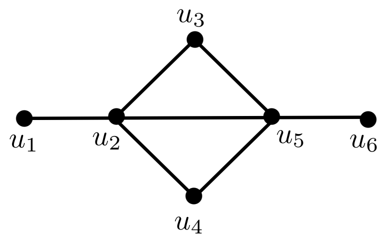

Figure 3: Graph G_4

Remark 5.1. *Every fully n-cover balanced graph is n-cover balanced. But the converse need not be true. For example, a wheel graph W_p n-cover balanced, but not fully n-cover balanced.*

Proposition 5.2.

(i) If a graph G is n-cover balanced, then $s\alpha_n(G) + w\alpha_n(G) \leq p$.

(ii) If a graph G is fully n-cover balanced, then $s\alpha_n(G) + w\alpha_n(G) = p$.

Proposition 5.3. *A graph G is fully n-cover balanced if, and only if, the following two conditions are satisfied.*

(i) $s\beta_n(G) + w\beta_n(G) = p$

(ii) There exists a $s\alpha_n$-set ($w\alpha_n$-set) which is a strong (weak) n-independent set of G.

Proof. Assume that G is fully n-cover balanced. Then, there exists a partition of vertex set $V(G) = S_1 \cup S_2$ such that S_1 is a $s\alpha_n$-set of G and S_2 is a $w\alpha_n$-set of G. This implies that, $s\beta_n(G) + w\beta_n(G) = p$. By Proposition 2.3, we have $V(G) - S_2 = S_1$ is a strong n-independent set of G. Thus, (ii) holds. Conversely, assume that the statements (i) and (ii) are true in G. Let S be a $s\alpha_n$-set ($w\alpha_n$-set) which is a strong (weak) n-independent set of G. Then, by Proposition 2.3, $V(G) - S$ is a weak n-covering set of G. Now, by (i) and Theorem 2.1, $|V(G) - S| = p - s\alpha_n(G) = w\beta_n(G) = p - s\beta_n(G) = w\alpha_n(G)$. That is, $V(G) - S$ $w\alpha_n$-set of G. Thus, G is fully n-cover balanced.

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