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A NOTE ON INTEGRAL REPRESENTATION OF (α, β, γ)-ORDER OF MEROMORPHIC FUNCTION

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Abstract: The classical growth indicators of entire and meromorphic functions are order and type, which are generalized by several authors during the past decades. Chyzhykov et al. have first introduced the generalized growth scale, namely the φ-order (see [3]) taking φ as an increasing unbounded function. But, Heittokangas et al. [5] have introduced another new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. Later, Belaïdi et al. [1] have extended the above ideas and have introduced the definition of (α, β, γ)-order of entire and meromorphic functions, where $\alpha \in L_1$-class, $\beta \in L_2$-class, $\gamma \in L_3$-class. In this paper, our motive is to develop the integral representations of (α, β, γ)-order and (α, β, γ)-lower order of a meromorphic function. We also investigate their equivalence relation under some certain conditions.

Keywords and Phrases: Meromorphic function, (α,β,γ)-order, (α,β,γ)-lower order, integral representation.

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1. Introduction

By the symbol $\mathbb{C}$, we denote the finite complex plane. The preliminary results in details are available in [2, 4, 9, 15, 16, 17]. Also, for a meromorphic function h, the definitions about $T_h(r)$, $m_h(r)$ and $N_h(r)$, one may see [4, p.4].

Our motive in the present paper, is to establish the integral representations of the definitions of (α,β,γ)-order and (α,β,γ)-lower order of f, taking f as a meromorphic function. We also establish the equivalence relations under certain conditions.

Now let L represents a class of functions β defined on $(-\infty,+\infty)$ which is continuous, non-negative and $\beta(y)=\beta(y_0)\geq 0$ for $y\leq y_0$ with $\beta(y)\uparrow+\infty$ as $y_0\leq y\to+\infty$. Further we assume that $\beta\in L_1$, if $\beta\in L$ with $\beta(p+n)\leq\beta(p)+\beta(n)+b$ for all $p,n\geq R_0$ and fixed $b\in(0,+\infty)$. We can say that $\beta\in L_2$, if $\beta\in L$ and $\beta(y+O(1))=(1+o(1))\beta(y)$ as $y\to+\infty$. Finally, $\beta\in L_3$, if $\beta\in L$ and $\beta(p+n)\leq\beta(p)+\beta(n)$ for all $p,n\geq R_0$, i.e., β is subadditive. Clearly $L_3\subset L_1$. Particularly, when $\beta\in L_3$, then one may check that $\beta(mr)\leq m\beta(r)$, where $m\geq 2$ is an integer. Up to a normalization, subadditivity is followed from concavity. Clearly, if $\beta(r)$ is concave on $[0,+\infty)$ and satisfies $\beta(0)\geq 0$, then for $x\in[0,1]$,

$$\begin{aligned}\beta(xy) &= \beta(xy+(1-x)\cdot 0)\\ &\geq x\beta(y)+(1-x)\beta(0)\geq x\beta(y),\end{aligned}$$

so that by choosing $x=\frac{p}{p+n}$ or $x=\frac{n}{p+n}$,

$$\begin{aligned}\beta(p+n) &= \frac{p}{p+n}\beta(p+n)+\frac{n}{p+n}\beta(p+n)\\ &\leq \beta\left(\frac{p}{p+n}(p+n)\right)+\beta\left(\frac{n}{p+n}(p+n)\right)\\ &= \beta(p)+\beta(n),\ p,n\geq 0.\end{aligned}$$

Since $\beta(r)$ is a non-decreasing, subadditive and unbounded function, it satisfies

$$\beta(r)\leq\beta(r+R_0)\leq\beta(r)+\beta(R_0)$$

for any $R_0\geq 0$ which implies that $\beta(r)\sim\beta(r+R_0)$ as $r\to+\infty$. Throughout this paper we assume $\alpha\in L_1$, $\beta\in L_2$, $\gamma\in L_3$.

During the past decades, several authors have investigated on the growth properties of entire and meromorphic functions in different directions using the concept of order, iterated p-order [8, 12], (p,q)-th order [6, 7], (p,q)-φ order [14] and achieved many valuable results. But in [3], Chyzhykov et al. showed that both definitions of iterated p-order and (p,q)-th order have the disadvantage that they do not cover arbitrary growth (see [3, Example 1.4]). They used more general scale, called the φ-order (see [3]). On the other hand, Heittokangas et al. [5] have introduced another new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. Extending this notion, Long et al. [10] have introduced the concepts of $(p,q)_{,\varphi}$-order which considerably extend and improve some earlier results. On the other hand, Mulyava et al. [11] have used the concepts of (α,β)-order or generalized order to investigate the properties of solutions of a heterogeneous differential equation of the second order and obtain some remarkable results. To study the growth of higher order linear differential equations, Belaïdi et al. [1] have extended the above ideas and have introduced the definitions of (α,β,γ)-order and (α,β,γ)-lower order of entire and meromorphic functions.

The (α,β,γ)-order and (α,β,γ)-lower order of a meromorphic function f defined by Belaïdi et al. [1], which are as follows:

Definition 1. [1] *For a meromorphic function f, the (α,β,γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$, is defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta\left(\ln(\gamma(r))\right)}.$$

Remark 1. *i. If we take $\alpha(r)=\beta(r)=\gamma(r)=r$, then the Definition 1 coincides with usual order.*

ii. If we take $\alpha(r)=\ln^{[p-1]} r$ $(p\geq 1)$, $\beta(r)=\gamma(r)=r$, then the Definition 1 coincides iterated p-order (see [8, 12]*).*

iii. If we take $\alpha(r)=\ln^{[p-1]} r$, $\beta(r)=\ln^{[q-1]} r$ $(p\geq q\geq 1)$, $\gamma(r)=r$, then the Definition 1 coincides iterated (p,q)-order (see [6, 7]*).*

iv. If we take $\alpha(r)=\varphi(e^r)$, $\beta(r)=\gamma(r)=r$, where $\varphi(r)$ is an increasing unbounded function in $[1,+\infty)$, then the Definition 1 coincides φ-order (see [3]*).*

v. If we take $\alpha(r)=\beta(r)=r$, $\gamma(r)=\varphi(r)$, where $\varphi:(R_0,+\infty)\to(0,+\infty)$ is a non-decreasing unbounded function satisfying the condition

$$\varphi(p+n)\leq\varphi(p)+\varphi(n) \text{ for all } p,n\geq R_0,$$

then the Definition 1 coincides φ-order introduced by Heittokangas et al. [5].

Remark 2. *Belaïdi et al.* [1] *have introduced the* (α,β,γ)*-order of an entire function* f*, denoted by* $\rho_{(\alpha,\beta,\gamma)}[f]$*, which is as follows:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\ln^{[2]}(M_f(r)))}{\beta\left(\ln(\gamma(r))\right)}.$$

Here, if we take α*,* β*,* $\gamma \in L$ *with* $\gamma(r) = r$*, then we get the definition of* (α,β)*-order introduced by Sheremeta* [13]*. In particular, if we take* $f(z) = \exp z$*,* $\alpha(r) = \ln r$*,* $\beta(r) = \ln r$*, then one can easily check*

$$\rho_{(\alpha,\beta)}[f] = +\infty.$$

Definition 2. *The growth indicator* $\rho_{(\alpha,\beta,\gamma)}[f]$ *is alternatively defined as: The integral* $\int_{r_0}^{\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}} dr$ $(r_0 > 0)$ *converges when* $k > \rho_{(\alpha,\beta,\gamma)}[f]$ *and diverges when* $k < \rho_{(\alpha,\beta,\gamma)}[f]$*.*

Definition 3. [1] *For a meromorphic function* f*, the* (α,β,γ)*-lower order denoted by* $\lambda_{(\alpha,\beta,\gamma)}[f]$*, is defined as:*

$$\lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta\left(\ln(\gamma(r))\right)}.$$

Definition 4. *The growth indicator* $\lambda_{(\alpha,\beta,\gamma)}[f]$ *is alternatively defined as: The integral* $\int_{r_0}^{\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}} dr$ $(r_0 > 0)$ *converges when* $k > \lambda_{(\alpha,\beta,\gamma)}[f]$ *and diverges when* $k < \lambda_{(\alpha,\beta,\gamma)}[f]$*.*

Example 1. Let us take $f(z) = \exp z$, $\alpha(r) = \ln r$, $\beta(r) = \ln r$, $\gamma(r) = r$, then one can easily check

$$\rho_{(\alpha,\beta,\gamma)}[f] = \lambda_{(\alpha,\beta,\gamma)}[f] = 1.$$

2. Main results

To establish the main results of this paper, firstly we establish a lemma which is necessary.

Lemma 1. *If the integral* $\int_{r_0}^{\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}} dr$ $(r_0 > 0)$ *is convergent for* $0 < k < +\infty$*, then*

$$\lim_{r\to+\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k}} = 0.$$

Proof. As the integral $\int\limits_{r_0}^{\infty}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}}dr$ converges for $0<k<+\infty$, so for given ε (>0) there exists a number $m=m(\varepsilon)$ such that

$$\int\limits_{r_0}^{\infty}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}}dr<\varepsilon \text{ for } r_0>m,$$

i.e., for $r_0>m$,

$$\int\limits_{r_0}^{r_0+r}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}}dr<\varepsilon.$$

Since $e^{[\alpha(\ln(T_f(r)))]}$ a increasing function of r, so

$$\int\limits_{r_0}^{r_0+e^{\beta(\ln(\gamma(r_0)))}}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}}dr\geq\frac{e^{[\alpha(\ln(T_f(r_0)))]}}{[e^{\beta(\ln(\gamma(r_0)))}]^{k+1}}\cdot e^{\beta(\ln(\gamma(r_0)))},$$

$$\text{i.e.,}\quad\int\limits_{r_0}^{r_0+e^{\beta(\ln(\gamma(r_0)))}}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}}dr\geq\frac{e^{[\alpha(\ln(T_f(r_0)))]}}{[e^{\beta(\ln(\gamma(r_0)))}]^{k}} \text{ for } r_0>m,$$

$$i.e.,\ \frac{e^{[\alpha(\ln(T_f(r_0)))]}}{[e^{\beta(\ln(\gamma(r_0)))}]^{k}}<\varepsilon \text{ for } r_0>m,$$

from which it is clear that

$$\lim_{r\to+\infty}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k}}=0.$$

This proves the lemma.

Theorem 1. *The Definition 1 implies and is implied by Definition 2, i.e., they are equivalent.*

Proof.

Case 1. $\rho_{(\alpha,\beta,\gamma)}[f]=+\infty$.

Definition 1⇒ Definition 2.

Since $\rho_{(\alpha,\beta,\gamma)}[f]=\infty$, by Definition 1 for arbitrary positive M, we have a sequence of real numbers r tending to infinity that

$$\alpha(\ln(T_f(r)))>M\cdot\beta\left(\ln(\gamma(r))\right),$$

$$i.e, e^{[\alpha(\ln(T_f(r)))]} > [e^{\beta(\ln(\gamma(r)))}]^M. \tag{2.1}$$

Let us suppose that the integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{M+1}} dr$ $(r_0 > 0)$ be convergent. Then by using Lemma 1,

$$\limsup_{r\to+\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^M} = 0.$$

So for all sufficiently large values of r,

$$e^{[\alpha(\ln(T_f(r)))]} < [e^{\beta(\ln(\gamma(r)))}]^M. \tag{2.2}$$

Now from (2.1) and (2.2) we reach at a contradiction.

Hence $\int_{r_0}^{\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{M+1}} dr$ $(r_0 > 0)$ is divergent whenever M is finite, which is Definition 2.

Definition 2⇒ Definition 1.

We choose any positive number M. As $\rho_{(\alpha,\beta,\gamma)}[f] = +\infty$, from Definition 2 the divergence of the integral $\int_{r_0}^{\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{M+1}} dr$ $(r_0 > 0)$ implies for arbitrarily chosen positive number ε and for a sequence of real numbers r tending to infinity,

$$e^{[\alpha(\ln(T_f(r)))]} > [e^{\beta(\ln(\gamma(r)))}]^{M-\varepsilon},$$

$$i.e, \ \alpha(\ln(T_f(r))) > (M-\varepsilon)\cdot\beta(\ln(\gamma(r))).$$

This gives that

$$\limsup_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} \geq (M-\varepsilon).$$

As $M > 0$ is arbitrarily chosen, it implies that

$$\limsup_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} = \infty.$$

Thus Definition 1 follows.

Case 2. $0 \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$.

Definition 1⇒ Definition 2.

Subcase (I). $0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$.

If $0<\rho_{(\alpha,\beta,\gamma)}[f]<\infty$, then for any arbitrarily chosen $\varepsilon(>0)$ and for all sufficiently large values of r,

$$\begin{aligned}\frac{\alpha(\ln(T_f(r)))}{\beta\left(\ln(\gamma(r))\right)} &< \rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon,\\ i.e,\ e^{[\alpha(\ln(T_f(r)))]} &< \left[e^{\beta(\ln(\gamma(r)))}\right]^{(\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon)},\\ i.e,\ \frac{e^{[\alpha(\ln(T_f(r)))]}}{\left[e^{\beta(\ln(\gamma(r)))}\right]^k} &< \frac{\left[e^{\beta(\ln(\gamma(r)))}\right]^{(\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon)}}{\left[e^{\beta(\ln(\gamma(r)))}\right]^k},\\ i.e,\ \frac{e^{[\alpha(\ln(T_f(r)))]}}{\left[e^{\beta(\ln(\gamma(r)))}\right]^k} &< \frac{1}{\left[e^{\beta(\ln(\gamma(r)))}\right]^{k-(\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon)}}.\end{aligned}$$

Therefore $\int_{r_0}^{\infty}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}}dr$ $(r_0>0)$ convergent when $k>\rho_{(\alpha,\beta,\gamma)}[f]$ and divergent when $k<\rho_{(\alpha,\beta,\gamma)}[f]$.

Subcase (II).

When $\rho_{(\alpha,\beta,\gamma)}[f]=0$, Definition 1 gives for all sufficiently large values of r that

$$\frac{\alpha(\ln(T_f(r)))}{\beta\left(\ln(\gamma(r))\right)}\le\varepsilon.$$

Then as previous we get that $\int_{r_0}^{\infty}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{k+1}}dr$ $(r_0>0)$ convergent when $k>0$ and divergent when $k<0$.

By Subcase (I) and Subcase (II),we get Definition 2.

Definition 2⇒ Definition 1.

By Definition 2, for arbitrary ε (>0) the integral $\int_{r_0}^{\infty}\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon+1}}dr$ converges. Then using Lemma 1, we get

$$\limsup_{r\to+\infty}\frac{e^{[\alpha(\ln(T_f(r)))]}}{\left[e^{\beta(\ln(\gamma(r)))}\right]^{\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon}}=0,$$

i.e, for all sufficiently large values of r,

$$\begin{aligned}\frac{e^{[\alpha(\ln(T_f(r)))]}}{\left[e^{\beta(\ln(\gamma(r)))}\right]^{\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon}} &< \varepsilon_0,\\ i.e,\ e^{[\alpha(\ln(T_f(r)))]} &< \varepsilon_0\cdot\left[e^{\beta(\ln(\gamma(r)))}\right]^{\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon},\\ i.e,\ \alpha(\ln(T_f(r))) &< \ln\varepsilon_0+(\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon)\cdot\beta\left(\ln(\gamma(r))\right),\end{aligned}$$

$$i.e,\ \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} \leq \frac{\ln \varepsilon_0}{\beta(\ln(\gamma(r)))} + (\rho_{(\alpha,\beta,\gamma)}[f] + \varepsilon)$$

$$i.e,\ \limsup_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} \leq \rho_{(\alpha,\beta,\gamma)}[f] + \varepsilon.$$

Since ε (> 0) is arbitrarily chosen, from above we get

$$\limsup_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} \leq \rho_{(\alpha,\beta,\gamma)}[f]. \tag{2.3}$$

As the integral $\int\limits_{r_0}^{\infty} \frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{\rho_{(\alpha,\beta,\gamma)}[f]-\varepsilon+1}} dr$ is divergent, so from Definition 2 we have a sequence of values of r tending to infinity for which

$$\frac{e^{[\alpha(\ln(T_f(r)))]}}{[e^{\beta(\ln(\gamma(r)))}]^{\rho_{(\alpha,\beta,\gamma)}[f]-\varepsilon+1}} > \frac{1}{[e^{\beta(\ln(\gamma(r)))}]^{1+\varepsilon}},$$

$$i.e,\ e^{[\alpha(\ln(T_f(r)))]} > (e^{\beta(\ln(\gamma(r)))})^{\rho_{(\alpha,\beta,\gamma)}[f]-2\varepsilon},$$

$$i.e,\ \alpha(\ln(T_f(r))) > (\rho_{(\alpha,\beta,\gamma)}[f] - 2\varepsilon) \cdot \beta(\ln(\gamma(r))),$$

$$i.e,\ \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} > (\rho_{(\alpha,\beta,\gamma)}[f] - 2\varepsilon).$$

As ε (> 0) is arbitrarily chosen, we have

$$\limsup_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} \geq \rho_{(\alpha,\beta,\gamma)}[f]. \tag{2.4}$$

Thus from (2.3) and (2.4) it follows that

$$\limsup_{r\to+\infty} \frac{\alpha(\ln(T_f(r)))}{\beta(\ln(\gamma(r)))} = \rho_{(\alpha,\beta,\gamma)}[f].$$

This is the Definition 1.

Hence by Case 1 and Case 2, we reach at the conclusion.

As Theorem 1, we can state Theorem 2 without its proof.

Theorem 2. *The Definition 3 and Definition 4 are equivalent.*

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