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A STRONG FORM OF GENERALIZED CLOSED SET IN A FUZZY TOPOLOGICAL SPACE

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Abstract: In this paper a strong form of fuzzy generalized closed set, viz., fg^*s-closed set is introduced and studied. With the help of this newly defined set, a new type of idempotent operator is introduced. Using this operator as a basic tool, here we introduce and study fg^*s-open and fg^*s-closed functions the class of which are strictly larger than that of fuzzy open (resp, fg-open) and fuzzy closed (resp., fg-closed) functions respectively and weaker than that of $fg\delta$-open and $fg\delta$-closed functions respectively. In the last section we introduce fg^*s-T_2-space the class of which is strictly larger than that of fuzzy T_2-space and some applications of fg^*s-open function are established here.

Keywords and Phrases: Fuzzy semiopen set, fuzzy regular open set, fuzzy δ-open set, fg-closed set, $fg\delta$-closed set, fg^*s-closed set, fg^*s-open function, fg^*s-closed function.

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1. Introduction

In 1965, L. A. Zadeh introduced fuzzy set [15] and in 1968, C. L. Chang introduced fuzzy topology [5]. Afterwards, many mathematicians have engaged themselves to introduce and study different types of fuzzy open-like sets. In 1981, K. K. Azad introduced fuzzy regular open and fuzzy semiopen set [1] and in [7], Ganguly and Saha introduced fuzzy δ-open set. In [2, 3], fuzzy generalized version of

closed set, viz., fg-closed set is introduced. Afterwards, several types of generalized version of fuzzy closed sets are introduced and studied. In [4], $fg\delta$-closed set is introduced. Here we introduce fg^*s-closed set, the class of which is strictly larger than that of fg-closed set, but smaller than $fg\delta$-closed set.
Lot of work has been done on quasi-coincidence and different weaker forms of closed and open sets. In this context, we have to mention [6, 11, 12, 13].

2. Preliminaries

Throughout this paper (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [5]. In [15], L. A. Zadeh introduced fuzzy set as follows: A fuzzy set A is a function from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [15] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$ [15]. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [15] while AqB means A is quasi-coincident (q-coincident, for short) with B, if there exists $x \in X$ such that $A(x) + B(x) > 1$ [10]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy point x_t and a fuzzy set A, $x_t \in A$ means $A(x) \geq t$, i.e., $x_t \leq A$. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [5] and fuzzy interior [5] of A respectively. A fuzzy set A is called a fuzzy neighbourhood (fuzzy nbd, for short) of a fuzzy point x_α if there exists a fuzzy open set U in X such that $x_\alpha \in U \leq A$ [10]. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_α [10]. A fuzzy set A is called a fuzzy quasi neighbourhood (fuzzy q-nbd, for short) [10] of a fuzzy point x_α in an fts X if there is a fuzzy open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open q-nbd [10] of x_α. A fuzzy set A in X is called fuzzy regular open [1] (resp., fuzzy semiopen [1]) if $A = int(clA)$ (resp., $A \leq cl(intA)$). The complement of a fuzzy semiopen set is called fuzzy semiclosed [1]. The intersection (resp., union) of all fuzzy semiclosed (resp.,fuzzy semiopen) sets containing (resp., contained in) a fuzzy set A is called fuzzy semiclosure [1] (resp., fuzzy semiinterior [1]) of A, to be denoted by $sclA$ (resp., $sintA$). The fuzzy δ-interior [7] of a fuzzy set A in X are defined as : $\delta intA = \bigvee\{W : W \text{ is fuzzy regular open in } X, W \leq A\}$. The collection of all fuzzy semiopen (resp., fuzzy semiclosed) sets in an fts (X, τ) is denoted by $FSO(X)$ (resp., $FSC(X)$).

2. fg^*s-Closed Set: Some Properties

In this section a new type of generalized version of fuzzy closed set is introduced which is in between fg-closed and $fg\delta$-closed sets.

Definition 3.1. *Let (X, τ) be an fts and $A \in I^X$. Then A is called fg^*s-closed set in X if $clsintA \leq U$ whenever $A \leq U \in \tau$.*

The complement of fg^*s-closed set is called fg^*s-open set in X.
The collection of all fg^*s-closed (resp., fg^*s-open) sets in an fts X is denoted by $FG^*SC(X)$ (resp., $FG^*SO(X)$).

Remark 3.2. *Union and intersection of two fg^*s-closed sets may not be so, as it seen from the following examples.*

Example 3.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.3$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U\}$ where $B \leq U \leq 1_X \setminus A$. Now consider two fuzzy sets C and D defined by $C(a) = 0.5, C(b) = 0, D(a) = 0, D(b) = 0.3$. Then clearly $C, D \in FG^*SC(X)$. Let $E = C \bigvee D$. Then $E(a) = 0.5, E(b) = 0.3$. Now $E \leq A \in \tau$. But $clsintE = 1_X \setminus A \not\leq A \Rightarrow E$ is not an fg^*s-closed set in (X, τ).

Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.2$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus B$, $B \leq V \leq 1_X \setminus A$. Now consider two fuzzy sets C and D defined by $C(a) = 0.6, C(b) = 0.2, D(a) = 0.3, D(b) = 0.7$. Clearly C and D are fg^*s-closed sets in (X, τ). Let $E = C \bigwedge D$. Then $E(a) = 0.3, E(b) = 0.2$. Now $E \leq B \in \tau$. But $clsintE = 1_X \setminus A \not\leq B \Rightarrow E$ is not an fg^*s-closed set in (X, τ).

Note 3.5. *So we can conclude that the set of all fg^*s-open sets in an fts (X, τ) does not form a fuzzy topology.*

Theorem 3.6. *Let (X, τ) be an fts and $A, B \in I^X$. If $A \leq B \leq clsintA$ and A is fg^*s-closed set in X, then B is also fg^*s-closed set in X.*
Proof. Let $U \in \tau$ be such that $B \leq U$. Then by hypothesis, $A \leq B \leq U$. Since A is fg^*s-closed set in X, $clsintA \leq U$. Then $clsintA \leq clsintB \leq clsint(clsintA) \leq clsintA \leq U \Rightarrow B$ is fg^*s-closed set in X.

Theorem 3.7. *Let (X, τ) be an fts and $A, B \in I^X$. If $intsclA \leq B \leq A$ and A is fg^*s-open set in X, then B is also fg^*s-open set in X.*
Proof. $intsclA \leq B \leq A \Rightarrow 1_X \setminus A \leq 1_X \setminus B \leq 1_X \setminus intsclA = clsint(1_X \setminus A)$ where $1_X \setminus A$ is fg^*s-closed set in X. By Theorem 3.6, $1_X \setminus B$ is fg^*s-closed set in $X \Rightarrow B$ is fg^*s-open set in X.

Theorem 3.8. *Let (X, τ) be an fts and $A \in I^X$. Then A is fg^*s-open set in X*

iff $K \leq intsclA$ *whenever* $K \leq A$ *and* K *is fuzzy closed set in* (X, τ).

Proof. Let $A(\in I^X)$ be fg^*s-open set in X and $K \leq A$ where K is fuzzy closed set in (X, τ). Then $1_X \setminus A \leq 1_X \setminus K$ where $1_X \setminus A$ is fg^*s-closed set in X and $1_X \setminus K$ is fuzzy open set in (X, τ). By hypothesis, $clsint(1_X \setminus A) \leq 1_X \setminus K \Rightarrow 1_X \setminus intsclA \leq 1_X \setminus K \Rightarrow K \leq intsclA$.

Conversely, let $K \leq intsclA$ whenever $K \leq A$, $K \in \tau^c$. Then $1_X \setminus A \leq 1_X \setminus K$ where $1_X \setminus K \in \tau$. By hypothesis, $1_X \setminus intsclA \leq 1_X \setminus K \Rightarrow clsint(1_X \setminus A) \leq 1_X \setminus K \Rightarrow 1_X \setminus A$ is fg^*s-closed set in $X \Rightarrow A$ is fg^*s-open set in X.

Theorem 3.9. *Let* (X, τ) *be an fts and* $A, B \in I^X$. *If* A *is* fg^*s*-closed set in* X *and* B *is fuzzy closed set in* (X, τ) *with* $A \not q B$. *Then* $clsintA \not q B$.

Proof. By hypothesis, $A \not q B \Rightarrow A \leq 1_X \setminus B \in \tau \Rightarrow clsintA \leq 1_X \setminus B \Rightarrow clsintA \not q B$.

Remark 3.10. *The converse of Theorem 3.9 may not be true, in general, as it is seen from the following example.*

Example 3.11. Consider Example 3.4. Here E is not fg^*s-closed set in X. Also $E \not q (1_X \setminus A)$ and $clsintE (= 1_X \setminus A) \not q (1_X \setminus A)$.

Now we recall the following definitions from [2, 3, 4] for ready references.

Definition 3.12. *Let* (X, τ) *be an fts and* $A \in I^X$. *Then* A *is called*
(i) fg*-closed set* [2, 3] *if* $clA \leq U$ *whenever* $A \leq U \in \tau$,
(ii) $fg\delta$*-closed set* [4] *if* $cl\delta intA \leq U$ *whenever* $A \leq U \in \tau$.

Remark 3.13. *It is clear from definitions that* fg*-closed set is* fg^*s*-closed set which implies* $fg\delta$*-closed set. But reverse implications are not necessarily true follow from the next examples.*

Example 3.14. fg^*s-closed set $\not\Rightarrow$ fg-closed set
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then $B \leq A \in \tau$. But $clB = 1_X \not\leq A \Rightarrow B$ is not fg-closed set in X. But $clsintB = 0_X \leq A \Rightarrow B$ is an fg^*s-closed set in X.

Example 3.15. $fg\delta$-closed set $\not\Rightarrow$ fg^*s-closed set
Consider Example 3.3. Here E is not fg^*s-closed set in X. Now $cl\delta intE = 0_X \leq A \Rightarrow E$ is $fg\delta$-closed set in X.

Definition 3.16. *An fts* (X, τ) *is called* fT_{g^*s}*-space (resp.* fgT_δ*-space* [4]*) if every* fg^*s*-closed (resp.,* $fg\delta$*-closed) set in* X *is fuzzy closed set in* X.

Note 3.17. *In* fT_{g^*s}*-space, every* fg^*s*-closed set is* fg*-closed set and in* fgT_δ*-*

space, every fuzzy $fg\delta$-closed set is fg^*s-closed.

Now we introduce a new type of generalized version of neighbourhood system in an fts.

Definition 3.18. *Let (X, τ) be an fts and x_α, a fuzzy point in X. A fuzzy set A is called fg^*s-neighbourhood (fg^*s-nbd, for short) of x_α, if there exists an fg^*s-open set U in X such that $x_\alpha \in U \leq A$. If, in addition, A is fg^*s-open set in X, then A is called an fg^*s-open nbd of x_α.*

Definition 3.19. *Let (X, τ) be an fts and x_α, a fuzzy point in X. A fuzzy set A is called fg^*s-quasi neighbourhood (fg^*s-q-nbd, for short) of x_α if there is an fg^*s-open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fg^*s-open set in X, then A is called an fg^*s-open q-nbd of x_α.*

Note 3.20. *(i) It is clear from definitions that every fg^*s-open set is an fg^*s-open nbd of each of its points. But every fg^*s-nbd of a fuzzy point x_α may not be an fg^*s-open set containing x_α follows from the following example.*
(ii) Also every fuzzy open nbd (resp., fuzzy open q-nbd) of a fuzzy point x_α is an fg^*s-open nbd (resp., fg^*s-open q-nbd) of x_α. But the converse is not necessarily true, in general, as it seen from the following example.

Example 3.21. Consider Example 3.14. Here B is fg^*s-open nbd of the fuzzy point $a_{0.4}$. But B is not fuzzy open nbd of $a_{0.4}$. Again B is fg^*s-open q-nbd of the fuzzy point $a_{0.6}$, but not a fuzzy open q-nbd of $a_{0.6}$.

Example 3.22. Consider Example 3.3 and the fuzzy set F defined by $F(a) = F(b) = 0.5$ and the fuzzy point $a_{0.4}$. Clearly F is fg^*s-closed as well as fg^*s-open set with $a_{0.4} \in F \leq 1_X \setminus E \notin FG^*SO(X)$. So $1_X \setminus E$ is an fg^*s-nbd of $a_{0.4}$ though it is not an fg^*s-open nbd of $a_{0.4}$.

4. fg^*s-Open and fg^*s-Closed Functions

In this section we first introduce fg^*s-closure operator which is seem to be an idempotent operator. Using this operator as a basic tool, we introduce and characterize fg^*s-open and fg^*s-closed functions, the classes of which are strictly larger than that of fuzzy open [14] and fuzzy closed [14] functions respectively.

Definition 4.1. *Let (X, τ) be an fts and $A \in I^X$. Then fg^*s-closure and fg^*s-interior of A, denoted by $fg^*scl(A)$ and $fg^*sint(A)$, are defined as follows:*

$fg^*scl(A) = \bigwedge\{F : A \leq F, F \text{ is } fg^*s\text{-closed set in } X\}$,

$fg^*sint(A) = \bigvee\{G : G \leq A, G \text{ is } fg^*s\text{-open set in } X\}$.

Remark 4.2. *It is clear from definition that for any $A \in I^X$, $A \leq fg^*scl(A)$. If A is fg^*s-closed set in an fts X, then $A = fg^*scl(A)$. Similarly, $fg^*sint(A) \leq A$.*

If A is fg^*s-open set in an fts X, then $A = fg^*sint(A)$. Again by Remark 3.2, we conclude that $fg^*scl(A)$ (resp., $fg^*sint(A)$) may not be fg^*s-closed (resp., fg^*s-open) set in an fts X.

Theorem 4.3. *Let (X, τ) be an fts and $A \in I^X$. Then for a fuzzy point x_t in X, $x_t \in fg^*scl(A)$ if and only if every fg^*s-open q-nbd U of x_t, UqA.*

Proof. Let $x_t \in fg^*scl(A)$ for any fuzzy set A in an fts X and F be any fg^*s-open q-nbd of x_t. Then $x_t qF \Rightarrow x_t \notin 1_X \setminus F$ which is fg^*s-closed set in X. Then by Definition 4.1, $A \not\leq 1_X \setminus F \Rightarrow$ there exists $y \in X$ such that $A(y) > 1 - F(y) \Rightarrow AqF$.

Conversely, let for every fg^*s-open q-nbd F of x_t, FqA. If possible, let $x_t \notin fg^*scl(A)$. Then by Definition 4.1, there exists an fg^*s-closed set U in X with $A \leq U$, $x_t \notin U$. Then $x_t q(1_X \setminus U)$ which being fg^*s-open set in X is fg^*s-open q-nbd of x_t. By assumption, $(1_X \setminus U)qA \Rightarrow (1_X \setminus A)qA$, a contradiction.

Theorem 4.4. *Let (X, τ) be an fts and $A, B \in I^X$. Then the following statements are true:*
(i) $fg^*scl(0_X) = 0_X$,
(ii) $fg^*scl(1_X) = 1_X$,
(iii) $A \leq B \Rightarrow fg^*scl(A) \leq fg^*scl(B)$,
(iv) $fg^*scl(A \bigvee B) = fg^*scl(A) \bigvee fg^*scl(B)$,
(v) $fg^*scl(A \wedge B) \leq fg^*scl(A) \wedge fg^*scl(B)$, equality does not hold, in general,
(vi) $fg^*scl(fg^*scl(A)) = fg^*scl(A)$.

Proof. (i), (ii) and (iii) are obvious.
(iv) From (iii), $fg^*scl(A) \bigvee fg^*scl(B) \leq fg^*scl(A \bigvee B)$.
To prove the converse, let $x_\alpha \in fg^*scl(A \bigvee B)$. Then by Theorem 4.3, for any fg^*s-open set U in X with $x_\alpha qU$, $Uq(A \bigvee B) \Rightarrow$ there exists $y \in X$ such that $U(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either $U(y) + A(y) > 1$ or $U(y) + B(y) > 1 \Rightarrow$ either UqA or $UqB \Rightarrow$ either $x_\alpha \in fg^*scl(A)$ or $x_\alpha \in fg^*scl(B) \Rightarrow x_\alpha \in fg^*scl(A) \bigvee fg^*scl(B)$.
(v) Follows from (iii) and equality does not hold, in general follows from Example 3.4.
(vi) Since $A \leq fg^*scl(A)$, for any $A \in I^X$, $fg^*scl(A) \leq fg^*scl(fg^*scl(A))$ (by (iii)).

Conversely, let $x_\alpha \in fg^*scl(fg^*scl(A)) = fg^*scl(B)$ where $B = fg^*scl(A)$. Let U be any fg^*s-open set in X with $x_\alpha qU$. Then UqB implies that there exists $y \in X$ such that $U(y) + B(y) > 1$. Let $B(y) = t$. Then $y_t qU$ and $y_t \in B = fg^*scl(A) \Rightarrow UqA \Rightarrow x_\alpha \in fg^*scl(A) \Rightarrow fg^*scl(fg^*scl(A)) \leq fg^*scl(A)$. Consequently, $fg^*scl(fg^*scl(A)) = fg^*scl(A)$.

Theorem 4.5. *Let (X, τ) be an fts and $A \in I^X$. Then the following statements*

hold:
(i) $fg^*scl(1_X \setminus A) = 1_X \setminus fg^*sint(A)$
(ii) $fg^*sint(1_X \setminus A) = 1_X \setminus fg^*scl(A)$.
Proof. (i). Let $x_t \in fg^*scl(1_X \setminus A)$ for a fuzzy set A in an fts (X, τ). If possible, let $x_t \notin 1_X \setminus fg^*sint(A)$. Then $1 - (fg^*sint(A))(x) < t \Rightarrow [fg^*sint(A)](x) + t > 1 \Rightarrow fg^*sint(A)qx_t \Rightarrow$ there exists at least one fg^*s-open set $F \leq A$ with $x_tqF \Rightarrow x_tqA$. As $x_t \in fg^*scl(1_X \setminus A), Fq(1_X \setminus A) \Rightarrow Aq(1_X \setminus A)$, a contradiction. Hence

$$fg^*scl(1_X \setminus A) \leq 1_X \setminus fg^*sint(A)...(1)$$

Conversely, let $x_t \in 1_X \setminus fg^*sint(A)$. Then $1 - [(fg^*sint(A)](x) \geq t \Rightarrow x_t \not{q}(fg^*sint(A)) \Rightarrow x_t \not{q}F$ for every fg^*s-open set F contained in A ... (2).
Let U be any fg^*s-closed set in X such that $1_X \setminus A \leq U$. Then $1_X \setminus U \leq A$. Now $1_X \setminus U$ is fg^*s-open set in X contained in A. By (2), $x_t \not{q}(1_X \setminus U) \Rightarrow x_t \in U \Rightarrow x_t \in fg^*scl(1_X \setminus A)$ and so

$$1_X \setminus fg^*sint(A) \leq fg^*scl(1_X \setminus A)...(3).$$

Combining (1) and (3), (i) follows.
(ii) Putting $1_X \setminus A$ for A in (i), we get $fg^*scl(A) = 1_X \setminus fg^*sint(1_X \setminus A) \Rightarrow fg^*sint(1_X \setminus A) = 1_X \setminus fg^*scl(A)$.

Let us now recall the following definitions from [3, 4, 14] for ready references.

Definition 4.6. *A function* $f : X \to Y$ *is called*
(i) fuzzy open [14] *(resp., fuzzy closed* [14]*) if* $f(U)$ *is fuzzy open (resp., fuzzy closed) set in* Y *for every fuzzy open (resp., fuzzy closed) set* U *in* X,
(ii) fg*-open* [3] *(*fg*-closed* [3]*) if* $f(U)$ *is* fg*-open (resp.,* fg*-closed) set in* Y *for every fuzzy open (resp., fuzzy closed) set* U *in* X,
(iii) $fg\delta$*-open* [4] *(*$fg\delta$*-closed* [4]*) if* $f(U)$ *is* $fg\delta$*-open (resp.,* $fg\delta$*-closed) in* Y *for every fuzzy open (resp., fuzzy closed) set* U *in* X.

Now we introduce the following concept.

Definition 4.7. *A function* $h : X \to Y$ *is called* fg^*s*-open function if* $h(U)$ *is* fg^*s*-open set in* Y *for every fuzzy open set* U *in* X.

Remark 4.8. *It is clear from definitions that*
(i) fuzzy open function is fg^*s*-open function,*
(ii) fg*-open function is* fg^*s*-open function,*
(iii) fg^*s*-open function is* $fg\delta$*-open function.*
But the converses need not be true, in general, as it is evidenced from the following examples.

Example 4.9. (i) fg^*s-open function $\not\Rightarrow$ fuzzy open function, fg-open function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = B(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in \tau_1, i(B) = B \leq A \in \tau_2$. Now $cl_{\tau_2} sint_{\tau_2} B = 0_X < A \Rightarrow B \in FG^*SC(X, \tau_2) \Rightarrow 1_X \setminus B = B \in FG^*SO(X, \tau_2) \Rightarrow i$ is an fg^*s-open function. But $B \notin \tau_2 \Rightarrow i$ is not fuzzy open function. Also $cl_{\tau_2} B = 1_X \not\leq A \Rightarrow B$ is not fg-closed as well as fg-open set in $(X, \tau_2) \Rightarrow i$ is not an fg-open function.

(ii) $fg\delta$-open function $\not\Rightarrow$ fg^*s-open function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, E\}$, $\tau_2 = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.3, E(a) = 0.5, E(b) = 0.7$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $E \in \tau_1, i(E) = E$. Now $1_X \setminus E \in \tau_1^c, i(1_X \setminus E) = 1_X \setminus E < A \in \tau_2$. But $cl_{\tau_2} sint_{\tau_2}(1_X \setminus E) = 1_X \setminus A \not\leq A \Rightarrow 1_X \setminus E \notin FG^*SC(X, \tau_2) \Rightarrow E \notin FG^*SO(X, \tau_2) \Rightarrow i$ is not fg^*s-open function. Now $cl_{\tau_2} \delta int_{\tau_2}(1_X \setminus E) = 0_X < A \Rightarrow 1_X \setminus E$ is $fg\delta$-closed and so E is $fg\delta$-open set in $(X, \tau_2) \Rightarrow i$ is an $fg\delta$-open function.

Theorem 4.10. *For a bijective function $h : X \to Y$, the following statements are equivalent:*
(i) h is fg^*s-open,
(ii) $h(intA) \leq fg^*sint(h(A))$, for all $A \in I^X$,
(iii) for each fuzzy point x_α in X and each fuzzy open set U in X containing x_α, there exists an fg^*s-open set V in Y containing $h(x_\alpha)$ such that $V \leq h(U)$.

Proof. (i) $\Rightarrow$ (ii). Let $A \in I^X$. Then $intA$ is a fuzzy open set in X. By (i), $h(intA)$ is fg^*s-open set in Y. Since $h(intA) \leq h(A)$ and $fg^*sint(h(A))$ is the union of all fg^*s-open sets contained in $h(A)$, we have $h(intA) \leq fg^*sint(h(A))$.
(ii) $\Rightarrow$ (i). Let U be any fuzzy open set in X. Then $h(U) = h(intU) \leq fg^*sint(h(U))$ (by (ii)) $\Rightarrow h(U)$ is fg^*s-open set in $Y \Rightarrow h$ is fg^*s-open function.
(ii) $\Rightarrow$ (iii). Let x_α be a fuzzy point in X, and U, a fuzzy open set in X such that $x_\alpha \in U$. Then $h(x_\alpha) \in h(U) = h(intU) \leq fg^*sint(h(U))$ (by (ii)). Then $h(U)$ is fg^*s-open set in Y. Let $V = h(U)$. Then $h(x_\alpha) \in V$ and $V \leq h(U)$.
(iii) $\Rightarrow$ (i). Let U be any fuzzy open set in X and y_α, any fuzzy point in $h(U)$, i.e., $y_\alpha \in h(U)$. Then there exists unique $x \in X$ such that $h(x) = y$ (as h is bijective). Then $[h(U)](y) \geq \alpha \Rightarrow U(h^{-1}(y)) \geq \alpha \Rightarrow U(x) \geq \alpha \Rightarrow x_\alpha \in U$. By (iii), there exists fg^*s-open set V in Y such that $h(x_\alpha) \in V$ and $V \leq h(U)$. Then $h(x_\alpha) \in V = fg^*sint(V) \leq fg^*sint(h(U))$. Since y_α is taken arbitrarily and $h(U)$ is the union of all fuzzy points in $h(U)$, $h(U) \leq fg^*sint(h(U)) \Rightarrow h(U)$ is fg^*s-open set in $Y \Rightarrow h$ is an fg^*s-open function.

Theorem 4.11. *If $h : X \to Y$ is fg^*s-open, bijective function, then the following*

statements are true:
(i) for each fuzzy point x_α *in* X *and each fuzzy open* q*-nbd* U *of* x_α *in* X*, there exists an* fg^*s*-open* q*-nbd* V *of* $h(x_\alpha)$ *in* Y *such that* $V \leq h(U)$*,*
(ii) $h^{-1}(fg^*scl(B)) \leq cl(h^{-1}(B))$*, for all* $B \in I^Y$*.*

Proof. (i) Let x_α be a fuzzy point in X and U be any fuzzy open q-nbd of x_α in X. Then $x_\alpha qU = intU \Rightarrow h(x_\alpha)qh(intU) \leq fg^*sint(h(U))$ (by Theorem 4.10 (i)⇒(ii)) implies that there exists at least one fg^*s-open q-nbd V of $h(x_\alpha)$ in Y with $V \leq h(U)$.
(ii) Let x_α be any fuzzy point in X such that $x_\alpha \notin cl(h^{-1}(B))$ for any $B \in I^Y$. Then there exists a fuzzy open q-nbd U of x_α in X such that $U \not{q} h^{-1}(B)$. Now

$$h(x_\alpha)qh(U)...(1)$$

where $h(U)$ is fg^*s-open set in Y. Now $h^{-1}(B) \leq 1_X \setminus U$ which is a fuzzy closed set in $X \Rightarrow B \leq h(1_X \setminus U)$ (as h is injective) $\leq 1_Y \setminus h(U) \Rightarrow B \not{q} h(U)$. Let $V = 1_Y \setminus h(U)$. Then $B \leq V$ which is fg^*s-closed set in Y. We claim that $h(x_\alpha) \notin V$. If possible, let $h(x_\alpha) \in V = 1_Y \setminus h(U)$. Then $1 - [h(U)](h(x)) \geq \alpha \Rightarrow h(U) \not{q} h(x_\alpha)$, contradicting (1). So $h(x_\alpha) \notin V \Rightarrow h(x_\alpha) \notin fg^*scl(B) \Rightarrow x_\alpha \notin h^{-1}(fg^*scl(B)) \Rightarrow h^{-1}(fg^*scl(B)) \leq cl(h^{-1}(B))$.

Theorem 4.12. *An injective function* $h : X \to Y$ *is* fg^*s*-open if and only if for each* $B \in I^Y$ *and* F*, a fuzzy closed set in* X *with* $h^{-1}(B) \leq F$*, there exists an* fg^*s*-closed set* V *in* Y *such that* $B \leq V$ *and* $h^{-1}(V) \leq F$*.*

Proof. Let $B \in I^Y$ and F, a fuzzy closed set in X with $h^{-1}(B) \leq F$. Then $1_X \setminus h^{-1}(B) \geq 1_X \setminus F$ where $1_X \setminus F$ is a fuzzy open set in $X \Rightarrow h(1_X \setminus F) \leq h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus B$ (as h is injective) where $h(1_X \setminus F)$ is an fg^*s-open set in Y. Let $V = 1_Y \setminus h(1_X \setminus F)$. Then V is fg^*s-closed set in Y such that $B \leq V$. Now $h^{-1}(V) = h^{-1}(1_Y \setminus h(1_X \setminus F)) = 1_X \setminus h^{-1}(h(1_X \setminus F)) \leq F$.

Conversely, let F be a fuzzy open set in X. Then $1_X \setminus F$ is a fuzzy closed set in X. We have to show that $h(F)$ is an fg^*s-open set in Y. Now $h^{-1}(1_Y \setminus h(F)) \leq 1_X \setminus F$ (as h is injective). By assumption, there exists an fg^*s-closed set V in Y such that

$$1_Y \setminus h(F) \leq V...(1)$$

and $h^{-1}(V) \leq 1_X \setminus F$. Therefore, $F \leq 1_X \setminus h^{-1}(V)$ implies that

$$h(F) \leq h(1_X \setminus h^{-1}(V)) \leq 1_Y \setminus V...(2)$$

(as h is injective). Combining (1) and (2), $h(F) = 1_Y \setminus V$ which is an fg^*s-open set in Y. Hence h is fg^*s-open function.

Definition 4.13. *A function* $h : X \to Y$ *is called* fg^*s*-closed function if* $h(A)$ *is*

fg^*s-closed set in Y for each fuzzy closed set A in X.

Remark 4.14. *It is clear from definitions that*
(i) fuzzy closed function is fg^*s-closed function,
(ii) fg-closed function is fg^*s-closed function,
(iii) fg^*s-closed function is $fg\delta$-closed function.
But the converses need not be true, as it is evidenced from the following examples.

Example 4.15. (i) fg^*s-closed function $\not\Rightarrow$ fuzzy closed function, fg-closed function.
Consider Example 4.9(i). Here i is not fuzzy closed function as well as fg-closed function, as $B = 1_X \setminus B \in \tau_1^c, i(B) \notin \tau_2^c$ and $cl_{\tau_2}B \not\leq A$. But as $B \in \tau_1^c \Rightarrow i(B) \in FG^*SC(X, \tau_2), i$ is fg^*s-closed function.

(ii) $fg\delta$-closed function $\not\Rightarrow$ fg^*s-closed function
Consider Example 4.9(ii). Here $1_X \setminus E \in \tau_1^c$ and $i(1_X \setminus E) = 1_X \setminus E \notin FG^*SC(X, \tau_2)$ $\Rightarrow i$ is not fg^*s-closed function. But $1_X \setminus E$ is $fg\delta$-closed set in $(X, \tau_2) \Rightarrow i$ $fg\delta$-closed function.

Theorem 4.16. *A bijective function $h : X \to Y$ is fg^*s-closed function if and only if $fg^*scl(h(A)) \leq h(clA)$, for all $A \in I^X$.*

Proof. Let us suppose that $h : X \to Y$ be an fg^*s-closed function and $A \in I^X$. Then $h(cl(A))$ is fg^*s-closed set in Y. Since $h(A) \leq h(clA)$ and $fg^*scl(h(A))$ is the intersection of all fg^*s-closed sets in Y containing $h(A)$, we have $fg^*scl(h(A)) \leq h(clA)$.

Conversely, let for any $A \in I^X$, $fg^*scl(h(A)) \leq h(clA)$. Let U be any fuzzy closed set in X. Then $h(U) = h(clU) \geq fg^*scl(h(U)) \Rightarrow h(U)$ is an fg^*s-closed set in $Y \Rightarrow h$ is an fg^*s-closed function.

Theorem 4.17. *If $h : X \to Y$ is an fg^*s-closed bijective function, then the following statements hold:*
(i) for each fuzzy point x_α in X and each fuzzy closed set U in X with $x_\alpha \not{q} U$, there exists an fg^*s-closed set V in Y with $h(x_\alpha) \not{q} V$ such that $V \geq h(U)$,
(ii) $h^{-1}(fg^*sint(B)) \geq int(h^{-1}(B))$, for all $B \in I^Y$.

Proof. (i). Let x_α be a fuzzy point in X and U be any fuzzy closed set in X with $x_\alpha \not{q} U = clU \Rightarrow h(x_\alpha) \not{q} h(clU) \geq fg^*scl(h(U))$ (by Theorem 4.16) $\Rightarrow h(x_\alpha) \not{q} V$ for some fg^*s-closed set V in Y with $V \geq h(U)$.
(ii). Let $B \in I^Y$ and x_α be any fuzzy point in X such that $x_\alpha \in int(h^{-1}(B))$. Then there exists a fuzzy open set U in X with $U \leq h^{-1}(B)$ such that $x_\alpha \in U$. Then $1_X \setminus U \geq 1_X \setminus h^{-1}(B) \Rightarrow h(1_X \setminus U) \geq h(1_X \setminus h^{-1}(B))$ where $h(1_X \setminus U)$ is an fg^*s-closed set in Y. Let $V = 1_Y \setminus h(1_X \setminus U)$. Then V is an fg^*s-open set in Y and $V = 1_Y \setminus h(1_X \setminus U) \leq 1_Y \setminus h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus (1_Y \setminus B) = B$ (as h is injective).

Now $U(x) \geq \alpha \Rightarrow x_\alpha \not q (1_X \setminus U) \Rightarrow h(x_\alpha) \not q h(1_X \setminus U) \Rightarrow h(x_\alpha) \leq 1_Y \setminus h(1_X \setminus U) = V \Rightarrow h(x_\alpha) \in V = fg^*sint(V) \leq fg^*sint(B) \Rightarrow x_\alpha \in h^{-1}(fg^*sint(B))$. Since x_α is taken arbitrarily, $int(h^{-1}(B)) \leq h^{-1}(fg^*sint(B))$, for all $B \in I^Y$.

Remark 4.18. *Composition of two fg^*s-closed (resp., fg^*s-open) functions need not be so, as it is evidenced from the following example.*

Example 4.19. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, E\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4$, $B(a) = B(b) = 0.3$, $E(a) = 0.5, E(b) = 0.7$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fg^*s-closed as well as fg^*s-open functions. Let $i_3 = i_2 \circ i_1 : (X, \tau_1) \to (X, \tau_3)$. We claim that i_3 is not fg^*s-closed function. Now $E \in \tau_1, 1_X \setminus E \in \tau_1^c$, $(i_2 \circ i_1)(1_X \setminus E) = 1_X \setminus E \leq A \in \tau_3$. But $cl_{\tau_3} sint_{\tau_3}(1_X \setminus E) = 1_X \setminus A \not\leq A \Rightarrow 1_X \setminus E$ is not an fg^*s-closed set in $(X, \tau_3) \Rightarrow i_2 \circ i_1$ is not an fg^*s-closed function. Again as $1_X \setminus E \notin FG^*SC(X, \tau_3) \Rightarrow E \notin FG^*SO(X, \tau_3) \Rightarrow i_2 \circ i_1$ is not fg^*s-open function.

Theorem 4.20. *If $h_1 : X \to Y$ is fuzzy closed (resp., fuzzy open) function and $h_2 : Y \to Z$ is fg^*s-closed (resp., fg^*s-open) function, then $h_2 \circ h_1 : X \to Z$ is fg^*s-closed (resp., fg^*s-open) function.*

Proof. Obvious.

5. fg^*s-T_2 Space and Some Applications of fg^*s-Open Function

In this section we first introduce a new type of separation axiom and then some applications of fg^*s-open function are established.

We first recall the definition and theorem from [8, 9] for ready references.

Definition 5.1. [8] *An fts (X, τ) is called fuzzy T_2-space if for any two distinct fuzzy points x_α and y_β; when $x \neq y$, there exist fuzzy open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1, U_1 \not q V_1$ and $x_\alpha q U_2, y_\beta \in V_2, U_2 \not q V_2$; when $x = y$ and $\alpha < \beta$ (say), there exist fuzzy open sets U and V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not q V$.*

Theorem 5.2. [9] *An fts (X, τ) is fuzzy T_2-space if and only if for any two distinct fuzzy points x_α and y_β in X; when $x \neq y$, there exist fuzzy open sets U, V in X such that $x_\alpha q U$, $y_\beta q V$ and $U \not q V$; when $x = y$ and $\alpha < \beta$ (say), x_α has a fuzzy open nbd U and y_β has a fuzzy open q-nbd V such that $U \not q V$.*

Now we introduce the following concept.

Definition 5.3. *An fts (X, τ) is called fg^*s-T_2-Space if for any two distinct fuzzy points x_α and y_β in X; when $x \neq y$, there exist fg^*s-open sets U, V in X such that $x_\alpha q U$, $y_\beta q V$ and $U \not q V$; when $x = y$ and $\alpha < \beta$ (say), x_α has an fg^*s-open nbd U and y_β has an fg^*s-open q-nbd V such that $U \not q V$.*

Remark 5.4. *Clearly fuzzy T_2-space is fg^*s-T_2-space, but the converse is not nec-*

essarily true, follows from the following example.

Example 5.5. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X\}$. Then (X, τ) is an fts. Clearly (X, τ) is not a fuzzy T_2-space. Here every fuzzy set in (X, τ) is fg^*s-open set in (X, τ). Clearly it is fg^*s-T_2-space.

Theorem 5.6. *If a bijective function $h : X \to Y$ is fg^*s-open function from a fuzzy T_2-space X onto an fts Y, then Y is fg^*s-T_2-space.*

Proof. Let z_α and w_β be two fuzzy points in Y. Since h is bijective, there exist unique x, y in X such that $h(x) = z, h(y) = w$, i.e., $h(x_\alpha) = z_\alpha, h(y_\beta) = w_\beta$.

Case I. Suppose $z \neq w$. Then $x \neq y$. Since X is fuzzy T_2-space, there exist fuzzy open sets U, V in X such that $x_\alpha q U, y_\beta q V$ and $U \not{q} V$. Then $h(x_\alpha)(= z_\alpha) q h(U), h(y_\beta)(= w_\beta) q V$ and $h(U) \not{q} h(V)$ where $h(U)$ and $h(V)$ are fg^*s-open sets in Y as h is an fg^*s-open function [Indeed, $h(U) q h(V) \Rightarrow$ there exists $t \in Y$ such that $[h(U)](t) + [h(V)](t) > 1 \Rightarrow U(h^{-1}(t)) + V(h^{-1}(t)) > 1$ where $h^{-1}(t) \in X \Rightarrow UqV$, a contradiction].

Case II. Suppose $z = w$ and $\alpha < \beta$ (say). Then $x = y$ and $\alpha < \beta$. Since X is fuzzy T_2-space, there exist a fuzzy open nbd U of x_α and a fuzzy open q-nbd V of y_β such that $U \not{q} V$. Then $h(x_\alpha) \in h(U)$, $h(y_\beta) q h(V)$ and $h(U) \not{q} h(V)$ where $h(U)$, $h(V)$ are fg^*s-open sets in Y, i.e., $h(U)$ is an fg^*s-open nbd of z_α, $h(V)$ is an fg^*s-open q-nbd of w_β and $h(U) \not{q} h(V)$. Consequently, Y is fg^*s-T_2-space.

In a similarly manner we can prove the following theorem easily.

Theorem 5.7. *If a bijective function $h : X \to Y$ is fg^*s-open function from a fuzzy T_2-space X onto an fT_{g^*s}-space Y, then Y is fuzzy T_2-space.*

6. Conclusion

Introducing a new type of generalized version of fuzzy closed set, here we study a new type of fuzzy open and fuzzy closed-like functions. Our next approach is to define some sort of fuzzy continuous-like functions and also new type of fuzzy separation axioms and fuzzy compactness. The applications of these types of functions are to be established.

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