South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 2 (2024), pp. 283-298

DOI: 10.56827/SEAJMMS.2024.2002.21 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

A TWO-POINT PROBLEM IN SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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(**Received:** *Feb. 23, 2024* **Accepted:** *Jul. 25, 2024* **Published:** *Aug. 30, 2024)*

Abstract: We investigate the correctness of a problem with local two-point conditions in the time variable and periodicity conditions in the spatial coordinates for systems of partial differential equations that are not solvable for the highest time derivative. We establish conditions for the existence and uniqueness of the solution and prove metric theorems to estimate the lower bounds of small denominators that arise during the construction of the solution.

Keywords and Phrases: Vector-valued functions, two-point conditions, systems of partial differential equations, Fourier series.

2020 Mathematics Subject Classification: 35M99.

1. Introduction

Interest in the study of multipoint problems for partial differential equations is related both to the significance of their physical interpretation, finding the process described by a given equation when the states of this process are known through several observations, and to the development of a general theory of boundary value problems for partial differential equations.

Multipoint problems for partial differential equations have been studied from various perspectives by many authors [5, 6, 8, 7, 9, 3, 2, 1]. Most of these works addressed cases where the problems were well-posed. However, such problems are generally conditionally well-posed, and their solvability is often associated with the problem of small denominators, for which the metric approach has proven effective.

In this paper, we investigate the conditions for the solvability of problems with two-point conditions in the time variable for systems of partial differential equations that are not solved with respect to the highest time derivative. These systems have constant coefficients and include an arbitrary elliptic operator involving the highest time derivative in bounded domains. Such problems are conditionally well-posed, and their solvability is connected to the problem of small denominators.

Using the method of separation of variables, we provide explicit formulas for the solutions of the considered problems in the form of series based on systems of orthogonal functions. Additionally, we establish conditions for the unique solvability of the problems in the corresponding functional spaces. Applying methods and results from metric number theory, we prove metric theorems on lower bounds for small denominators that arise during the study of the considered problems.

2. Notations

Throughout the paper, we will employ the following notations.

1. Ω is the p-dimensional torus formed by formed by identifying the opposite faces of the cube $\{x \in \mathbb{R}^p \mid 0 \leq x_r \leq 2\pi, r = 1, 2, \cdots p\}$.

2. $\mathbb{Z}^p(\mathbb{Z}_+{}^p)$ is the set of points in $\mathbb{R}^p$ with integer (non-negative integer) coordinates.

3. $D = (0, T) \times \Omega$

4. $C^r(\bar{D})$ is the Banach space of continuous functions along with all derivatives up to order r over a domain D, equipped with the maximum norm.

5. $C^{(n,m)}(\bar{D})$ is the Banach space of C^m-functions in the variable t, which are C^n in the variable $x \in \mathbb{R}^p$, equipped with the maximum norm.

6. $A^\beta_\delta(\Omega), \delta > 0, \beta > 0$, is the Banach space of 2π-periodic complex functions in $x_1, \cdots, x_p$ with the norm $||\phi||_{A^\beta_\delta(\Omega)} := \sum_{k\geq 0} |\phi_k| \exp(\delta|k|^\beta)$

7. $C^n([0, T], A^\beta_\delta(\Omega))$ is the Banach space of C^n-functions in t defined on $\bar{D}$ such that, for each fixed $t \in [0, T]$, the partial derivatives with respect to x belong to $A^\beta_\delta(\Omega)$.

8. $\widetilde{C}^r(\bar{D})$, $\widetilde{A}^\beta_\delta(\Omega)$, and $\widetilde{C}^n([0, T], \widetilde{A}^\beta_\delta(\Omega))$ are the corresponding spaces of vector-valued functions.

3. Preliminaries

In this paper, we consider the following two-point problem in the domain D:

$$\left(\frac{\partial}{\partial t}\right)^2 L\left(\frac{\partial}{\partial x}\right) u(t,x) + \sum_{|s|\le m} A^s \frac{\partial^{|s|} u(t,x)}{\partial x_1^{s_1}\cdots \partial x_p^{s_p}} = f(t,x), \tag{1}$$

$$u(t_1,x) = \varphi_1(x), u(t_2,x) = \varphi_2(x), \qquad 0 \le t_1 < t_2 \le T, \tag{2}$$

where $u(t,x) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$; $f(t,x) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$; $\varphi_j(x) = \begin{bmatrix} \varphi_{j1} \\ \varphi_{j2} \end{bmatrix}$, $j = 1, 2$. Moreover, $A^s = \left[a_{rq}^s\right]_{p,q=1,2}$, $a_{rq} \in \mathbb{R}$, $B^s = \left[b_{rq}^s\right]_{p,q=1,2}$, $l \ge 1$, $b_{rq}^s \in \mathbb{R}$, and

$$L\left(\frac{\partial}{\partial x}\right) \equiv \sum_{|s|\le \ell} B^s \frac{\partial^{|s|}}{\partial x_1^{s_1}\cdots \partial x_p^{s_p}}$$

is an elliptic matrix differential expression.

We focus exclusively on the case where $m > l$. The complementary case will be addressed in a forthcoming paper.

Let us assume the following condition:

$$A := detA^\circ \ne 0. \tag{3}$$

The choice of the domain D implies that the functions $u_q(t,x)$, $f_q(t,x)$, $\varphi_{1q}(x)$, and $\varphi_{q2}(x)$ (for $q = 1, 2$) are 2π-periodic with respect to $x_1, \cdots, x_p$.

We look for the solution to the problem (1)-(2) in the form of the following vector series:

$$u(t,x) = \sum_{|k|\ge 0} u_k(t) \exp(i(k,x)), \quad \text{where } u_k(t) = \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix}. \tag{4}$$

Let us define the functions:

$$\begin{aligned} f(t,x) &= \sum_{|k|\ge 0} f_k(t) \exp(i(k,x)),\ f_k(t) = \tfrac{1}{(2\pi)^p} \int_\Omega f(t,x) \exp(-i(k,x)) dx, \\ \varphi_j(x) &= \sum_{|k|\ge 0} \varphi_{jk} \exp(i(k,x)),\ \varphi_{jk} = \tfrac{1}{(2\pi)^p} \int_\Omega \varphi(t,x) \exp(-i(k,x)) dx. \end{aligned} \tag{5}$$

By substituting the function $u(t,x)$ from (4) into Equation (1) and Conditions (2), we obtain that each of the vector functions $u_k(t)$, for $k \in \mathbb{Z}^p$, will be the solution to the two-point problem for the following system of ordinary differential equations:

$$N(k)u_k(t) := L(ik)u_k^{(2)}(t) + \sum_{|s|\le m} A^s (ik_1)^{s_1} \cdots (ik_p)^{s_p} u_k(t) = f_k(t), \tag{6}$$

$$u_k(t_1) = \varphi_{1k}, \quad u_k(t_2) = \varphi_{2k}, \tag{7}$$

where $\varphi_{jk} = \begin{bmatrix}\varphi_{kj1}\\ \varphi_{kj2}\end{bmatrix}$, $j = 1, 2$, $f_k(t) = \begin{bmatrix}f_{1k}\\ f_{2k}\end{bmatrix}$.

The solution of the problem (6)-(7) needs to be in the following form:

$$u_k(t) = w_k(t) + v_k(t), \; k \in \mathbb{Z}^p \tag{8}$$

where $w_k(t) = \begin{bmatrix}w_{1k}\\ w_{2k}\end{bmatrix}$, $v_k(t) = \begin{bmatrix}v_{1k}\\ v_{2k}\end{bmatrix}$ are the corresponding solutions to the following problems:

$$N(k)w_k(t) = 0, \quad w_k(t_1) = \varphi_{1k}, \quad w_k(t_2) = \varphi_{2k}, \tag{9}$$

$$N(k)v_k(t) = f_k(t), \quad v_k(t_1) = 0, \quad v_k(t_2) = 0. \tag{10}$$

From now on, we will assume that for all $k \in \mathbb{Z}^p$:

$$\det\, L(ik) \equiv \det \left[\sum_{|s|\le \ell} B^s (ik_1)^{s_1} \cdots (ik_p)^{s_p}\right] \neq 0. \tag{11}$$

Lemma 3.1. *There exist constants C_0 and $K(C_0)$ such that for all $k \in \mathbb{Z}^p$, $|k| > K(C_0)$, the following estimate is true*

$$det\, L(ik) \ge C_0 |k|^{4l}. \tag{12}$$

Proof. The proof relies on the ellipticity of the operator L and is identical to the proof of [8, Lemma 1], so we omit its details.

Suppose that for all $k \in \mathbb{Z}^p$, the roots $\lambda_j(k)$, $j = 1, 2$, of the following equation

$$\Lambda(\lambda, k) := \det \left[L(ik)\lambda + \sum_{|s|\le m} A^s (ik_1)^{s_1} \cdots (ik_p)^{s_p}\right] = \sum_{j=0}^{2} \Lambda_j(k)\lambda^j = 0, \tag{13}$$

are distinct. It follows from Condition (3) and (11) that these roots are different from zero.

By Lemma 3.1 and Inequality (12), we obtain the following estimate for the roots of the polynomial $\Lambda(\lambda, k)$ in (13):

$$|\lambda_j(k)| \le A_1 |k|^{2(m-2l)}, \; j = 1, 2, \; A_1 > 0, \tag{14}$$

here A_1 is a positive constant that does not depend on k.

The fundamental system of solutions of the equation $N(k)\omega_k(t) = 0$ is the following:

$$Y_1(t) = \begin{bmatrix} g(\lambda_1)\exp(\mu_1(k)t) & g(\lambda_2)\exp(\mu_2(k)t) \\ \exp(\mu_1(k)t) & \exp(\mu_2(k)t) \end{bmatrix},$$

$$Y_2(t) = \begin{bmatrix} g(\lambda_1)\exp(-\mu_1(k)t) & g(\lambda_2)\exp(-\mu_2(k)t) \\ \exp(-\mu_1(k)t) & \exp(-\mu_2(k)t) \end{bmatrix},$$

where $\mu_j = \sqrt{|\lambda_j|}\exp(i\arg(\lambda_j/2))$, $j = 1, 2$, and

$$g(\lambda_j) = -\frac{\lambda_j \sum\limits_{|s|\le 2l} b_{12}^s (ik_1)\cdots(ik_p)^{s_p} + \sum\limits_{|s|\le 2l} a_{12}^s (ik_1)\cdots(ik_p)^{s_p}}{\lambda_j \sum\limits_{|s|\le 2l} b_{11}^s (ik_1)\cdots(ik_p)^{s_p} + \sum\limits_{|s|\le 2l} a_{11}^s (ik_1)\cdots(ik_p)^{s_p}}, \; j = 1, 2. \tag{15}$$

Then, for each $k \in \mathbb{Z}^p$, the components of the solution of the problem (9) are represented by the following formulas:

$$\begin{cases} w_{k1}(t) = \sum\limits_{j=1}^{2}(C_{kj}g(\lambda_j)\exp(\mu_j(k)t) + C_{k,2+j}g(\lambda_j)\exp(-\mu_j(k)t)), \\ w_{k2}(t) = \sum\limits_{j=1}^{2}(C_{kj}\exp(\mu_j(k)t) + C_{k,2+j}\exp(-\mu_j(k)t)), \end{cases} \tag{16}$$

where the coefficients $C_{km}, m = 1, 2, 3, 4$, are determined from the following system of equations:

$$\begin{cases} \sum\limits_{j=1}^{2}(C_{kj}g(\lambda_j)\exp(\mu_j(k)t_q) + C_{k,2+j}g(\lambda_j)\exp(-\mu_j(k)t_q)) = \varphi_{kq1}, \; q = 1, 2 \\ \sum\limits_{j=1}^{2}(C_{kj}\exp(\mu_j(k)t_q) + C_{k,2+j}\exp(-\mu_j(k)t_q)) = \varphi_{kq2}, \; q = 1, 2. \end{cases} \tag{17}$$

The determinant of the system (17) is calculated by

$$\Delta(k) = (g(\lambda_1) - g(\lambda_2))^2 \prod_{j=1}^{2}(\exp(-\mu_j(k)(t_2 - t_1)) - \exp(\mu_j(k)(t_2 - t_1))). \tag{18}$$

Theorem 3.2. *The solution of the problem (1)-(2) in the space* $\overline{C}^{(2,m)}(\overline{D})$ *is unique if and only if the following conditions are satisfied:*

$$(\forall k \in \mathbb{Z}^p) \quad 1 - \exp(2\mu_j(k)T) \neq 0, \; j = 1, 2, \tag{19}$$

$$g(\lambda_1) - g(\lambda_2) \neq 0. \tag{20}$$

Proof. Assume that the solution of the problem (1)-(2) in the space $\overline{C}^{(2,m)}(\overline{D})$ is unique. If for some $k = \overline{k} \in \mathbb{Z}^p$ at least one of the conditions (19) or (20) does not hold, then $\Delta(\overline{k}) = 0$, and the homogeneous problem corresponding to (9) has nontrivial solutions $w_{\overline{k}}(t)$, which are represented by Formulas (16). Here $C_{\overline{k}m}$, $m = 1, 2, 3, 4$, is the solution for $k = \overline{k}$ of the system of homogeneous equations corresponding to the system (17). Then the homogeneous problem associated with the problem (1)-(2) has non-trivial solutions of the form $u(t, x) = u_{\overline{k}}(t) \exp(i(\overline{k}, x))$. Consequently, the solution of the non-homogeneous problem (1)-(2), if it exists, will not be unique, leading to a contradiction.

Conversely, suppose that (19) and (20) are satisfied. If there are two different solutions u_1 and u_2 to the problem (1)-(2) in the space $\overline{C}^{(2,m)}(\overline{D})$, then the vector-valued function $\tilde{u}(t, x) = u_1(t, x) - u_2(t, x)$, which belongs to the space $\overline{C}^{(2,m)}(\overline{D})$, is a solution of a homogeneous problem corresponding to (1)-(2), and it has the Fourier series representation of the form (4).

The vector-valued series for $N\tilde{u}(t, x)$ coincides with the series obtained by applying the operator N to the series for the vector-valued function $\tilde{u}(t, x)$. From Parseval's equality (see [4]) for the component of the vector-valued functions $N\tilde{u}(t, x)$, $\tilde{u}(t_1, x)$, and $\tilde{u}(t_2, x)$, it follows that each of the Fourier coefficients $\tilde{u}_k(t)$ of $\tilde{u}(t, x)$ is a solution of the homogeneous problem which corresponds to the problem (9). Thus, by conditions (19)-(20), for all $k \in \mathbb{Z}^p$ we get $\tilde{u}_k(t) \equiv 0$. Thus, from Parseval equality for the components of $\tilde{u}_k(t)$ and their continuity, it follows that $\tilde{u}(t, x) = 0$.

By the uniqueness of the expansion of a periodic function into the Fourier series, we conclude that $\tilde{u}(t, x) = 0$, which implies that $u_1(t, x) = u_2(t, x)$. This leads to a contradiction.

3.1. Main Results

Assume that conditions (19) and (20) are satisfied. Then, the solution of the problems (1)-(2) is unique. Consequently, for each $k \in \mathbb{Z}^p$ the system of equations (17) has a unique solution, and there is also a unique solution to the problem (9). Moreover, there is a unique Green's matrix $G_k(t, \tau)$ of the homogeneous problem corresponding to the problem (10), which allows us to represent the solution of the problem (10) as follows:

$$v_k(t) = \int_0^T \overline{G}_k(t, \tau) f_k(\tau) d\tau, \tag{21}$$

$$G_k(t,\tau)=\begin{cases} g_k(t,\tau)-M(k)/[(g(\lambda_1)-g(\lambda_2))\prod_{j=1}^{2}P(\mu_j,t_1,t_2)], & 0<\tau<t_1,\\ g_k(t,\tau)-H(k)/[(g(\lambda_1)-g(\lambda_2))\prod_{j=1}^{2}P(\mu_j,t_1,t_2)], & t_1<\tau<t_2,\\ g_k(t,\tau)+M(k)/[(g(\lambda_1)-g(\lambda_2))\prod_{j=1}^{2}P(\mu_j,t_1,t_2)], & t_2<\tau<T,\end{cases} \quad (22)$$

$$g_k(t,\tau)=\frac{\operatorname{sgn}\,(t-\tau)}{(g(\lambda_1)-g(\lambda_2))}\times$$

$$\begin{bmatrix} g(\lambda_1)Q(\mu_1,t,\tau)-g(\lambda_2)Q(\mu_2,t,\tau) & g(\lambda_1)g(\lambda_2)(Q(\mu_2,t,\tau)-Q(\mu_1,t,\tau)) \\ Q(\mu_1,t,\tau)-Q(\mu_2,t,\tau) & g(\lambda_1)Q(\mu_2,t,\tau)-g(\lambda_2)Q(\mu_1,t,\tau)\end{bmatrix}, \quad (23)$$

$$M(k)\coloneqq\left[m_{ij}(k)\right]_{i,j=1,2}=\begin{bmatrix} g(\lambda_1)S(k)-g(\lambda_2)R(k) & g(\lambda_1)g(\lambda_2)(S(k)-R(k)) \\ S(k)-R(k) & g(\lambda_1)S(k)-g(\lambda_2)R(k)\end{bmatrix}, \quad (24)$$

$$H(k)\coloneqq\left[h_{ij}(k)\right]_{i,j=1,2}=\begin{bmatrix} g(\lambda_1)B(k)-g(\lambda_2)C(k) & g(\lambda_1)g(\lambda_2)(B(k)-C(k)) \\ B(k)-C(k) & g(\lambda_1)B(k)-g(\lambda_2)C(k)\end{bmatrix}, \quad (25)$$

$$\begin{aligned}
&C(k)=P(\mu_1,t_1,t_2)\left(P(\mu_2,\tau+t_2,t+t_1)-P(\mu_2,\tau+t_1,t+t_2)+2P(\mu_2,t_1+t_2,t+\tau)\right),\\
&B(k)=P(\mu_2,t_1,t_2)\left(P(\mu_1,\tau+t_1,t+t_2)+P(\mu_1,\tau+t_2,t+t_1)+2P(\mu_1,t_1+t_2,t+\tau)\right),\\
&S(k)=P(\mu_2,t_1,t_2)\left(P(\mu_1,\tau+t_1,t+t_2)-P(\mu_1,\tau+t_2,t+t_1)\right),\\
&R(k)=P(\mu_1,t_1,t_2)\left(P(\mu_2,\tau+t_2,t+t_1)-P(\mu_2,\tau+t_1,t+t_2)\right),\\
&P(\mu,\xi,\eta)=\exp(-\mu(\eta-\xi))-\exp(\mu(\eta-\xi)),\\
&Q(\mu,\xi,\eta)=\exp(-\mu(\eta-\xi))+\exp(\mu(\eta-\xi)).
\end{aligned}$$

The components of the solution of the problem (9) have the form

$$\begin{aligned}
w_{k1}=&\frac{1}{g(\lambda_1)-g(\lambda_2)}\left(g(\lambda_1)\frac{P(\mu_1,t,t_2)}{P(\mu_1,t_1,t_2)}-g(\lambda_2)\frac{P(\mu_2,t,t_2)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k11}+\\
&+g(\lambda_1)g(\lambda_2)\left(\frac{P(\mu_1,t,t_2)}{P(\mu_1,t_1,t_2)}+\frac{P(\mu_2,t,t_2)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k12}\\
&+\left(g(\lambda_1)\frac{Q(\mu_1,t,t_1)}{P(\mu_1,t_1,t_2)}+g(\lambda_2)\frac{Q(\mu_2,t,t_1)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k21}\\
&+g(\lambda_1)g(\lambda_2)\left(\frac{Q(\mu_1,t,t_2)}{P(\mu_1,t_1,t_2)}+\frac{Q(\mu_2,t,t_1)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k22}.
\end{aligned} \quad (26)$$

$$\begin{aligned}
w_{k2} =& \frac{1}{g(\lambda_1)-g(\lambda_2)}\left(\frac{P(\mu_1,t,t_2)}{P(\mu_1,t_1,t_2)}-\frac{P(\mu_2,t,t_2)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k11}+ \\
&+\left(g(\lambda_2)\frac{P(\mu_1,t,t_2)}{P(\mu_1,t_1,t_2)}+g(\lambda_1)\frac{P(\mu_2,t,t_2)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k12} \\
&+\left(\frac{Q(\mu_1,t,t_1)}{P(\mu_1,t_1,t_2)}+\frac{Q(\mu_2,t,t_1)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k21} \\
&+\left(g(\lambda_2)\frac{Q(\mu_1,t,t_1)}{P(\mu_1,t_1,t_2)}+g(\lambda_1)\frac{Q(\mu_2,t,t_1)}{P(\mu_2,t_1,t_2)}\right)\varphi_{k22}. \qquad (27)
\end{aligned}$$

Based on Formulas (4) and (8), we obtain the formal representation of the solution of the problem (1) and (2) in the form of the following series:

$$u(t,x)=\sum_{|k|\geq 0}(w_k(t)+v_k(t))\exp(i(k,x)), \qquad (28)$$

where $w_k(t)$ and $v_k(t)$ and $k\in\mathbb{Z}^p$, are defined by Formulas (21)-(27), respectively. In general, the series (28) is divergent since the non-zero quantities $P(\mu_j,t_1,t_2)$, $j=1,2$, and $g(\lambda_1)-g(\lambda_2)$, which appear in the denominators of the expressions for the functions $\omega_k(t)$ and $v_k(t)$, $k\in\mathbb{Z}^p$, can become arbitrarily small in modulus for an infinite set of vectors $k\in\mathbb{Z}^p$. Therefore, the question of the existence of a solution to the problem (1), (2) is related to the problem of small denominators.

Denote by $\tilde{A}_\delta(\Omega)$, $\tilde{C}([0,T]$, and $\tilde{A}_\delta(\Omega))$ the spaces of vector functions corresponding to the spaces $A_\delta(\Omega)$, $C([0,T]$, and $A_\delta(\Omega))$.

Theorem 3.3. *Assume that there exist positive constants β_1, β_2 such that the following inequalities hold for all (except for a finite number of) vectors $k\in\mathbb{Z}^p$.*

$$|\exp(-\mu_j(t_2-t_1))-\exp(\mu_j(t_2-t_1))|>|k|^{-\beta_1}exp(-\nu|k|), \quad =(m-2l)\ j=1,2. \qquad (29)$$

$$\left|\frac{g(\lambda_1)g(\lambda_2)}{g(\lambda_1)-g(\lambda_2)}\right|<|k|^{\beta_2}. \qquad (30)$$

If $\varphi_j\in\tilde{A}^\gamma_\delta$, $j=1.2$, $\tilde{C}([0,T],\tilde{A}^\gamma_\delta)$, $\delta>2AT+\nu$, then there exists a solution of the problem (1), (2) in the space $\tilde{C}^{2,m}(\overline{D})$, which depends continuously on the functions $\varphi_j(x)$, $j=1,2$, and $f(t,x)$.

Proof. From (24), (25) and the estimate (14) it follows that for all (except for a finite number of) vectors $k\in\mathbb{Z}^p$, the following inequalities hold

$$|m_{ij}(k)|\leq C_1\exp\left(2A^{1/2}|k|^\gamma\right),|h_{ij}(k)|\leq C_2\exp\left(2A^{1/2}|k|^\gamma\right),\ i,j=1,2,\ C_1,C_2>0. \qquad (31)$$

From Formulas (21)-(28) and Inequalities (29), (30), (14) it follows that there exists a solution $u(t,x)$ such that

$$||u(t,x)||_{\tilde{C}^{2,m}(\overline{D})} \leq \sum_{j=1}^{2} ||u_j(t,x)||_{\tilde{C}^{2,m}(\overline{D})} \leq \sum_{|k|\geq 0} \left(\sum_{q=1}^{2} C_3 |\varphi_{qk}| \exp(\delta |k|^{\gamma}) + C_4 \tilde{f}_k \exp(\delta |k|^{\gamma}) \right)$$

$$\leq C_5 \left(\sum_{q=1}^{2} ||\varphi_q(x)||_{\tilde{A}^{\gamma}_{\delta}} + ||f(t,x)||_{\tilde{C}([0,T],\tilde{A}^{\gamma}_{\delta})} \right) < \infty, \quad \text{where } \tilde{f}_k = \max_{0\leq t\leq T} |f_k|, k \in \mathbb{Z}^p.$$

Therefore, $u(t,x) \in \tilde{C}^{2,m}(\overline{D})$.

Next, we will examine when Inequalities (29) and (30) hold. Note that

$$|\exp(-\mu_j(t_2-t_1)) - \exp(\mu_j(t_2-t_1))| = |\exp(-\mu_j(t_2-t_1))||1-\exp(2\mu_j(t_2-t_1))|.$$

If $\operatorname{Re}\mu_j(k) \neq 0,\ j = 1.2$, then, taking into account (14), we will get

$$|\exp(-\mu_j(t_2-t_1)) - \exp(\mu_j(t_2-t_1))| > \exp(-A^{1/2}|k|^{(m-2l)}T)|\operatorname{Re}\mu_j(k)(t_2-t_1)|. \tag{32}$$

Conversely, since for all $x \in [0; \frac{\pi}{2}]$, $\sin x \geq 2x/\pi$, we will obtain the following estimate:

$$\begin{aligned} &|\exp(-\mu_j(t_2-t_1)) - \exp(\mu_j(t_2-t_1))| > \left| \sin \left| \tfrac{\operatorname{Im}\mu_j(k)(t_2-t_1)}{\pi} - m_k \right| \pi \right| > \\ &> 4(t_2-t_1)|k|^{(m-2l)} \left| \tfrac{\operatorname{Im}\mu_j(k)}{\pi|k|^{(m-2l)}} - \tfrac{m_k}{(t_2-t_1)|k^{(m-2l)}|} \right|, \end{aligned} \tag{33}$$

where $m_k \in \mathbb{Z}$ is such that $\left| \frac{\operatorname{Im}\mu_j(k)(t_2-t_1)}{\pi} - m_k \right| < \frac{1}{2}$.

It follows from [8, Lemma 1] that for almost all numbers $T = t_2 - t_1$ with respect to the Lebesgue measure, denoted as **mess**, the following inequalities

$$\left| \frac{\operatorname{Im}\mu_j(k)}{\pi|k|^{m-2l}} - \frac{m_k}{T|k|^{m-2l}} \right| > \frac{1}{|k|^{p+(m-2l)+\varepsilon}},\ j = 1,2;\ 0 < \varepsilon < 1 \tag{34}$$

are valid for all (except for a finite number of) pairs (m,k), $m \in \mathbb{Z}$, $k \in \mathbb{Z}^p$.

Let us represent the constant term Λ_0 of Equation (13) as follows:

$$\Lambda_0 = \det \left[\sum_{|s|\leq m} A^s (ik_1)^{s_1} \cdots (ik_p)^{s_p} \right] = \sum_{|\nu|\leq 2m} a^{\nu} (ik_1)^{\nu_1} \cdots (ik_p)^{\nu_p}, \tag{35}$$

where

$$a^{\nu} = \sum_{\sum_{s=1}^{2} s_{\nu}^{(h)}=\nu_q,\, q=1,\cdots,p} \det\left[a_{rq}^{s}(h)\right]_{r,h=1,2},$$

and $a_{rh}^{s(h)}$ $(r, h = 1, 2)$ are the elements of the h-th column of a matrix A^s, $|s| \leq m$ and $s_q^{(h)}$ is q-th component of the multi-index of this matrix.

We denote by $\overline{a} \in \mathbb{R}^{\sigma}$ the vector which consists of the coefficients a^{ν} of a polynomial (35), where σ is the number of solutions of the inequality $\nu \leq 2m$.

Lemma 3.4. *For almost all vectors $\overline{a}$ (with respect to the Lebesgue measure in $\mathbb{R}^{\sigma}$), the following inequality holds:*

$$|\lambda_j(k)| > |k|^{-p-2m-\varepsilon}, \quad j = 1, 2, \varepsilon > 0, \tag{36}$$

for all (except for a finite number of) vectors $k \in \mathbb{Z}^p$.

Proof. From Formula (35) it follows that

$$|\Lambda_0(k)| \geq \max\left(|Re\Lambda_0(k)|,\ |Im\Lambda_0(k)|\right) \geq |Re\Lambda_0(k)|.$$

We will show that for almost all vectors $\overline{a}$ (with respect to the Lebesgue measure in $\mathbb{R}^{\sigma}$), the following inequality holds:

$$|Re\Lambda_0(k)| \geq |k|^{-p-\varepsilon}. \tag{37}$$

We denote by W the set of those vectors $\overline{a}$ that belong to some σ-dimensional parallelepiped $\Pi_{\sigma} = [\alpha_1, \beta_1] \times \Pi_{\sigma-1}$, for which the inequality

$$|Re\Lambda_0(k)| < |k|^{-p-\varepsilon}. \tag{38}$$

has an infinite number of solutions $k \in \mathbb{Z}^p$.

Let's fix k and $a_2, \ldots, a_{\sigma}$. Then the set $W_k(a_2, \ldots, a_{\sigma})$ of those $a_1 \in [\alpha_1, \beta_1]$ for which Inequality (38) holds, satisfies the following inequality:

$$\textbf{mess}\, W_k(a_2, \ldots, a_{\sigma}) < |k|^{-p-\varepsilon}. \tag{39}$$

By integrating the estimate (39) over the parallelepiped $\Pi_{\sigma-1}$, we obtain that the measure of the set W_k of those vectors $a \in \Pi_{\sigma}$ for a fixed k satisfies the inequality:

$$\textbf{mess}\, W_k < 2C|k|^{-p-\varepsilon},$$

where C is the volume of the parallelepiped $\Pi_{\sigma-1}$. Since the series $\sum_{|k|>0} 2C|k|^{-p-\varepsilon}$ converges, it follows from [10, Theorem 1] that $\textbf{mess}\, W = 0$. Therefore, for almost all $\overline{a} \in \Pi_\sigma$, Inequality (37) holds. Furthermore, Since the space $\mathbb{R}^\sigma$ can be covered by the countable number of parallelepipeds Π_δ, it follows that Inequality (37) holds for almost all $\overline{a} \in \Pi_\sigma$.

By Inequalities (14), (37), and the estimate $\det L(ik) \leq C|k|^4 l$, we obtain

$$|\lambda_1| = \frac{\Lambda_0(k)}{|\det L(ik)|\,|\lambda_2|} > C_\sigma \frac{|k|^{-p-\varepsilon}}{|k|^{4l}|k|^{2(m-2l)}} \geq C_\sigma |k|^{-p-2m-\varepsilon}.$$

That concludes the proof.

The following theorem follows from Inequalities (31)–(34), and Lemma 3.4.

Theorem 3.5. *I. If* $\operatorname{Re}\mu_j(k) \neq 0,\ j = 1.2$, *then for almost all vectors* $\bar{a}$ *(with respect to the Lebesgue measure in* $\mathbb{R}^\sigma$*), Inequalities (29) hold for all (except for a finite number of) vectors* $k \in \mathbb{Z}^p$ *when* $\beta_1 > \frac{-p-m+2l}{2}$ *and* $\nu = A^{1/2}$.

II. If $\operatorname{Re}\mu_j(k) = 0,\ j = 1.2$, *then for almost all numbers* T *(with respect to the Lebesgue measure in* $\mathbb{R}$*), Inequalities (29) hold for all (except for a finite number of) vectors* $k \in \mathbb{Z}^p$ *if* $\beta_1 > p$.

We denote by $b = (b_1, \ldots, b_p)$ the vector with the coordinates $b_r = \det\left[B^{\gamma(r)}\right]$, $\gamma(r) = (\underbrace{0, \ldots, 0}_{r-1}, 2l, 0, \ldots, 0)$, $r = 1, \ldots, p$, and by $h = (h_1, \ldots, h_p)$, where $h_r = \det\left[A^{\gamma(r)}\right]$, $r = 1, \ldots, p$ $\gamma(r) = (\underbrace{0, \ldots, 0}_{r-1}, m, 0, \ldots, 0)$.

Theorem 3.6. *For almost all vectors* h *(with respect to the Lebesgue measure in* $\mathbb{R}^p$*) the inequality*

$$|\lambda_1(k) - \lambda_2(k)| > H|k|^\eta,\ \eta = -\frac{p}{2} + m - 2l - \varepsilon,\ H > 0,\ \varepsilon > 0 \tag{40}$$

is hold for all (except for a finite number of) vectors $k \in \mathbb{Z}^p$.

Proof. For the discriminant $D(\Lambda)$ of the polynomial $\Lambda(\lambda, k)$ (see (14)) the following representations are valid:

$$D(\Lambda) = \Lambda_2^2(k)(\lambda_1(k) - \lambda_2(k))^2, \tag{41}$$

$$D(\Lambda) = -\frac{1}{\Lambda_2(k)} \begin{vmatrix} \Lambda_2(k) & \Lambda_1(k) & \Lambda_0(k) \\ 2\Lambda_2(k) & \Lambda_1(k) & 0 \\ 0 & 2\Lambda_2(k) & \Lambda_1(k) \end{vmatrix}, \tag{42}$$

where $\Lambda_2(k) = L(ik)$. We will show that for almost all vectors h that belong to the parallelepiped $\Pi_p = [\alpha, \beta] \times \Pi_{p-1} \subset \mathbb{R}^p$ the inequality

$$|\operatorname{Re} D(\Lambda)| > |k|^\mu,\ \mu < -p + 2(m + 2l) \tag{43}$$

is fulfilled for all but a finite number of vectors $k \in \mathbb{Z}^p$.

We denote by W the set of vectors h for which inequality

$$|\mathrm{Re}\, D(\Lambda)| < |k|^{\mu},\ \mu < -p + 2(m + 2l) \tag{44}$$

holds for an infinite number of vectors $k \in \mathbb{Z}^p$. We denote by W_k the set of vectors h for which Inequality (44) is valid for a fixed $k \in \mathbb{Z}^p$. Without loss of the generality, we assume that $h_1 \neq 0$ and $|k_1| = \max_{1 \le i \le p} |k_i|$. From Formula (41) we have

$$D(\Lambda) = -4\Lambda_0\Lambda_2 + \Lambda_1^2.$$

Taking into account that

$$\Lambda_0 = k_1^{2m} \det \left[a_{rh}^{(m,0,\dots,0)}\right]_{r,h=1,2} + \sum_{|\nu| \le 2m}{}' a^{\nu}(ik_1)^{\nu_1} \dots (ik_p)^{\nu_p},$$

where the symbol "$'$" above the summation $\sum$ indicates that the summation is excluded for $\nu = (2m, 0, \dots, 0)$, we get

$$\mathrm{Re}\, D(\Lambda) = -4h_1 k_1^{2m}\, \mathrm{Re}\, \Lambda_2(k) + \mathrm{Re}\, (\Lambda_1)^2 \tag{45}$$

It follows from Equality (45) and Estimate (12) that for all $k \in \mathbb{Z}^p$, $|k| > K(C_0)$

$$\left|\frac{\partial \mathrm{Re}\, D(\Lambda)}{\partial h_1}\right| > \frac{4C_0}{\sqrt{2}}|k_1|^{2m}|k|^{4e} > C(p)|k|^{2(m+2l)}.$$

Then, by [8, Lemma 2, ch. 1], the measure of the set $W_k(h_1)$ of those values $h_1 \in [\alpha, \beta]$ that satisfy Inequality (44) (for a fixed $h_1, \dots, h_p$) has the following estimate:

$$|W_k(h_1)| \le C\,|k|^{-p-\varepsilon},\ \varepsilon > 0. \tag{46}$$

By integrating the estimate (46) with respect to the variables $h_2, \dots, h_p$ over the parallelepiped Π_{p-1}, we get

$$|W_k| \le C\,|k|^{-p-\varepsilon},\ \varepsilon > 0. \tag{47}$$

The estimate (47) implies the convergence of the series $\sum_{|k|>0} |W_k|$. Therefore, from [10, Lemma 2.1], we conclude that the measure of the set W is zero. Since the space $\mathbb{R}^p$ can be covered by the countable number of parallelepipeds Π_p, Inequality

(40) is thus proved. From the inequality $|D(\Lambda)| \geq |\mathrm{Re}\, D(\Lambda)|$ and (43) and (41), we obtain that for almost all vectors h, the following estimate:

$$|\lambda_1(k) - \lambda_2(k)| > H\,|k|^{-(\frac{p}{2}+m-2l)-\varepsilon},\ \varepsilon > 0,$$

is valid for all (except for a finite number of) vectors $k \in \mathbb{Z}^p$. We denote by

$$d^\nu = \sum_{\sum\limits_{n=1}^{2} s_q^{(h)}=\nu_q,\ q=1,\dots,p} \det\left[d_{rh}^{s(h)}\right]_{r,h=1,2},$$

where $d_{rh}^{s(h)}$, $r = 1, 2$, are the elements of the h-th column of any matrix $\left[d_{rh}^{s}\right]_{r,h=1,2}$ with elements $d_{11}^s = b_{11}^{s_1}$, $d_{12}^s = b_{11}^{s_1}$, $|s_1| \leq 2l$; $d_{21}^s = a_{12}^{s_2}$, $d_{22}^s = a_{11}^{s_2}$, $|s_2| \leq m$, and $s_q^{(h)}$ is the q-th component multi-index from of this matrix; $\bar{d} = (d_1, \dots, d_p)$ is the vector with coordinates

$$d_2 = \det\begin{bmatrix} d_{11}^{\gamma_1(r)} & d_{12}^{\gamma_1(r)} \\ d_{12}^{\gamma_2(r)} & d_{22}^{\gamma_2(r)} \end{bmatrix},\ r = 1, \dots, p,$$

where $\gamma_1(r) = (\underbrace{0, \dots, 0}_{r-1}, 2l, 0, \dots, 0)$, $\gamma_2(r) = (\underbrace{0, \dots, 0}_{r-1}, m, 0, \dots, 0)$, $r = 1, \dots, p$, and

$$P(ik) = \sum_{|\nu| \leq m+2l} d^\nu (ik_1)^{\nu_1} \dots (ik_p)^{\nu_p}.$$

Lemma 3.7. *For almost all vectors $\bar{d}$ (with respect to the Lebesgue measure in $\mathbb{R}^p$) inequality*

$$|P(ik)| > C\,|k|^{\xi},\ \xi = -p + m + 2l - \varepsilon \tag{48}$$

is true for all (except for a finite number of) vectors $k \in \mathbb{Z}^p$.

Proof. The proof is nearly identical to the proof of Theorem 4.4 in [8].

Theorem 3.8. *For almost all vectors (h, d) (with respect to the Lebesgue measure in $\mathbb{R}^{2p}$) and for almost all vectors $\bar{a}$ (with respect to the Lebesgue measure in $\mathbb{R}^\sigma$), Inequality (30) holds for $\beta_2 > 2(m - 2l) + \frac{3}{2}p$, for all (except for a finite number) of vectors $k \in \mathbb{Z}^p$.*

Proof. From Equality (15) and Estimate (14) it follows that

$$\left|\frac{g(\lambda_1)g(\lambda_2)}{g(\lambda_1) - g(\lambda_2}\right| < \frac{|k|^{4(m-l)}}{|\lambda_1(k) - \lambda_2(k)||P(ik)|}.$$

This inequality, Theorem 3.6, and Lemma 3.4 conclude the proof.

4. Conclusion

The article addresses the complexities associated with a specific class of partial differential equations that incorporate local two-point conditions in time and periodicity conditions in space.

We establish criteria for the existence and uniqueness of solutions. Furthermore, the metric theorems developed offer critical insights into the behaviour of small denominators encountered during solution construction, ensuring that our findings are robust and applicable to broader contexts within the field.

Acknowledgment

The authors appreciate the unknown referee's valuable and profound comments.

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