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ON L-FUZZY TOPOLOGIES INDUCED BY L-G-FILTERS

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Abstract: This paper addresses L-fuzzy topologies induced by L-G-filters and studies the categorical relations between L-G-filter spaces and L-fuzzy topological spaces. Three functors from the category of L-G-filter spaces to the category of L-fuzzy topological spaces are obtained. Having introduced the concept of monotone L-fuzzy topologies, the study inquires into the sum, subspace, product, quotient and the lattice structure of such topologies.

Keywords and Phrases: Residuated lattice, Functor, Monotone.

2020 Mathematics Subject Classification: 54A40, 18A40.

1. Introduction

In 1968, Chang [4] introduced the concept of fuzzy topological spaces. Later, Höhle [6] developed the idea of fuzzification of topological spaces. Subsequently Kubiak [16] and Šostak [19] independently developed the notion of L-fuzzy topological spaces. Later Kubiak and Šostak [17] extended this notion to LM-fuzzy topological spaces. In 2007, Yue [21] defined product, sum and quotient space of LM-fuzzy topological spaces and studied several subcategories of LM-fuzzy topological spaces.

Many authors studied the relationship between fuzzy topologies and filters. In 1977, Lowen [18] developed the idea of filters in I^X, called prefilters to discuss convergence in fuzzy topological spaces. In 1999 Burton et al. [3] introduced the concept of generalized filters as a map from 2^X to I. Subsequently Höhle and Šostak [8] developed the notion of L-filters and stratified L-filters on a complete quasi-monoidal lattice. Later, in 2013 Jäger [9] developed the theory of stratified LM-filters which generalizes the theory of stratified L-filters by introducing stratification mapping, where L and M are frames.

In [10], the authors introduced the concept of LM-G-filter spaces as a generalization of LM-filter spaces on a complete residuated lattice. Some subcategories of LM-**G**, the category of LM-G-filter spaces have been identified by introducing the concepts of catalyzed LM-G-filter spaces in [11] and weak and strong LM-G-filter spaces in [14]. Images of LM-G-filter spaces and LM-G-filterbases induced by functions are investigated and some of their properties are derived in [15]. Moreover, the categorical connections of L-G-filters with L-filters, L-interior operators and L-fuzzy pre-proximity spaces, L-fuzzy grills, L-closure operators and L-fuzzy cotopologies are identified in [12] and [13].

In this paper, we identify L-fuzzy topologies induced by L-G-filters and study categorical relations between L-G-filter spaces and L-fuzzy topological spaces. The study obtains three functors from the category of L-G-filter spaces to the category of L-fuzzy topological spaces. The concept of monotone L-fuzzy topologies is introduced and lattice structure, subspace, quotient, product and sum of monotone L-fuzzy topologies are investigated.

2. Preliminaries

Throughout this paper X stands for a non-empty ordinary set. For the notions of category theory, the readers can refer to [1].

Definition 2.1. [2] *An algebra* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is called a complete residuated lattice if it satisfies the following properties:*

(C1) $(L, \leq, \vee, \wedge, 0, 1)$ *is a complete lattice where* 1 *is the universal upper bound and* 0 *denotes the universal lower bound;*

(C2) $(L, \odot, 1)$ *is a commutative monoid;*

(C3) $x \odot y \leq z$ *if and only if* $x \leq y \rightarrow z$ *for* $x, y, z \in L$.

Unless otherwise specified, in this paper, we assume that $(L, \leq, \odot)$ is a complete residuated lattice.

Remark 2.2. *The following lattices are complete residuated lattices.*

(1) Complete locally finite BL-Algebra.

(2) Every completely distributive lattice.

(3) Complete locally finite MV-Algebra.

Lemma 2.3. [2, 5, 20] *Let L be a complete residuated lattice. Then for each $x, y, z, x_i, y_i, w \in L$, we have the following properties.*

(1) $x \to y = \bigvee\{z | z \odot x \leq y\}$.

(2) $1 \to x = x$, $0 \odot x = 0$ *and* $x \leq y$ *if and only if* $x \to y = 1$.

(3) If $y \leq z$*, then* $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ *and* $z \to x \leq y \to x$.

(4) $x \odot y \leq x, y$, $x \odot y \leq x \wedge y$.

(5) $x \odot (\bigwedge_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \odot y_i)$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$.

(7) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *and* $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

All algebraic operations on L can be extended pointwise to L^X as $A \leq B$ if and only if $A(x) \leq B(x)$ and $(A \odot B)(x) = A(x) \odot B(x)$ for all $x \in X$. For all $\alpha \in L$, α_X is defined by $\alpha_X(x) = \alpha$ for all $x \in X$.

Lemma 2.4. [2] *For a given set X, define a binary map $S : L^X \times L^X \to L$ by* $S(A, B) = \bigwedge_{x\in X}(A(x) \to B(x))$*. Then for each $A, B, C, D \in L^X$, the following properties hold.*

(1) $S(A, B) \odot S(B, C) \leq S(A, C)$ *;*

(2) $A \leq B$ *if and only if* $S(A, B) = 1$*;*

(3) If $A \leq B$*, then* $S(C, A) \leq S(C, B)$ *and* $S(A, C) \geq S(B, C)$*;*

(4) $S(A, B) \odot S(C, D) \leq S(A \odot C, B \odot D)$.

For the rest of the paper, S represents the map defined in the above Lemma.

Definition 2.5. [1] *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are categories, then a functor* F *from* $\boldsymbol{A}$ *to* $\boldsymbol{B}$ *is a function that assigns to each* $\boldsymbol{A}$*-object* A*, a* $\boldsymbol{B}$*-object* $F(A)$*, and to each* $\boldsymbol{A}$*-morphism* $f : A \to A'$*, a* $\boldsymbol{B}$*-morphism* $F(f) : F(A) \to F(A')$*, in such a way that*

(i.) F *preserves composition; i.e.,* $F(f \circ g) = F(f) \circ F(g)$ *whenever* $f \circ g$ *is defined, and*

(ii.) F *preserves identity morphisms; i.e.,* $F(id_A) = id_F(A)$ *for each* $\boldsymbol{A}$*-object* A*.*

Definition 2.6. [10] *An L-G-filter on a set* X *is defined to be a mapping* $G : L^X \to L$ *satisfying:*

(G1) $G(1_X) = 1$*;*

(G2) For every $A, B \in L^X$ *such that* $A \leq B$*,* $G(A) \leq G(B)$*;*

(G3) For every $A, B \in L^X, G(A \odot B) \geq G(A) \odot G(B)$*.*

The pair (X, G) *is called an L-G-filter space. In addition to the above axioms, if* $(\mathbf{G}4) : G(0_X) = 0$ *is also satisfied, then* (X, G) *becomes an L-filter space* [8].
The pair (X, G) *is called stratified L-G-filter space if* $G(\alpha_X \odot A) \geq G(A) \odot \alpha$ *for all* $A \in L^X$ *and* $\alpha \in L$*.*
Let (X, G_1) *and* (Y, G_2) *be L-G-filter spaces. A map* $f : (X, G_1) \to (Y, G_2)$ *is called an L-G-filter map if* $G_1(f^{\leftarrow}(B)) \geq G_2(B),\ \forall B \in L^Y$*.*

Remark 2.7. *An L-G-filter space* (X, G) *is stratified if and only if* $G(\alpha_X) \geq \alpha$ *for all* $\alpha \in L$*.*

Definition 2.8. [17] *A mapping* $\tau : L^X \to M$ *is called an LM-fuzzy topology on a set* X *if it satisfies the following properties:*

(T1) $\tau(0_X) = \tau(1_X) = 1$*;*

(T2) $\tau(A \odot B) \geq \tau(A) \odot \tau(B)$ *for all* $A, B \in L^X$*;*

(T3) $\tau(\bigvee_{i\in I} A_i) \geq \bigwedge_{i\in I} \tau(A_i)$ *for each arbitrary family* $\{A_i \in L^X; i \in I\}$*.*

The pair (X, τ) *is called LM-fuzzy topological space. An LM-fuzzy topological space* (X, τ) *is called stratified if* $\tau(\alpha_X) = 1$ *for all* $\alpha \in L$*. When* $L = M$*, the pair* (X, τ) *is called L-fuzzy topological space* [8].

Let (X, τ_1) and (Y, τ_2) be LM-fuzzy topological spaces. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is called a continuous map if $\tau_1(f^{\leftarrow}(B)) \geq \tau_2(B),\ \forall B \in L^Y$.

Definition 2.9. [7] *Let (X, τ) be an L-fuzzy topological space, $Y \subseteq X$ and $\tau|_Y$ be the L-fuzzy topology defined on Y by $(\tau|_Y)(B) = \bigvee\{\tau(A)|A \in L^X, A|_Y = B\}$ for all $B \in L^Y$. Then $(Y, \tau|_Y)$ is called the subspace of (X, τ).*

Y. Yue [21] defined quotient, product and sum of LM-fuzzy topological spaces in a completely distributive lattice as:

Definition 2.10. [21] *Let (X, τ) be an LM-fuzzy topological space and $f : X \to Y$ be a surjective mapping. Then the LM-fuzzy topology, τ/f defined on Y by $(\tau/f)(B) = \tau(f^{\leftarrow}(B))$ for all $B \in L^Y$ is called quotient LM-fuzzy topology of τ with respect to f.*

Definition 2.11. [21] *Let $\{(X_j, \tau_j)\}_{j\in J}$ be a family of LM-fuzzy topological spaces, $X = \prod_{j\in J} X_j$ and $p_j : \prod_{j\in J} X_j \to X_j$ be the projection map. Then the product of $\{(X_j, \tau_j)\}_{j\in J}$ is defined as $(\prod_{j\in J} \tau_j)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{\sqcap C_{\lambda_\beta} = B_\lambda} \bigwedge_{\beta} \bigvee_{j\in J} \bigvee_{p_j^{\leftarrow}(D) = C_{\lambda_\beta}} \tau_j(D)$ for all $A \in L^X$ with $(\sqcap)$ standing for finite intersection.*

Definition 2.12. [21] *Let $\{(X_j, \tau_j)\}_{j\in J}$ be a family of LM-fuzzy topological spaces, X_j's be pairwise disjoint and $X = \bigcup_{j\in J} X_j$. Then the LM-fuzzy topology, $\bigoplus_{j\in J} \tau_j$ defined on X by $(\bigoplus_{j\in J} \tau_j)(A) = \bigwedge_{j\in J} \tau_j(A|_{X_j})$ for all $A \in L^X$ is called sum LM-fuzzy topology of $\{\tau_j\}_{j\in J}$.*

Remark 2.13. *The above definitions of subspace, quotient, product and sum space are valid in the case of L-fuzzy topological spaces where L is a complete residuated lattice.*

3. Some Functors from L-G-Filter Spaces to L-Fuzzy Topological Spaces

This section identifies two functors from the category of L-G-filter spaces to the category of L-fuzzy topological spaces.

Notation 3.1. *Let L-**G** denotes the category of L-G-filter spaces and L-**FTop** denotes the category of L-fuzzy topological spaces.*

The following two theorems suggest a functor from L-**G** to L-**FTop**.

Theorem 3.2. *Let $G : L^X \to L$ be an L-G-filter on X. Then $\eta(G) : L^X \to L$ defined by $\eta(G)(A) = S(A, G(A) \odot A)$ is an L-fuzzy topology on X.*

Proof. (T1) is obvious.

(T2) For all $A, B \in L^X$,

$$\begin{array}{rcl} \eta(G)(A) \odot \eta(G)(B) & = & S\big(A, G(A) \odot A\big) \odot S\big(B, G(B) \odot B\big) \\ & \leq & S\big(A \odot B, G(A) \odot A \odot G(B) \odot B\big) \\ & \leq & S\big(A \odot B, G(A \odot B) \odot A \odot B\big) \\ & = & \eta(G)(A \odot B). \end{array}$$

(T3) For each family $\{A_i \in L^X; i \in I\}$,

$$\begin{array}{rcl} \eta(G)(\bigvee_{i\in I} A_i) & = & S\Big(\bigvee_{i\in I} A_i, G(\bigvee_{i\in I} A_i) \odot \bigvee_{i\in I} A_i\Big) \\ & \geq & S\Big(\bigvee_{i\in I} A_i, \bigvee_{i\in I}(G(A_i) \odot A_i)\Big) \\ & = & \bigwedge_{x\in X} \Big((\bigvee_{i\in I} A_i)(x) \to \bigvee_{i\in I} \big(G(A_i) \odot A_i\big)(x)\Big) \\ & \geq & \bigwedge_{x\in X} \bigwedge_{i\in I} \Big(A_i(x) \to \big(G(A_i) \odot A_i\big)(x)\Big) \\ & = & \bigwedge_{i\in I} S\big(A_i, G(A_i) \odot A_i\big) \\ & = & \bigwedge_{i\in I} \eta(G)(A_i). \end{array}$$

Corollary 3.3. *If (X, G) is a stratified L-G-filter space and L satisfies the idempotency condition, i.e. $a \odot a = a$ for all $a \in L$, then $(X, \eta(G))$ is a stratified L-fuzzy topological space.*

Theorem 3.4. *Let (X, G_1) and (Y, G_2) be L-G-filter spaces such that $f : (X, G_1) \to (Y, G_2)$ be an L-G-filter map. Then $f : (X, \eta(G_1)) \to (Y, \eta(G_2))$ is a continuous map.*

Proof. For all $B \in L^Y$,

$$\begin{array}{rcl} \eta(G_1)(f^{\leftarrow}(B)) & = & S\Big(f^{\leftarrow}(B), G_1(f^{\leftarrow}(B)) \odot f^{\leftarrow}(B)\Big) \\ & = & \bigwedge_{x\in X} \Big(f^{\leftarrow}(B)(x) \to \big(G_1(f^{\leftarrow}(B)) \odot f^{\leftarrow}(B)\big)(x)\Big) \\ & = & \bigwedge_{x\in X} \Big(B(f(x)) \to \big(G_1(f^{\leftarrow}(B)) \odot B(f(x))\big)\Big) \\ & \geq & \bigwedge_{y\in Y} \Big(B(y) \to \big(G_1(f^{\leftarrow}(B)) \odot B(y)\big)\Big) \end{array}$$

$$
\begin{aligned}
&\geq \bigwedge_{y\in Y}\Big(B(y)\to \big(G_2(B)\odot B\big)(y)\Big)\\
&= S\big(B, G_2(B)\odot B\big)\\
&= \eta(G_2)(B).
\end{aligned}
$$

Corollary 3.5. *$\eta : L$-$\boldsymbol{G}$ to L-$\boldsymbol{FTop}$ is a functor.*

Further, the following two theorems provide yet another functor from L-**G** to L-**FTop**.

Theorem 3.6. *Let $G : L^X \to L$ be an L-G-filter on X. Then $\zeta(G) : L^X \to L$ defined by $\zeta(G)(A) = \Big(\bigvee_{x\in X} A(x)\Big) \to G(A)$ is an L-fuzzy topology on X.*

Proof. (T1) is obvious.

(T2) For all $A, B \in L^X$,

$$
\begin{aligned}
\zeta(G)(A)\odot\zeta(G)(B) &= \bigg(\Big(\bigvee_{x\in X} A(x)\Big)\to G(A)\bigg)\odot\bigg(\Big(\bigvee_{x\in X} B(x)\Big)\to G(B)\bigg)\\
&\leq \bigg(\Big(\bigvee_{x\in X} A(x)\Big)\odot\Big(\bigvee_{x\in X} B(x)\Big)\bigg)\to\Big(G(A)\odot G(B)\Big)\\
&\leq \Big(\bigvee_{x\in X}(A\odot B)(x)\Big)\to G(A\odot B)\\
&= \zeta(G)(A\odot B).
\end{aligned}
$$

(T3) For each family $\{A_i \in L^X; i \in I\}$

$$
\begin{aligned}
\zeta(G)(\bigvee_{i\in I} A_i) &= \Big(\bigvee_{x\in X}(\bigvee_{i\in I} A_i)(x)\Big)\to G(\bigvee_{i\in I} A_i)\\
&\geq \Big(\bigvee_{i\in I}(\bigvee_{x\in X} A_i(x))\Big)\to\bigvee_{i\in I} G(A_i)\\
&\geq \bigwedge_{i\in I}\Big((\bigvee_{x\in X} A_i(x))\to G(A_i)\Big)\\
&= \bigwedge_{i\in I}\zeta(G)(A_i).
\end{aligned}
$$

Corollary 3.7. *If (X, G) is a stratified L-G-filter space, then $(X, \zeta(G))$ is a stratified L-fuzzy topological space.*

Theorem 3.8. *Let (X, G_1) and (Y, G_2) be L-G-filter spaces such that $f : (X, G_1) \to$*

(Y, G_2) be an L-G-filter map. Then $f : (X, \zeta(G_1)) \to (Y, \zeta(G_2))$ is a continuous map.

Proof. For all $B \in L^Y$,

$$\begin{aligned} \zeta(G_1)(f^{\leftarrow}(B)) &= \Big(\bigvee_{x\in X} f^{\leftarrow}(B)(x)\Big) \to G_1(f^{\leftarrow}(B)) \\ &\geq \Big(\bigvee_{x\in X} B(f(x))\Big) \to G_2(B) \\ &\geq \Big(\bigvee_{y\in Y} B(y)\Big) \to G_2(B) \\ &= \zeta(G_2)(B). \end{aligned}$$

Corollary 3.9. *$\zeta : L$-**G** to L-**FTop** is a functor.*

4. Monotone L-Fuzzy Topological Spaces

This section obtains a functor from the category of L-G-filter spaces to the category of L-fuzzy topological spaces and introduces the notion of monotone L-fuzzy topological spaces. Moreover, properties like lattice structure, subspace, quotient, product and sum of monotone L-fuzzy topologies are also examined.

The following two theorems give rise to a functor from L-**G** to L-**FTop**.

Theorem 4.1. *Let $G : L^X \to L$ be an L-G-filter on X. Then $\mu(G) : L^X \to L$ defined by $\mu(G)(0_X) = 1$ and $\mu(G)(A) = G(A)$ for all $A \in L^X$ such that $A \neq 0_X$ is an L-fuzzy topology on X.*

Proof.

(T1) By definition $\mu(G)(0_X) = 1$ and $\mu(G)(1_X) = G(1_X) = 1$.

(T2) For all $A, B \in L^X$ such that $A \odot B \neq 0_X$, it is clear that $A \neq 0_X$ and $B \neq 0_X$. Therefore, $\mu(G)(A \odot B) = G(A \odot B) \geq G(A) \odot G(B) = \mu(G)(A) \odot \mu(G)(B)$.

(T3) For an arbitrary family $\{A_i \in L^X; i \in I\}$ containing atleast one non zero element A,

$$\begin{aligned} \mu(G)(\bigvee_{i\in I} A_i) &= G(\bigvee_{i\in I} A_i) \\ &\geq G(A) \\ &= \mu(G)(A) \\ &\geq \bigwedge_{i\in I} \mu(G)(A_i). \end{aligned}$$

Theorem 4.2. *Let (X, G_1) and (Y, G_2) be L-G-filter spaces such that $f : (X, G_1) \to (Y, G_2)$ be an L-G-filter map. Then $f : (X, \mu(G_1)) \to (Y, \mu(G_2))$ is a continuous map.*

Proof. For all $B \in L^Y$ such that $f^{\leftarrow}(B) \neq 0_X$,

$$\begin{aligned} \mu(G_1)(f^{\leftarrow}(B)) &= G_1(f^{\leftarrow}(B)) \\ &\geq G_2(B) \\ &= \mu(G_2)(B). \end{aligned}$$

Corollary 4.3. *μ : L-**G** to L-**FTop** is a functor.*

Proof. Proof follows from Theorem 4.1 and Theorem 4.2.

Theorem 4.1 above motivates the following notion of monotone L-fuzzy topological spaces.

Definition 4.4. *An L-fuzzy topology τ satisfying $\tau(A) \leq \tau(B)$ for all non zero $A, B \in L^X$ such that $A \leq B$ is called monotone L-fuzzy topology.*

Remark 4.5. *If G is an L-G-filter, then $\mu(G)$ is a monotone L-fuzzy topology.*

Notation 4.6. *Let L-**FTop(X)** denotes the lattice of set of all L-fuzzy topologies on a set X and **M**-L-**FTop(X)** denotes the set of all monotone L-fuzzy topologies on a set X.*

Theorem 4.7. ***M**-L-**FTop(X)** is a complete sublattice of L-**FTop(X)**.*

Proof. It is easy to observe that arbitrary meet of a subfamily family of monotone L-fuzzy topologies $\{\tau_j; j \in J\}$ on a set X defined by $\tau(A) = \bigwedge_{j \in J} \tau_j(A)$ for all $A \in L^X$ is a monotone L-fuzzy topology on X. $\tau : L^X \to L$ defined by $\tau(A) = 1$ for all $A \in L^X$ is the greatest element in L-**FTop(X)** and it is monotone. Therefore **M**-L-**FTop(X)** is a complete sublattice of L-**FTop(X)**.

Theorem 4.8. *Let (X, τ) be a monotone LM-fuzzy topological space and $Y \subseteq X$. Then the subspace, $(Y, \tau|_Y)$ is a monotone LM-fuzzy topological space.*

Proof. Let $A, B \in L^Y$ such that $A \leq B$ and $A \neq 0_Y$. Consider $C \in L^X$ such that $C|_Y = A$. Define $D \in L^X$ by $D(x) = C(x)$ if $x \in X \setminus Y$ and $D(x) = B(x)$ if $x \in Y$. Clearly $D|_Y = B$ and $C \leq D$. Hence $\tau(C) \leq \tau(D)$ so that $(\tau|_Y)(A) \leq (\tau|_Y)(B)$. Therefore, $\tau|_Y$ a monotone L-fuzzy topology on Y.

Theorem 4.9. *Let (X, τ) be a monotone L-fuzzy topological space and $f : X \to Y$ be a surjective mapping. Then the quotient L-fuzzy topology, τ/f is a monotone L-fuzzy topology on Y.*

Proof. Let $B_1, B_2 \in L^Y$ such that $B_1 \leq B_2$ and $B_1 \neq 0_Y$. Since $B_1 \leq B_2$,

$f^{\leftarrow}(B_1) \leq f^{\leftarrow}(B_2)$. As $B_1 \neq 0_Y$, there exists $y \in Y$ such that $B_1(y) \neq 0$. Since $f : X \to Y$ is surjective, there exists $x \in X$ such that $f(x) = y$. Thus $f^{\leftarrow}(B_1)(x) = B_1(y) \neq 0$. As a result, $f^{\leftarrow}(B_1) \neq 0_Y$ whenever $B_1 \neq 0_Y$. Therefore, $(\tau/f)(B_1) = \tau(f^{\leftarrow}(B_1)) \leq \tau(f^{\leftarrow}(B_2)) = (\tau/f)(B_2)$. Hence τ/f is a monotone L-fuzzy topology on Y.

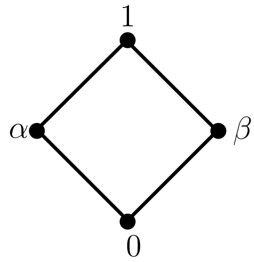

Figure 1: The diamond type lattice

Remark 4.10. *Let $\{(X_j, \tau_j)\}_{j\in J}$ be a family of monotone L-fuzzy topological spaces, X_j's be pairwise disjoint and $X = \bigcup_{j\in J} X_j$. Then the sum L-fuzzy topology, $\bigoplus_{j\in J} \tau_j$ need not be a monotone L-fuzzy topological space. For example, let $X_1 = \{1,2,3\}, X_2 = \{4,5,6,7\}$ and L be the lattice the shown in Figure 1. Let $A_1 \in L^{X_1}$ be defined by $A_1(1) = A_1(2) = 1$ and $A_1(3) = 0$ and $B_1 \in L^{X_2}$ be defined by $B_1(4) = B_1(5) = 1$ and $B_1(6) = B_1(7) = 0$. τ_1, $\tau_2 : L^X \to L$ defined by*

$$\tau_1(A) = \begin{cases} 1 & \text{if } A = 0_{X_1} \text{ or } 1_{X_1}, \\ \alpha & \text{if } A \geq A_1 \text{ and } A \neq 1_{X_1}, \\ 0 & \text{otherwise.} \end{cases} \qquad \tau_2(B) = \begin{cases} 1 & \text{if } B = 0_{X_2} \text{ or } 1_{X_2}, \\ \beta & \text{if } B \geq B_1 \text{ and } B \neq 1_{X_2}, \\ 0 & \text{otherwise.} \end{cases}$$

are monotone L-fuzzy topologies on X_1 and X_2 respectively. Let $X = X_1 \bigcup X_2$ and τ be the sum L-fuzzy topology on X. Then $\tau(\{4,5\}) = \tau_1(\phi) \wedge \tau_2(\{4,5\}) = \beta$ and $\tau(\{1,2,4,5\}) = \tau_1(\{1,2\}) \wedge \tau_2(\{4,5\}) = \alpha \wedge \beta = 0_X$. It is clear that τ is not monotone. Therefore, the sum of monotone L-fuzzy topologies need not be monotone.

Remark 4.11. *Let $\{(X_j, \tau_j)\}_{j\in J}$ be a family of monotone L-fuzzy topological spaces and $X = \prod_{j\in J} X_j$. Then the product L-fuzzy topology, $\prod_{j\in J} \tau_j$ need not be a monotone L-fuzzy topology on X. For example, let $X = \{x_1, x_2\}, Y = \{y_1, y_2\}$ and $L = \{0,1\}$. Then $\tau_1 = \{\phi, \{x_1\}, \{x_1, x_2\}\}$ and $\tau_2 = \{\phi, \{y_1\}, \{y_1, y_2\}\}$ are monotone L-fuzzy topologies on X and Y respectively. Then the product topology τ*

on $X \times Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$ *is defined by* $\tau = \{\phi, \{(x_1, y_1)\}, \{(x_1, y_1), (x_1, y_2)\}, \{(x_1, y_1), (x_1, y_2), (x_2, y_1)\}, \{(x_1, y_1), (x_2, y_1)\}, X \times Y\}$. *Thus* $\{(x_1, y_1)\} \in \tau$ *but* $\{(x_1, y_1), (x_2, y_2)\} \notin \tau$. *Therefore, product of monotone L-fuzzy topologies need not be monotone.*

5. Conclusion

The study has identified three different functors from the category of L-G-filter spaces to the category of L-fuzzy topological spaces. Having introduced the concept of monotone L-fuzzy topological spaces, certain properties of the same have been investigated. It is observed that the quotient and subspace of monotone L-fuzzy topological spaces are again monotone whereas sum and product need not be so.

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