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# UNICITY THEOREMS CONCERNING A L-FUNCTION AND A MEROMORPHIC FUNCTION

**Harina P. Waghamore and Preetham N. Raj**

Department of Mathematics,
Jnanabharathi Campus, Bangalore University,
Bengaluru - 560056, Karnataka, INDIA

E-mail : harinapw@gmail.com, preethamnraj@gmail.com

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**Abstract:** Inspired by a lot of studies on the uniqueness results of a $L$-function with a meromorphic function, in this article, we examine the uniqueness of two differential polynomials, one generated by a meromorphic function with finitely many poles and another by a $L$-function, when they share two values with some weight. The results of our examination extend, generalize as well as improve the results of Hao and Chen [2, 3].

**Keywords and Phrases:** L-function, linear differential polynomial, sharing values, finite weight, Nevanlinna theory.

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## 1. Introduction

Let $\mathbb{C}$ represent the complex plane, $\mathbb{N}$ represent the set of natural numbers, $\mathbb{W} = \mathbb{N} \cup \{0\}$, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\underline{\mathbb{C}} = \mathbb{C} \backslash \{0\}$.

We assume that the readers are well aware of the standard notations and definitions used in the Nevanlinna value distribution theory such as $T(r, \mathsf{f})$, $m(r, \mathsf{f})$, $N(r, \mathsf{f})$, $\overline{N}(r, \mathsf{f})$ etc. The reader can refer ([4], [15], [16]) for basics of Nevanlinna theory.

Let $\mathscr{F} = \{\mathsf{f}(z) | \mathsf{f}(z)$ is a non-constant meromorphic function in $\mathbb{C}\}$ and let $\mathscr{G} = \{\mathsf{g}(z) | \mathsf{g}(z)$ is a non-constant meromorphic function $\mathbb{C}$ with finitely many poles$\}$. For $\mathsf{f}_1, \mathsf{f}_2 \in \mathscr{F}$ and $b \in \overline{\mathbb{C}}$, if $\mathsf{f}_1 - b$ and $\mathsf{f}_2 - b$ have identical zeros taking into account the multiplicities then we say, $\mathsf{f}_1(z)$ and $\mathsf{f}_2(z)$ share $b$ CM (counting multiplicities), if multiplicities are not taken into account then we say $\mathsf{f}_1(z)$ and $\mathsf{f}_2(z)$ share $b$ IM (ignoring multiplicities).

In general for $\mathsf{f} \in \mathscr{F}$, $m(r, \mathsf{f})$ denotes the proximity function of $\mathsf{f}$, $N(r, \mathsf{f})$ denotes the counting function of poles of $\mathsf{f}(z)$, whose multiplicities are taken into account (respectively $\overline{N}(r, \mathsf{f})$ denotes the reduced counting function when multiplicities are ignored). $N(r, b; \mathsf{f})$ $\left(\text{notation inter-changable with } N\left(r, \frac{1}{\mathsf{f}-b}\right)\right)$ denotes the counting function of $b$-points of $\mathsf{f}(z)$, whose multiplicities are taken into account (respectively $\overline{N}(r, b; \mathsf{f})$ denotes the reduced counting function when multiplicities are neglected). $T(r, \mathsf{f})$ represents the characteristic function of $\mathsf{f}$. $S(r, \mathsf{f})$ denotes any quantity satisfying $S(r, \mathsf{f}) = o(T(r, \mathsf{f}))$ as $r \to \infty$ outside a possible exceptional set of finite linear measure. A meromorphic function $\eta(z)$ is said to be a small function of $\mathsf{f}$, if $T(r, \eta) = S(r, \mathsf{f})$.

Below we give some definitions which are required for our paper.

**Definition 1.1.** [5, 6] *Let $\mathsf{f}_1, \mathsf{f}_2 \in \mathscr{F}$ and $p \in \mathbb{W} \cup \{\infty\}$. For $b \in \overline{\mathbb{C}}$, we represent by $\mathbb{E}_p(b; \mathsf{f}_1)$ the set of all zeros of $\mathsf{f}_1 - b$ where a zero of multiplicity $s$ is counted $s$ times if $s \leq p$ and $p + 1$ times if $s > p$. If $\mathbb{E}_p(b; \mathsf{f}_1) = \mathbb{E}_p(b; \mathsf{f}_2)$, we say that $\mathsf{f}_1$, $\mathsf{f}_2$ share $b$ with weight $p$.*

**Definition 1.2.** [5] *Let $q \in \mathbb{W}$ and $\mathsf{f}_1 \in \mathscr{F}$, then we set $N_q(r, b; \mathsf{f}_1)$ as the counting function of $b$-points of $\mathsf{f}_1$, where any $m$ multiplicity $b$-point of $\mathsf{f}_1$ is counted $m$ times if $m \leq q$ and $q$ times if $m > q$.*

**Definition 1.3.** [5] *Suppose $\mathsf{f}_1$ and $\mathsf{f}_2$ share the value $b$ IM. Then we set $\overline{N}_*(r, b; \mathsf{f}_1, \mathsf{f}_2)$ as the reduced counting function of those $b$-points of $\mathsf{f}_1$ whose multiplicities differ from the multiplicities of the corresponding $b$-points of $\mathsf{f}_2$.*

In modern number theory, $L$-functions play a very important role. The value distributions of the $L$-functions provides valuable insights into the algebraic structure that is not available through the use of the elementary algebraic techniques. Specifically, the distribution of zeros of $L$-functions holds particular significance for numerous multiplicative number theory problems. One illustration is the Riemann hypothesis in the right half of the critical strip for a non-vanishing Riemann zeta-function and its impact on the distribution of prime numbers.

One feature all $L$-functions have in common is that they can be described by an Euler product. Therefore all $L$-function can be described as a product taken over prime numbers. Taking into account unique prime factorization of integers we can express $L$-functions as Dirichlet series. We may regard the well-known Riemann zeta-function as the prototype, i.e., $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_p \left(1 - \frac{1}{p^z}\right)^{-1}$, where $z = x + iy$, $x > 1$ and $p$ stands for a prime integer and the product is taken over all prime numbers.

Towards the end of twentieth century, in an effort to summarize the core properties of classical $L$-functions, Selberg [12] gave an axiomatic characterization of what would be called general $L$-functions. A $L$-function $\mathcal{L}$ means a Selberg class function with the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ as the prototype and the Selberg class $\mathcal{S}$ of $L$-function is defined as follows:

**Definition 1.4.** [12] *The Selberg class $\mathcal{S}$ consists of the functions $\mathcal{L}$ satisfying the following axioms:*

1. *(Dirichlet series) $\mathcal{L}(z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^z}$, absolutely convergent for $\sigma > 1$.*
2. *(Analytic continuation) There exists an integer $m$ such that $(z-1)^m \mathcal{L}(z)$ is an entire function of finite order.*
3. *(Functional equation) There exist an integer $r \geq 0$, positive real numbers $Q, \lambda_j$, complex numbers $\mu_j$ with $Re\, \mu_j \geq 0$ and $\omega$ with $|\omega| = 1$, such that the function $\Lambda(z)$ defined by*

$$\Lambda(z) = Q^z \prod_{j=1}^{r} \Gamma(\lambda_j z + \mu_j) \mathcal{L}(z) = \gamma(z) \mathcal{L}(z),$$

   *satisfies the functional equation $\Lambda(z) = \omega \overline{\Lambda}(1-z)$. We would call the function $\gamma(z)$ the $\gamma$-factor.*

4. *(Ramanujan conjecture) For every $\epsilon > 0$, $a(n) = O(n^{\epsilon})$.*
5. *(Euler product) $a(1) = 1$, and $\log \mathcal{L}(z) = \sum_{n \geq 1} \frac{b(n)}{n^z}$, where $b(n) = 0$ unless $n$ is a prime power, and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.*

By the comment on the order of a function, we can choose $m$ in axiom (2) to be the order of the pole of $\mathcal{L}$ at $z = 1$.

Since $L$-functions are analytically continued as meromorphic functions, we can study the value distribution and uniqueness results for a $L$-function, similar to any arbitrary meromorphic function using the Nevanlinna value distribution theory and the fact that $L$-functions has the only pole at $z = 1$ helps us in this.

So, by utilizing this fact in 2017, Liu et. al [8] studied the uniqueness results of a differential polynomial of a $L$-function with the same of a meromorphic function sharing a non-zero finite value and obtained the following result.

**Theorem A.** [8] *Let* $\mathsf{f} \in \mathscr{F}$ *and* $\mathcal{L}$ *be a L-function. Let* $n, l \in \mathbb{N}$ *such that* $n > 3l+6$. *Suppose that* $[\mathsf{f}^n]^{(l)}$ *and* $[\mathcal{L}^n]^{(l)}$ *share* 1 *CM, then* $\mathsf{f} \equiv t\mathcal{L}$ *for a constant* $t$ *satisfying* $t^n = 1$.

In 2018, Hao and Chen [2] generalized the differential monomial of Liu et. al [8], as well as reduced the sharing value from CM to IM and obtained the following results.

**Theorem B.** [2] *Let* $\mathsf{f} \in \mathscr{G}$ *and* $\mathcal{L}$ *be a L-function. Let* $n, m, l \in \mathbb{N}$. *Suppose that* $[\mathsf{f}^n(\mathsf{f}-1)^m]^{(l)}$ *and* $[\mathcal{L}^n(\mathcal{L}-1)^m]^{(l)}$ *share* 1 *IM. If* $n > 7l + 4m + 11$, $l \geq 2$, *then* $\mathsf{f} \equiv \mathcal{L}$ *or* $\mathsf{f}^n(\mathsf{f}-1)^m \equiv \mathcal{L}^n(\mathcal{L}-1)^m$.

**Theorem C.** [2] *Let* $\mathsf{f} \in \mathscr{G}$ *and* $\mathcal{L}$ *be a L-function. Let* $n, l \in \mathbb{N}$. *Suppose that* $[\mathsf{f}^n]^{(l)}$ *and* $[\mathcal{L}^n]^{(l)}$ *share* 1 *IM. If* $n > 7l + 11$, *then* $\mathsf{f} \equiv t\mathcal{L}$ *for a constant* $t$ *satisfying* $t^n = 1$.

Once again in 2018, Hao and Chen [3], studied the uniqueness result of a $L$-function with an arbitrary meromorphic function as follows.

**Theorem D.** [3] *Let* $\mathcal{L}$ *be a L-function and* $\mathsf{f} \in \mathscr{G}$. *Let* $\alpha_1, \alpha_2 \in \mathbb{C}$ *and* $\kappa_1, \kappa_2 \in \mathbb{N}$ *satisfying* $\kappa_1\kappa_2 > 1$. *If* $\mathbb{E}_{\kappa_i}(\alpha_i, \mathcal{L}) = \mathbb{E}_{\kappa_i}(\alpha_i, \mathsf{f})$, *for* $i = 1, 2$, *then* $\mathcal{L} \equiv \mathsf{f}$.
The main motivation to this paper are the following questions,

(i) whether we can reduce the condition for $n$ in Theorem B and C?

(ii) whether we can reduce the weights of sharing $\kappa_1, \kappa_2$ in Theorem D?

(iii) whether a similar uniqueness result holds when we consider, an additional linear differential polynomial along with the polynomial of $\mathcal{L}$ as defined below?

**Definition 1.5.** *Let* $\mathsf{f} \in \mathscr{F}$. *Then we define its linear differential polynomial* $d_\kappa[\mathsf{f}]$ *as*

$$d_\kappa[\mathsf{f}] = a_0\mathsf{f} + a_1\mathsf{f}' + a_2\mathsf{f}'' + \cdots + a_\kappa\mathsf{f}^{(\kappa)}, \tag{1.1}$$

*where* $a_0, a_1, a_2, ..., a_{\kappa-1}$ *and* $a_\kappa \neq 0$ *are complex constants.*

## 2. Main Results

As a positive answer to the above questions, we give the following results.

**Theorem 2.1.** *Let* $\mathcal{L}$ *be a L-function and* $\mathsf{f} \in \mathscr{G}$. *Let* $\mathsf{F}^* = (\mathsf{f}^n(\mathsf{f}-1)^m d_\kappa[\mathsf{f}])^{(l)}$ *and* $\mathcal{L}^* = (\mathcal{L}^n(\mathcal{L}-1)^m d_\kappa[\mathcal{L}])^{(l)}$. *Let* $a_1 = 1, a_2 = \infty$ *and* $\kappa_1, \kappa_2 \in \mathbb{W} \cup \{\infty\}$. *If* $\mathbb{E}_{\kappa_1}(a_1, \mathcal{L}^*) = \mathbb{E}_{\kappa_1}(a_1, \mathsf{F}^*)$ *and* $\mathbb{E}_{\kappa_2}(a_2, \mathcal{L}^*) = \mathbb{E}_{\kappa_2}(a_2, \mathsf{F}^*)$, *such that*

*(i)* $n > \kappa + 2l + m + 5$, *when* $\kappa_1 \geq 2$ *and* $0 \leq \kappa_2 \leq \infty$ *or,*

*(ii)* $n > \kappa + \frac{5l+3m+13}{2}$, *when* $\kappa_1 = 1$ *and* $\kappa_2 = 0$ *or,*

*(iii)* $n > \kappa + 5l + 4m + 11$, *when* $\kappa_1 = 0$ *and* $\kappa_2 = 0$,

*then we have* $\mathsf{f}^n(\mathsf{f}-1)^m d_\kappa[\mathsf{f}] \equiv \mathcal{L}^n(\mathcal{L}-1)^m d_\kappa[\mathcal{L}]$. *Further, if* $\kappa = 0$, *then we have one of the following conclusions*

*(i)* $\mathsf{f} \equiv t\mathcal{L}$, *for a constant* $t$ *satisfying* $t^d = 1$, *where* $d = GCD(n+m+1, n+m, n+m-1, ..., n+1)$.

*(ii)* $\mathsf{f}$ *and* $\mathcal{L}$ *satisfy the algebraic equation* $R(\mathsf{f}, \mathcal{L}) = 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^{n+1}(z)(\omega_1(z)-1)^m - \omega_2^{n+1}(z)(\omega_2(z)-1)^m.$$

**Example.** Let $\mathsf{f} = \frac{1}{z-1} + 1$ and $\mathcal{L} = \sum_{n=1}^{\infty} \frac{1}{n^z} + 1$. Then it is easy to see that $\mathsf{f}$ and $\mathcal{L}$ share the values 1 and $\infty$ CM. Suppose $l = 0$, $m = 0$ and $\kappa = 0$, then again $\mathsf{F}^*$ and $\mathcal{L}^*$ shares 1 and $\infty$ CM, but none of the conclusions of Theorem 2.1 holds, which shows that the conditions given in the theorem are necessary, but not sufficient.

**Corollary 2.1.** *Let* $\mathcal{L}$ *be a L-function and* $\mathsf{f} \in \mathscr{G}$. *Let* $\mathsf{F}^* = (\mathsf{f}^n d_\kappa[\mathsf{f}])^{(l)}$ *and* $\mathcal{L}^* = (\mathcal{L}^n d_\kappa[\mathcal{L}])^{(l)}$. *Let* $a_1 = 1, a_2 = \infty$ *and* $\kappa_1, \kappa_2 \in \mathbb{W} \cup \{\infty\}$. *If* $\mathbb{E}_{\kappa_1}(a_1, \mathcal{L}^*) = \mathbb{E}_{\kappa_1}(a_1, \mathsf{F}^*)$ *and* $\mathbb{E}_{\kappa_2}(a_2, \mathcal{L}^*) = \mathbb{E}_{\kappa_2}(a_2, \mathsf{F}^*)$, *such that*

*(i)* $n > \kappa + 2l + 5$, *when* $\kappa_1 \geq 2$ *and* $0 \leq \kappa_2 \leq \infty$ *or,*

*(ii)* $n > \kappa + \frac{5l+13}{2}$, *when* $\kappa_1 = 1$ *and* $\kappa_2 = 0$ *or,*

*(iii)* $n > \kappa + 5l + 11$, *when* $\kappa_1 = 0$ *and* $\kappa_2 = 0$,

*then we have* $\mathsf{f}^n d_\kappa[\mathsf{f}] \equiv \mathcal{L}^n d_\kappa[\mathcal{L}]$. *Further, if* $\kappa = 0$, *then we have* $\mathsf{f} \equiv s\mathcal{L}$, *for a constant* $s$ *satisfying* $s^{n+1} = 1$.

**Theorem 2.2.** *Let* $\mathcal{L}$ *be a L-function and* $\mathsf{f} \in \mathscr{G}$. *Let* $\mathsf{F}^* = (\mathsf{f}^n(\mathsf{f}-1)^m)^{(l)}$ *and* $\mathcal{L}^* = (\mathcal{L}^n(\mathcal{L}-1)^m)^{(l)}$. *Let* $a_1 = 1, a_2 = \infty$ *and* $\kappa_1, \kappa_2 \in \mathbb{W} \cup \{\infty\}$. *If* $\mathbb{E}_{\kappa_1}(a_1, \mathcal{L}^*) = \mathbb{E}_{\kappa_1}(a_1, \mathsf{F}^*)$ *and* $\mathbb{E}_{\kappa_2}(a_2, \mathcal{L}^*) = \mathbb{E}_{\kappa_2}(a_2, \mathsf{F}^*)$, *such that*

*(i)* $n > 2l + m + 4$, *when* $\kappa_1 \geq 2$ *and* $0 \leq \kappa_2 \leq \infty$ *or,*

*(ii)* $n > \frac{5l+3m+9}{2}$, *when* $\kappa_1 = 1$ *and* $\kappa_2 = 0$ *or,*

*(iii)* $n > \kappa + 5l + 4m + 7$, *when* $\kappa_1 = 0$ *and* $\kappa_2 = 0$,

*then we have one of the following conclusions*

*(i)* $\mathsf{f} \equiv t\mathcal{L}$, *for a constant* $t$ *satisfying* $t^d = 1$, *where* $d = GCD(n+m, n+m-1, n+m-2, ..., n)$.

*(ii)* $\mathsf{f}$ *and* $\mathcal{L}$ *satisfy the algebraic equation* $R(\mathsf{f},\mathcal{L})=0$, *where*

$$R(\omega_1,\omega_2)=\omega_1^n(z)(\omega_1(z)-1)^m-\omega_2^n(z)(\omega_2(z)-1)^m.$$

**Corollary 2.2.** *Let* $\mathcal{L}$ *be a L-function and* $\mathsf{f}\in\mathscr{G}$. *Let* $a_1=1, a_2=\infty$ *and* $\kappa_1,\kappa_2\in\mathbb{W}\cup\{\infty\}$. *If* $\mathbb{E}_{\kappa_1}(a_1,[\mathcal{L}^n]^{(l)})=\mathbb{E}_{\kappa_1}(a_1,[\mathsf{f}^n]^{(l)})$ *and* $\mathbb{E}_{\kappa_2}(a_2,[\mathcal{L}^n]^{(l)})=\mathbb{E}_{\kappa_2}(a_2[\mathsf{f}^n]^{(l)})$, *such that*

*(i)* $n>2l+4$, *when* $\kappa_1\geq 2$ *and* $0\leq\kappa_2\leq\infty$ *or,*

*(ii)* $n>\frac{5l+9}{2}$, *when* $\kappa_1=1$ *and* $\kappa_2=0$ *or,*

*(iii)* $n>\kappa+5l+7$, *when* $\kappa_1=0$ *and* $\kappa_2=0$,

*then we have one* $\mathsf{f}\equiv t\mathcal{L}$, *for a constant* $t$ *satisfying* $t^n=1$.

**Remarks.** *By additionally considering that the differential polynomials of* $\mathsf{f}$ *and* $\mathcal{L}$ *share* $\infty$ *with weight* $\kappa_2$ *we have obtained Theorems 2.1, 2.2 and Corollaries 2.1, 2.2. We can see that Theorem 2.2 and Corollary 2.2 are improvements of the Theorem B and C respectively, where as Theorem 2.1 and Corollary 2.1 are generalization as well as improvements of Theorem D.*

## 3. Lemmas

Here we provide all the lemmas which we will be using. Let $\mathscr{F}=\{\mathsf{f}(z)|\mathsf{f}(z)$ is a non-constant meromorphic function in $\mathbb{C}\}$. For any $\mathsf{f}_1,\mathsf{f}_2\in\mathscr{F}$, let $\Omega$ be defined as:

$$\Omega\equiv\left(\frac{\mathsf{f}_1''}{\mathsf{f}_1'}-\frac{2\mathsf{f}_1'}{\mathsf{f}_1-1}\right)-\left(\frac{\mathsf{f}_2''}{\mathsf{f}_2'}-\frac{2\mathsf{f}_2'}{\mathsf{f}_2-1}\right). \tag{3.1}$$

**Lemma 3.1.** [15] *Let* $\mathsf{f}\in\mathscr{F}$ *and* $n\in\mathbb{N}$. *Let* $\mathscr{P}_n(\mathsf{f})=a_n\mathsf{f}^n+a_{n-1}\mathsf{f}^{n-1}+\cdots+a_1\mathsf{f}$, *where* $a_\kappa$ *for* $\kappa=1,2,...,n$ *are meromorphic functions such that* $T(r,a_\kappa)=S(r,\mathsf{f})$ *for* $\kappa=1,2,...,n$ *and* $a_\kappa\not\equiv 0$. *Then*

$$T(r,\mathscr{P}_n(\mathsf{f}))=nT(r,\mathsf{f})+S(r,\mathsf{f}).$$

**Lemma 3.2.** [17] *Let* $\mathsf{f}_1,\mathsf{f}_2\in\mathscr{F}$ *and* $a(z)(\not\equiv 0,\infty)$ *be a small function of* $\mathsf{f}_1$ *and* $\mathsf{f}_2$. *Suppose* $\mathsf{f}_1$ *and* $\mathsf{f}_2$ *share* $a(z)$ *IM, then one of the following three cases holds:*

*(i)*

$$\begin{aligned}T(r,\mathsf{f}_1)\leq{}& N_2(r,0;\mathsf{f}_1)+N_2(r,\infty;\mathsf{f}_1)+N_2(r,0;\mathsf{f}_2)+N_2(r,\infty;\mathsf{f}_2)+2(\overline{N}(r,0;\mathsf{f}_1)\\&+\overline{N}(r,\infty;\mathsf{f}_1))+\overline{N}(r,0;\mathsf{f}_2)+\overline{N}(r,\infty;\mathsf{f}_2)+S(r,\mathsf{f}_1)+S(r,\mathsf{f}_2),\end{aligned}$$

*and a similar inequality holds for* $T(r,\mathsf{f}_2)$, *(ii)* $\mathsf{f}_1\mathsf{f}_2 \equiv 1$,

*(iii)* $\mathsf{f}_1 \equiv \mathsf{f}_2$.

**Lemma 3.3.** [7] *Let* $\mathsf{f} \in \mathscr{F}$ *and* $\kappa, q \in \mathbb{N}$. *Then*

$$N_q(r, 0; \mathsf{f}^{(\kappa)}) \leq T(r, \mathsf{f}^{(\kappa)}) - T(r, \mathsf{f}) + N_{\kappa+q}(r, 0; \mathsf{f}) + S(r, \mathsf{f}),$$

*and*

$$N_q(r, 0; \mathsf{f}^{(\kappa)}) \leq N_{\kappa+q}(r, 0; \mathsf{f}) + \kappa\overline{N}(r, \infty; \mathsf{f}) + S(r, \mathsf{f}).$$

**Lemma 3.4.** [1] *Let* $\mathsf{f}_1, \mathsf{f}_2 \in \mathscr{F}$. *Suppose* $\mathsf{f}_1$, $\mathsf{f}_2$ *share* $(1, 2)$ *and* $(\infty, \kappa)$, *where* $0 \leq \kappa \leq \infty$ *and* $\Omega \not\equiv 0$. *Then*

$$\begin{aligned} T(r, \mathsf{f}_1) \leq\ & N_2(r, 0; \mathsf{f}_1) + N_2(r, 0; \mathsf{f}_2) + \overline{N}(r, \infty; \mathsf{f}_1) + \overline{N}(r, \infty; \mathsf{f}_2) + \overline{N}_*(r, \infty; \mathsf{f}_1, \mathsf{f}_2) \\ & + S(r, \mathsf{f}_1) + S(r, \mathsf{f}_2). \end{aligned}$$

**Lemma 3.5.** [11] *Let* $\mathsf{f}_1, \mathsf{f}_2 \in \mathscr{F}$. *Suppose* $\mathsf{f}_1$, $\mathsf{f}_2$ *share* $(1, 1)$ *and* $(\infty, 0)$, *and* $\Omega \not\equiv 0$. *Then*

$$\text{(i)}\quad \begin{aligned} T(r, \mathsf{f}_1) \leq\ & N_2(r, 0; \mathsf{f}_1) + N_2(r, 0; \mathsf{f}_2) + \frac{3}{2}\overline{N}(r, \infty; \mathsf{f}_1) + \overline{N}(r, \infty; \mathsf{f}_2) + \frac{1}{2}\overline{N}(r, 0; \mathsf{f}_1) \\ & + \overline{N}_*(r, \infty; \mathsf{f}_1, \mathsf{f}_2) + S(r, \mathsf{f}_1) + S(r, \mathsf{f}_2); \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} T(r, \mathsf{f}_2) \leq\ & N_2(r, 0; \mathsf{f}_1) + N_2(r, 0; \mathsf{f}_2) + \overline{N}(r, \infty; \mathsf{f}_1) + \frac{3}{2}\overline{N}(r, \infty; \mathsf{f}_2) + \frac{1}{2}\overline{N}(r, 0; \mathsf{f}_2) \\ & + \overline{N}_*(r, \infty; \mathsf{f}_2, \mathsf{f}_1) + S(r, \mathsf{f}_1) + S(r, \mathsf{f}_2). \end{aligned}$$

**Lemma 3.6.** [11] *Let* $\mathsf{f}_1, \mathsf{f}_2 \in \mathscr{F}$. *Suppose* $\mathsf{f}_1$, $\mathsf{f}_2$ *share* $(1, 0)$ *and* $(\infty, 0)$, *and* $\Omega \not\equiv 0$. *Then*

$$\text{(i)}\quad \begin{aligned} T(r, \mathsf{f}_1) \leq\ & N_2(r, 0; \mathsf{f}_1) + N_2(r, 0; \mathsf{f}_2) + 3\overline{N}(r, \infty; \mathsf{f}_1) + 2\overline{N}(r, \infty; \mathsf{f}_2) + 2\overline{N}(r, 0; \mathsf{f}_1) \\ & + \overline{N}(r, 0; \mathsf{f}_2) + \overline{N}_*(r, \infty; \mathsf{f}_1, \mathsf{f}_2) + S(r, \mathsf{f}_1) + S(r, \mathsf{f}_2); \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} T(r, \mathsf{f}_2) \leq\ & N_2(r, 0; \mathsf{f}_1) + N_2(r, 0; \mathsf{f}_2) + 2\overline{N}(r, \infty; \mathsf{f}_1) + 3\overline{N}(r, \infty; \mathsf{f}_2) + \overline{N}(r, 0; \mathsf{f}_1) \\ & + 2\overline{N}(r, 0; \mathsf{f}_2) + \overline{N}_*(r, \infty; \mathsf{f}_2, \mathsf{f}_1) + S(r, \mathsf{f}_1) + S(r, \mathsf{f}_2). \end{aligned}$$

**Lemma 3.7.** [13] *Let* $\mathcal{L}$ *be a L-function with degree* $d$. *Then*

$$T(r, \mathcal{L}) = \frac{d}{\pi} r \log r + O(r).$$

**Lemma 3.8.** [9] *Let* $\mathcal{L}$ *be an L-function. Then* $N(r, \infty; \mathcal{L}) = S(r, \mathcal{L})$.

**Lemma 3.9.** [15] *Let* $\mathsf{f} \in \mathscr{F}$. *Then* $m(r, \infty; \mathsf{f}'/\mathsf{f}) = S(r, \mathsf{f})$. *Since, we can write* $\frac{\mathsf{f}^{(\kappa)}}{\mathsf{f}}$ *as* $\frac{\mathsf{f}^{(\kappa)}}{\mathsf{f}^{(\kappa-1)}} \cdot \frac{\mathsf{f}^{(\kappa-1)}}{\mathsf{f}^{(\kappa-2)}} \cdots \frac{\mathsf{f}'}{\mathsf{f}}$, *we get*

$$m\left(r, \infty; \frac{\mathsf{f}^{(\kappa)}}{\mathsf{f}}\right) = S(r, \mathsf{f}).$$

**Lemma 3.10.** *Suppose* $\mathsf{f} \in \mathscr{F}$ *and* $d_\kappa[\mathsf{f}]$ *is a linear differential polynomial of* $\mathsf{f}$. *Then*

$$N(r, \infty; d_\kappa[\mathsf{f}]) \leq N(r, \infty; \mathsf{f}) + \kappa \overline{N}(r, \infty; \mathsf{f}) + S(r, \mathsf{f}).$$

**Proof.** It is easy to see that, the poles of $d_\kappa[\mathsf{f}]$ occurs at the poles of $\mathsf{f}$. Suppose, $z_*$ is a pole of $\mathsf{f}$ of order $r$, then $z_*$ is a pole of $d_\kappa[\mathsf{f}]$ of order at most $r + \kappa$. Hence, we have

$$N(r, \infty; d_\kappa[\mathsf{f}]) \leq N(r, \infty; \mathsf{f}) + \kappa \overline{N}(r, \infty; \mathsf{f}) + S(r, \mathsf{f}).$$

Thus the proof.

**Lemma 3.11.** *Suppose* $\mathsf{f} \in \mathscr{F}$ *and* $d_\kappa[\mathsf{f}]$ *is a linear differential polynomial of* $\mathsf{f}$. *Then*

$$\begin{aligned} T(r, d_\kappa[\mathsf{f}]) &\leq T(r, \mathsf{f}) + \kappa \overline{N}(r, \infty; \mathsf{f}) + S(r, \mathsf{f}), \\ N(r, 0; d_\kappa[\mathsf{f}]) &\leq T(r, d_\kappa[\mathsf{f}]) - T(r, 1/\mathsf{f}) + N(r, 0; \mathsf{f}) + S(r, \mathsf{f}), \\ N(r, 0; d_\kappa[\mathsf{f}]) &\leq N(r, 0; \mathsf{f}) + \kappa \overline{N}(r, \mathsf{f}) + S(r, \mathsf{f}). \end{aligned}$$

**Proof.** From the Nevanlinna's Fundamental Theorem-I, we have

$$N(r, 0; d_\kappa[\mathsf{f}]) = T(r, d_\kappa[\mathsf{f}]) - m(r, 0; d_\kappa[\mathsf{f}]) + O(1). \tag{3.2}$$

Also, we have

$$m(r, 0; \mathsf{f}) \leq m\left(r, \infty; \frac{d_\kappa[\mathsf{f}]}{\mathsf{f}}\right) + m\left(r, 0; d_\kappa[\mathsf{f}]\right),$$

this implies that

$$m(r, 0; \mathsf{f}) \leq m(r, 0; d_\kappa[\mathsf{f}]) + S(r, \mathsf{f}),$$

which further implies that

$$-m(r, 0; d_\kappa[\mathsf{f}]) \leq -m(r, 0; \mathsf{f}) + S(r, \mathsf{f}). \tag{3.3}$$

Using (3.3) in (3.2), we get

$$\begin{aligned} N(r, 0; d_\kappa[\mathsf{f}]) &\leq T(r, d_\kappa[\mathsf{f}]) - m(r, 0; \mathsf{f}) + S(r, \mathsf{f}) \\ N(r, 0; d_\kappa[\mathsf{f}]) &\leq T(r, d_\kappa[\mathsf{f}]) - T(r, 1/\mathsf{f}) + N(r, 0; \mathsf{f}) + S(r, \mathsf{f}). \end{aligned} \tag{3.4}$$

Since,

$$\begin{aligned} T(r, d_\kappa[\mathsf{f}]) &= m(r,\infty; d_\kappa[\mathsf{f}]) + N(r,\infty; d_\kappa[\mathsf{f}]) \\ &\leq m\left(r,\infty; \frac{d_\kappa[\mathsf{f}]}{\mathsf{f}}\right) + m(r,\infty;\mathsf{f}) + N(r,\infty; d_\kappa[\mathsf{f}]) \\ &\leq m(r,\infty;\mathsf{f}) + N(r,\infty;\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}) \\ &\leq T(r,\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}). \end{aligned} \tag{3.5}$$

Substituting (3.5) in (3.4), we get

$$\begin{aligned} N(r,0;d_\kappa[\mathsf{f}]) &\leq T(r,\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) - T(r,1/\mathsf{f}) + N(r,0;\mathsf{f}) + S(r,\mathsf{f}) \\ &\leq N(r,0;\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}). \end{aligned} \tag{3.6}$$

Thus the proof.

**Lemma 3.12.** *Suppose* $\mathsf{f} \in \mathscr{F}$, $q \in \mathbb{N}$ *and* $d_\kappa[\mathsf{f}]$ *is a linear differential polynomial of* $\mathsf{f}$*. Then*

$$N_q(r,0;d_\kappa[\mathsf{f}]) \leq (\kappa+1)\left(q+\frac{\kappa}{2}\right)\overline{N}(r,0;\mathsf{f}) + \frac{\kappa(\kappa+1)}{2}\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}).$$

**Proof.** We know that the number of zeros of a polynomial are less than the sum of the number of zeros of the constituent monomials and hence

$$\begin{aligned} N_q(r,0;d_\kappa[\mathsf{f}]) &\leq \sum_{i=0}^{\kappa} N_q(r,0;\mathsf{f}^{(i)}) \\ &\leq \sum_{i=0}^{\kappa} \left[N_{q+i}(r,0;\mathsf{f}) + i\overline{N}(r,\infty;\mathsf{f})\right] + S(r,\mathsf{f}) \\ &\leq \sum_{i=0}^{\kappa}(q+i)\overline{N}(r,0;\mathsf{f}) + \sum_{i=0}^{\kappa} i\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}) \\ &\leq (\kappa+1)\left(q+\frac{\kappa}{2}\right)\overline{N}(r,0;\mathsf{f}) + \frac{\kappa(\kappa+1)}{2}\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}). \end{aligned}$$

Thus the proof.

**Lemma 3.13.** *Suppose* $\mathsf{f} \in \mathscr{F}$ *and* $d_\kappa[\mathsf{f}]$ *is a linear differential polynomial of* $\mathsf{f}$*. Then*

$$N\left(r,\infty;\frac{d_\kappa[\mathsf{f}]}{\mathsf{f}}\right) \leq \kappa\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}).$$

**Proof.** We know that,

$$\frac{d_\kappa[\mathsf{f}]}{\mathsf{f}} = \sum_{i=0}^{\kappa} \frac{\mathsf{f}^{(i)}}{\mathsf{f}}.$$

If $z_*$ is a pole of $\mathsf{f}$ of order $r$, then $z_*$ is a pole of $\frac{\mathsf{f}'}{\mathsf{f}}$ of order 1 and a pole of $\frac{\mathsf{f}''}{\mathsf{f}}$ of order 2 and so on. Hence $z_*$ is a pole of $\frac{\mathsf{f}^{(\kappa)}}{\mathsf{f}}$ of order $\kappa$. Thus,

$$N\left(r, \infty; \frac{d_\kappa[\mathsf{f}]}{\mathsf{f}}\right) \leq \kappa \overline{N}(r, \infty; \mathsf{f}) + S(r, \mathsf{f}).$$

Thus the proof.

**Lemma 3.14.** *Let* $\mathsf{f} \in \mathscr{F}$. *Let* $\mathsf{F}_1 = \mathsf{f}^n(\mathsf{f}-1)^m d_\kappa[\mathsf{f}]$, *where* $n, m (\geq 0)$ *are positive integers. Then*

$$(n+m+1-\kappa)T(r, \mathsf{f}) \leq T(r, \mathsf{F}_1) + S(r, \mathsf{f}).$$

**Proof.** From Lemmas 3.1, 3.9, 3.13 and the Nevanlinna's Fundamental Theorem -I, we have

$$\begin{aligned}
(n+m+1)T(r, \mathsf{f}) &= T(r, \mathsf{f}^n(\mathsf{f}-1)^m \mathsf{f}) + S(r, \mathsf{f}) \\
&\leq T\left(r, \frac{\mathsf{F}_1 \mathsf{f}}{d_\kappa[\mathsf{f}]}\right) + S(r, \mathsf{f}) \\
&\leq T(r, \mathsf{F}_1) + T\left(r, \frac{d_\kappa[\mathsf{f}]}{\mathsf{f}}\right) + S(r, \mathsf{f}) \\
&\leq T(r, \mathsf{F}_1) + N\left(r, \infty; \frac{d_\kappa[\mathsf{f}]}{\mathsf{f}}\right) + m\left(r, \infty; \frac{d_\kappa[\mathsf{f}]}{\mathsf{f}}\right) + S(r, \mathsf{f}) \\
&\leq T(r, \mathsf{F}_1) + \kappa \overline{N}(r, \infty; \mathsf{f}) + S(r, \mathsf{f}).
\end{aligned}$$

Thus, $(n+m+1-\kappa)T(r, \mathsf{f}) \leq T(r, \mathsf{F}_1) + S(r, \mathsf{f})$.

## 4. Proof of Theorems

### 4.1. Proof of Theorem 2.1.

Let,

$$\mathsf{F}_1 = \mathsf{f}^n(\mathsf{f}-1)^m d_\kappa[\mathsf{f}] \quad \text{and} \quad \mathcal{L}_1 = \mathcal{L}^n(\mathcal{L}-1)^m d_\kappa[\mathcal{L}],$$

$$\mathsf{F}^* = \mathsf{F}_1^{(l)} \quad \text{and} \quad \mathcal{L}^* = \mathcal{L}_1^{(l)},$$

$$\Omega = \left(\frac{\mathsf{F}^{*\prime\prime}}{\mathsf{F}^{*\prime}} - \frac{2\mathsf{F}^{*\prime}}{\mathsf{F}^* - 1}\right) - \left(\frac{\mathcal{L}^{*\prime\prime}}{\mathcal{L}^{*\prime}} - \frac{2\mathcal{L}^{*\prime}}{\mathcal{L}^* - 1}\right). \tag{4.1}$$

From the hypothesis we have $\mathsf{F}^*$ and $\mathcal{L}^*$ share $(1, \kappa_1)$ and also share $(\infty, \kappa_2)$. We now discuss the following two cases.

**Case 1.** We assume that $\Omega \not\equiv 0$. Now we consider the following three subcases.

**Subcase 1.1.** Suppose that $\kappa_1 \geq 2$ and $0 \leq \kappa_2 \leq \infty$, then using Lemmas 3.3, 3.4, 3.8, 3.11 and 3.14 we obtain

$$\begin{aligned} T(r,\mathsf{F}^*) &\leq N_2(r,0;\mathsf{F}^*) + N_2(r,0;\mathcal{L}^*) + \overline{N}(r,\infty;\mathsf{F}^*) + \overline{N}(r,\infty;\mathcal{L}^*) + \overline{N}_*(r,\infty;\mathsf{F}^*,\mathcal{L}^*) \\ &\quad + S(r,\mathsf{F}^*) + S(r,\mathcal{L}^*) \\ &\leq T(r,\mathsf{F}^*) - T(r,\mathsf{F}_1) + N_{l+2}(r,0;\mathsf{F}_1) + l\overline{N}(r,\infty;\mathcal{L}_1) + N_{l+2}(r,0;\mathcal{L}_1) \\ &\quad + S(r,\mathsf{f}) + S(r,\mathcal{L}). \end{aligned}$$

This implies,

$$\begin{aligned} (n+m+1-\kappa)T(r,\mathsf{f}) &\leq (l+2)\overline{N}(r,0;\mathsf{f}) + mN(r,0;\mathsf{f}) + N(r,0;\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) \\ &\quad + (l+2)\overline{N}(r,0;\mathcal{L}) + mN(r,0;\mathcal{L}) + N(r,0;\mathcal{L}) \\ &\quad + \kappa\overline{N}(r,\infty;\mathcal{L}) + S(r,\mathsf{f}) + S(r,\mathcal{L}). \end{aligned} \tag{4.2}$$

By similar calculations, we get

$$\begin{aligned} (n+m+1-\kappa)T(r,\mathcal{L}) &\leq (l+2)\overline{N}(r,0;\mathcal{L}) + mN(r,0;\mathcal{L}) + N(r,0;\mathcal{L}) \\ &\quad + \kappa\overline{N}(r,\infty;\mathcal{L}) + (l+2)\overline{N}(r,0;\mathsf{f}) + mN(r,0;\mathsf{f}) \\ &\quad + N(r,0;\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}) + S(r,\mathcal{L}). \end{aligned} \tag{4.3}$$

Now, from combining the inequalities (4.2) and (4.3), we get

$$\begin{aligned} (n+m+1-\kappa)[T(r,\mathsf{f}) + T(r,\mathcal{L})] &\leq 2(l+2+m+1)[N(r,0;\mathsf{f}) + N(r,0;\mathcal{L})] \\ &\quad + S(r,\mathsf{f}) + S(r,\mathcal{L}), \end{aligned}$$

which contradicts $n > \kappa + 2l + m + 5$.

**Subcase 1.2.** Suppose $\kappa_1 = 1$ and $\kappa_2 = 0$, then using Lemmas 3.3, 3.5, 3.8, 3.11 and 3.14 we obtain

$$\begin{aligned} T(r,\mathsf{F}^*) &\leq N_2(r,0;\mathsf{F}^*) + N_2(r,0;\mathcal{L}^*) + \frac{3}{2}\overline{N}(r,\infty;\mathsf{F}^*) + \overline{N}(r,\infty;\mathcal{L}^*) \\ &\quad + \overline{N}_*(r,\infty;\mathsf{F}^*,\mathcal{L}^*) + \frac{1}{2}\overline{N}(r,0;\mathsf{F}^*) + S(r,\mathsf{F}^*) + S(r,\mathcal{L}^*) \\ &\leq T(r,\mathsf{F}^*) - T(r,\mathsf{F}_1) + N_{l+2}(r,0;\mathsf{F}_1) + N_{l+2}(r,0;\mathcal{L}_1) + l\overline{N}(r,\infty;\mathcal{L}_1) \\ &\quad + \frac{1}{2}[N_{l+1}(r,0;\mathsf{F}_1) + lN(r,\infty;\mathsf{F}_1)] + S(r,\mathsf{f}) + S(r,\mathcal{L}), \end{aligned}$$

which implies that

$$\begin{aligned}(n+m+1-\kappa)T(r,\mathsf{f}) \leq{}& (l+2)\overline{N}(r,0;\mathsf{f})+mN(r,0;\mathsf{f})+N(r,0;\mathsf{f})+\kappa\overline{N}(r,\infty;\mathsf{f})\\ &+(l+2)\overline{N}(r,0;\mathcal{L})+mN(r,0;\mathcal{L})+N(r,0;\mathcal{L})+\kappa\overline{N}(r,\infty;\mathcal{L})\\ &+\frac{1}{2}(l+1)\overline{N}(r,0;\mathsf{f})+\frac{m}{2}N(r,0;\mathsf{f})+N(r,0;\mathsf{f})+\kappa\overline{N}(r,\infty;\mathsf{f})\\ &+S(r,\mathsf{f})+S(r,\mathcal{L}). \end{aligned} \tag{4.4}$$

By similar calculations, we get

$$\begin{aligned}(n+m+1-\kappa)T(r,\mathcal{L}) \leq{}& (l+2)\overline{N}(r,0;\mathsf{f})+mN(r,0;\mathsf{f})+N(r,0;\mathsf{f})+\kappa\overline{N}(r,\infty;\mathsf{f})\\ &+(l+2)\overline{N}(r,0;\mathcal{L})+mN(r,0;\mathcal{L})+N(r,0;\mathcal{L})+\kappa\overline{N}(r,\infty;\mathcal{L})\\ &+\frac{1}{2}(l+1)\overline{N}(r,0;\mathcal{L})+\frac{m}{2}N(r,0;\mathcal{L})+N(r,0;\mathcal{L})\\ &+\kappa\overline{N}(r,\infty;\mathcal{L})+S(r,\mathsf{f})+S(r,\mathcal{L}). \end{aligned} \tag{4.5}$$

Now, by combining the inequalities (4.4) and (4.5), we get

$$\begin{aligned}(n+m+1-\kappa)[T(r,\mathsf{f})+T(r,\mathcal{L})] \leq{}& 2(l+2+m+1)[N(r,0;\mathsf{f})+N(r,0;\mathcal{L})]\\ &+\left(\frac{l+1}{2}+\frac{m}{2}+1\right)[N(r,0;\mathsf{f})+N(r,0;\mathcal{L})]\\ &+S(r,\mathsf{f})+S(r,\mathcal{L}), \end{aligned}$$

which contradicts $n>\kappa+\frac{5l+3m+13}{2}$.

**Subcase 1.3.** Suppose $\kappa_1=0$ and $\kappa_2=0$, then using Lemmas 3.3, 3.6, 3.8, 3.11 and 3.14 we obtain

$$\begin{aligned}T(r,\mathsf{F}^*) \leq{}& N_2(r,0;\mathsf{F}^*)+N_2(r,0;\mathcal{L}^*)+3\overline{N}(r,\infty;\mathsf{F}^*)+2\overline{N}(r,\infty;\mathcal{L}^*)+2\overline{N}(r,0;\mathsf{F}^*)\\ &+\overline{N}_*(r,\infty;\mathsf{F}^*,\mathcal{L}^*)+\overline{N}(r,0;\mathcal{L}^*)+S(r,\mathsf{F}^*)+S(r,\mathcal{L}^*)\\ \leq{}& T(r,\mathsf{F}^*)-T(r,\mathsf{F}_1)+N_{l+2}(r,0;\mathsf{F}_1)+N_{l+2}(r,0;\mathcal{L}_1)+l\overline{N}(r,\mathcal{L}_1)\\ &+2N_{l+1}(r,0;\mathsf{F}_1)+l\overline{N}(r,\mathsf{F}_1)+N_{l+1}(r,0;\mathcal{L}_1)+l\overline{N}(r,\infty;\mathcal{L}_1)\\ &+S(r,\mathsf{F}_1)+S(r,\mathcal{L}_1), \end{aligned}$$

which implies that,

$$\begin{aligned}(n+m+1-\kappa)T(r,\mathsf{f}) \leq{}& (l+2)\overline{N}(r,0;\mathsf{f})+mN(r,0;\mathsf{f})+N(r,0;\mathsf{f})+\kappa\overline{N}(r,\infty;\mathsf{f})\\ &+(l+2)\overline{N}(r,0;\mathcal{L})+mN(r,0;\mathcal{L})+N(r,0;\mathcal{L})+\kappa\overline{N}(r,\infty;\mathcal{L})\\ &+2(l+1)\overline{N}(r,0;\mathsf{f})+2mN(r,0;\mathsf{f})+2N(r,0;\mathsf{f})+2\kappa\overline{N}(r,\infty;\mathsf{f})\\ &+(l+1)\overline{N}(r,0;\mathcal{L})+mN(r,0;\mathcal{L})+N(r,0;\mathcal{L})+\kappa\overline{N}(r,\infty;\mathcal{L})\\ &+S(r,\mathsf{f})+S(r,\mathcal{L}). \end{aligned} \tag{4.6}$$

By similar calculations, we get

$$\begin{aligned}(n+m+1-\kappa)T(r,\mathcal{L}) \leq\ & (l+2)\overline{N}(r,0;\mathsf{f}) + mN(r,0;\mathsf{f}) + N(r,0;\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f})\\ &+ (l+2)\overline{N}(r,0;\mathcal{L}) + mN(r,0;\mathcal{L}) + N(r,0;\mathcal{L}) + \kappa\overline{N}(r,\infty;\mathcal{L})\\ &+ 2(l+1)\overline{N}(r,0;\mathcal{L}) + 2mN(r,0;\mathcal{L}) + 2N(r,0;\mathcal{L})\\ &+ 2\kappa\overline{N}(r,\infty;\mathcal{L}) + (l+1)\overline{N}(r,0;\mathsf{f}) + mN(r,0;\mathsf{f}) + N(r,0;\mathsf{f})\\ &+ \kappa\overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}) + S(r,\mathcal{L}). \qquad (4.7)\end{aligned}$$

Now, by combining the inequalities (4.6) and (4.7), we get

$$\begin{aligned}(n+m+1-\kappa)[T(r,\mathsf{f}) + T(r,\mathcal{L})] \leq\ & 2(l+2+m+1)[N(r,0;\mathsf{f}) + N(r,0;\mathcal{L})]\\ &+ (3l+3m+6)[N(r,0;\mathsf{f}) + N(r,0;\mathcal{L})]\\ &+ S(r,\mathsf{f}) + S(r,\mathcal{L}),\end{aligned}$$

which contradicts $n > \kappa + 5l + 4m + 11$.

**Case 2.** We now assume that $\Omega \equiv 0$. Then

$$\left(\frac{\mathsf{F}^{*\prime\prime}}{\mathsf{F}^{*\prime}} - \frac{2\mathsf{F}^{*\prime}}{\mathsf{F}^{*}-1}\right) \equiv \left(\frac{\mathcal{L}^{*\prime\prime}}{\mathcal{L}^{*\prime}} - \frac{2\mathcal{L}^{*\prime}}{\mathcal{L}^{*}-1}\right).$$

By integrating twice the both sides of the above equality we get,

$$\frac{1}{\mathsf{F}^{*}-1} = \frac{a_1}{\mathcal{L}^{*}-1} + a_2, \qquad (4.8)$$

where $a_1(\neq 0)$ and $a_2$ are constants. (4.8) obviously says that $\mathsf{F}^{*}$, $\mathcal{L}^{*}$ share the value 1 CM and hence they share the value 1 with weight $\kappa_1 = 2$, and therefore, $n > \kappa + 2l + m + 5$.

Now, let us discuss the three following subcases separately.

**Subcase 2.1.** If possible $a_2 \neq 0$ and $a_1 = a_2$, then from (4.8), we deduce

$$\frac{1}{\mathsf{F}^{*}-1} = \frac{a_2\mathcal{L}^{*}}{\mathcal{L}^{*}-1} \qquad (4.9)$$

If $a_2 = -1$, then from (4.9), we obtain

$$\mathsf{F}^{*}\mathcal{L}^{*} = 1,$$

i.e.,

$$[\mathsf{f}^{n}(\mathsf{f}-1)^{m}d_{\kappa}[\mathsf{f}]]^{(l)}[\mathcal{L}^{n}(\mathcal{L}-1)^{m}d_{\kappa}[\mathcal{L}]]^{(l)} \equiv 1,$$

which implies

$$[\mathsf{f}^n(\mathsf{f}-1)^m d_\kappa[\mathsf{f}]]^{(l)} \equiv \frac{1}{[\mathcal{L}^n(\mathcal{L}-1)^m d_\kappa[\mathcal{L}]]^{(l)}}. \tag{4.10}$$

Since $\mathsf{F}^*$ and $\mathcal{L}^*$ share the poles, (4.10) is not possible.
If $a_2 \neq -1$, then from (4.9), we have

$$\frac{1}{\mathsf{F}^*} = \frac{a_2\mathcal{L}^*}{(a_2+1)\mathcal{L}^*-1} \quad \text{and so} \quad \overline{N}\left(r, \frac{1}{1+a_2}; \mathcal{L}^*\right) = \overline{N}(r, 0; \mathsf{F}^*).$$

From Nevanlinna's Fundamental Theorem -II, we have

$$\begin{aligned} T(r,\mathcal{L}_1) &\leq T(r,\mathcal{L}^*) + S(r,\mathcal{L}^*) \\ &\leq \overline{N}(r,0;\mathcal{L}^*) + \overline{N}\left(r, \frac{1}{1+a_2}; \mathcal{L}^*\right) + \overline{N}(r,\infty;\mathcal{L}^*) + S(r,\mathcal{L}^*) \\ &\leq \overline{N}(r,0;\mathsf{F}^*) + \overline{N}(r,0;\mathcal{L}^*) + S(r,\mathcal{L}^*) \end{aligned}$$

Using Lemmas 3.3, 3.11 and 3.14, we have

$$\begin{aligned} (n+m+1-\kappa)T(r,\mathcal{L}) \leq\; & (l+1)\overline{N}(r,0;\mathsf{f}) + mN(r,0;\mathsf{f}) + N(r,0;\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) \\ & + (l+1)\overline{N}(r,0;\mathcal{L}) + mN(r,0;\mathcal{L}) + N(r,0;\mathcal{L}) + \kappa\overline{N}(r,\infty;\mathcal{L}) \\ & + S(r,\mathsf{f}) + S(r,\mathcal{L}) \end{aligned}$$

Similarly, we have for $T(r,\mathsf{f})$

$$\begin{aligned} (n+m+1-\kappa)T(r,\mathsf{f}) \leq\; & (l+1)\overline{N}(r,0;\mathsf{f}) + mN(r,0;\mathsf{f}) + N(r,0;\mathsf{f}) + \kappa\overline{N}(r,\infty;\mathsf{f}) \\ & + (l+1)\overline{N}(r,0;\mathcal{L}) + mN(r,0;\mathcal{L}) + N(r,0;\mathcal{L}) + \kappa\overline{N}(r,\infty;\mathcal{L}) \\ & + S(r,\mathsf{f}) + S(r,\mathcal{L}) \end{aligned}$$

Thus by combining the above two inequalities, we get

$$(n+m+1-\kappa)\{T(r,\mathsf{f})+T(r,\mathcal{L})\} \leq (2l+2m+4)\{N(r,0;\mathsf{f})+N(r,0;\mathcal{L})\}+S(r,\mathsf{f})+S(r,\mathcal{L})$$

which contradicts $n > \kappa + 2l + m + 5$.

**Subcase 2.2.** Suppose $a_2 \neq 0$ and $a_1 \neq a_2$. Then by (4.8), we get

$$\mathsf{F}^* = \frac{(a_2+1)\mathcal{L}^* - (a_2-a_1+1)}{a_1\mathcal{L}^* + (a_1-a_2)} \quad \text{and so} \quad \overline{N}\left(r, \frac{a_2-a_1+1}{a_2+1}; \mathcal{L}^*\right) = \overline{N}(r,0;\mathcal{L}^*).$$

Proceeding in a manner similar to subcase 2.1, we can arrive at a contradiction.

**Subcase 2.3.** Let $a_2 = 0$ and $a_1 \neq 0$. Then from (4.8), we get

$$\mathsf{F}^* = \frac{\mathcal{L}^* + a_1 - 1}{a_1} \quad \text{and} \quad \mathcal{L}^* = a_1\mathsf{F}^* - (a_1 - 1).$$

If $a_1 \neq 1$, it follows that,

$$\overline{N}\left(r, \frac{a_1 - 1}{a_1}; \mathsf{F}^*\right) = \overline{N}(r, 0; \mathcal{L}^*) \quad \text{and} \quad \overline{N}(r, 1 - a_1; \mathcal{L}^*) = \overline{N}(r, 0; \mathsf{F}^*).$$

Following an argument as in subcase 2.1, we obtain a contradiction. Thus $a_1 = 1$, which implies $\mathsf{F}^* = \mathcal{L}^*$, and therefore,

$$(\mathsf{f}^n(\mathsf{f} - 1)^m d_\kappa[\mathsf{f}])^{(l)} = (\mathcal{L}^n(\mathcal{L} - 1)^m d_\kappa[\mathcal{L}])^{(l)}. \tag{4.11}$$

Integrating the above equation for $l$ times, we get

$$(\mathsf{f}^n(\mathsf{f} - 1)^m d_\kappa[\mathsf{f}]) = (\mathcal{L}^n(\mathcal{L} - 1)^m d_\kappa[\mathcal{L}]) + b(z), \tag{4.12}$$

where $b(z)$ is a polynomial of degree atmost $l - 1$.
Suppose $b(z) \neq 0$, then we get

$$\frac{\mathsf{f}^n(\mathsf{f} - 1)^m d_\kappa[\mathsf{f}]}{b(z)} = \frac{\mathcal{L}^n(\mathcal{L} - 1)^m d_\kappa[\mathcal{L}]}{b(z)} + 1, \tag{4.13}$$

*i.e.*,

$$\frac{\mathsf{F}_1}{b(z)} = \frac{\mathcal{L}_1}{b(z)} + 1. \tag{4.14}$$

By the Nevanlinna's Fundamental Theorem -II and Lemma 3.14, we have

$$T(r, \mathsf{F}_1) \leq \overline{N}\left(r, \infty; \frac{\mathsf{F}_1}{b(z)}\right) + \overline{N}\left(r, 0; \frac{\mathsf{F}_1}{b(z)}\right) + \overline{N}\left(r, 0; \frac{\mathcal{L}_1}{b(z)}\right) + S(r, \mathsf{F}_1)$$
$$\begin{aligned}(n + m + 1 - \kappa)T(r, \mathsf{f}) \leq\ & \overline{N}(r, 0; \mathsf{f}) + mN(r, 0; \mathsf{f}) + N(r, 0; \mathsf{f}) + \kappa\overline{N}(r, \infty; \mathsf{f}) \\ & + \overline{N}(r, 0; \mathcal{L}) + mN(r, 0; \mathcal{L}) + N(r, 0; \mathcal{L}) + \kappa\overline{N}(r, \infty; \mathcal{L}) \\ & + S(r, \mathsf{f}) + S(r, \mathcal{L}).\end{aligned} \tag{4.15}$$

Similarly, we have

$$\begin{aligned}(n + m + 1 - \kappa)T(r, \mathsf{f}) \leq\ & \overline{N}(r, 0; \mathcal{L}) + mN(r, 0; \mathcal{L}) + N(r, 0; \mathcal{L}) + \kappa\overline{N}(r, \infty; \mathcal{L}) \\ & + \overline{N}(r, 0; \mathsf{f}) + mN(r, 0; \mathsf{f}) + N(r, 0; \mathsf{f}) + \kappa\overline{N}(r, \infty; \mathsf{f}) \\ & + S(r, \mathsf{f}) + S(r, \mathcal{L}).\end{aligned} \tag{4.16}$$

combining inequalities (4.15) and (4.16), we get

$$(n+m+1-\kappa)[T(r,\mathsf{f})+T(r,\mathcal{L})] \leq (2m+4)[N(r,0;\mathsf{f})+N(r,0;\mathcal{L})] + S(r,\mathsf{f})+S(r,\mathcal{L}),$$

which contradicts $n > \kappa + 2l + m + 5$. Therefore $b(z) = 0$. Hence from (4.12), we have

$$(\mathsf{f}^n(\mathsf{f}-1)^m d_\kappa[\mathsf{f}]) = (\mathcal{L}^n(\mathcal{L}-1)^m d_\kappa[\mathcal{L}]), \tag{4.17}$$

which is the required conclusion. Further if $\kappa = 0$, then from (4.17), we have

$$\mathsf{f}^{n+1}(\mathsf{f}-1)^m = \mathcal{L}^{n+1}(\mathcal{L}-1)^m \tag{4.18}$$

Let $t = \frac{\mathsf{f}}{\mathcal{L}}$. We shall consider two subcases of subcase 2.3.

**Subcase 2.3.1.** If $t(z)$ is a constant function, then by substitution of $\mathsf{f} = t\mathcal{L}$ in (4.18), we obtain

$$\mathsf{f}^{n+1}[\mathsf{f}^m - m\mathsf{f}^{m-1} + \cdots + (-1)^m] = \mathcal{L}^{n+1}[\mathcal{L}^m - m\mathcal{L}^{m-1} + \cdots + (-1)^m] \tag{4.19}$$

substituting $\mathsf{f} = t\mathcal{L}$ in (4.19), we will have

$$\mathcal{L}^{n+m+1}[t^{n+m+1}-1] - m\mathcal{L}^{n+m}[t^{n+m}-1] + \cdots + (-1)^m\mathcal{L}^{n+1}[t^{n+1}-1] = 0, \tag{4.20}$$

which implies $t^d = 1$, where $d = GCD(n+m+1, n+m, n+m-1, ..., n+1)$. Thus we get the conclusion $\mathsf{f} \equiv t\mathcal{L}$, where $t$ is a constant such that $t^d = 1$.

**Subcase 2.3.2.** Suppose, $t(z)$ is not a constant, then $\mathsf{f}$ and $\mathcal{L}$ satisfy the algebraic equation $R(\mathsf{f}, \mathcal{L}) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^{n+1}(z)(\omega_1(z)-1)^m - \omega_2^{n+1}(z)(\omega_2(z)-1)^m.$$

This completes the proof of Theorem 2.1.

### 4.2. Proof of Corollary 2.1, Theorem 2.2 and Corollary 2.2.

Corollary 2.1, Theorem 2.2 and Corollary 2.2. can be proved easily in a similar way as Theorem 2.1.

## 5. Conclusion

We have examined the uniqueness of differential polynomials of $\mathsf{f}$ and $\mathcal{L}$, when they share the values 1 with weight $\kappa_1$ and $\infty$ with weight $\kappa_2$. By fixing the values $\alpha_1 = 1$ and $\alpha_2 = \infty$ and additionally considering the linear differential polynomial, our results extend as well as reduce the weights of sharing $\kappa_1, \kappa_2$ in the result of Hao and Chen [3], as well as reduce the condition for $n$ in their results [2].

Also, we can pose the following open questions.

**Open Questions:**

1. What happens to Theorem 2.1 and Corollary 2.1, if we replace the linear differential polynomial $d_\kappa[\mathsf{f}]$ by a homogeneous and non-homogeneous differential polynomials $H[\mathsf{f}]$ as defined in [14], as well as by difference differential polynomial $P[\mathsf{f}]$ as defined in [10]?

2. Can the condition for $n$ in Theorems 2.1, 2.2 and Corollaries 2.1, 2.2 be still reduced?

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