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TRIPLED FIXED POINT THEOREMS IN N-CONE METRIC SPACE UNDER F-INVARIANT SET

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Abstract: A notion of an F-invariant in N-cone metric space is introduced and we prove some fixed point results for mappings satisfying certain contractive conditions under the concept of c-distance. Our results complement and extend well known results in the literature.

Keywords and Phrases: N-cone metric space, contractive mappings, fixed point.

2020 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

The notion of cone metric space was introduced in [6]. Huang and Zhang replaced the real numbers by ordering Banach space and defined cone metric space. They also gave an example of function which is contraction in the category of cone metric but not contraction if considered over metric spaces and hence by proving fixed point theorem in cone metric spaces ensured that this map must have a unique fixed point.

Subsequently, Rezapour and Halbarani [14] omitted the assumption of normality in cone metric space. After that a series of articles in cone metric space started to appear (see, [5, 8, 9, 11, 13, 16, 17, 18, 19] and references therein).

In [3], Bhaskar and Lakshmikanthan introduced the concept of mixed monotone property and proved fixed point in partially ordered metric spaces. Then they have evidenced coupled fixed point theorems for mappings that satisfy mixed monotone property and applied their theorems to produce some applications in the problems of existence and uniqueness of solution for a periodic boundary value problem.

In 2011, Beride et al. [2] introduced the definition of mixed monotone property and the definition of tripled fixed point for mapping $F : X \times X \times X \to X$ and proved tripled fixed point theorems for contractive type mappings having that property in partially ordered metric spaces.

In this paper, we prove some tripled fixed point theorems under the concept of c-distance by using the idea of F-invariant in N-cone metric spaces. Our results have several consequences including generalisations of comparable results in the literature (see, [1, 7, 10, 15] and references therein).

2. N-cone metric spaces

Let E be a real Banach space and P be a subset of E. P is called a cone if

(1) P is closed, nonempty and $P \neq \{0\}$;

(2) $ax + by \in P$, for all $x, y \in P$ and non-negative real numbers a, b;

(3) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$, $x < y$ will stand for $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there is a number $N > 0$ such that for all $x, y \in E$, $\| x \| \leq N \| y \|$. The least positive number satisfying the above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded above is convergent, that is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, there is $x \in E$ such that $\lim_{n\to\infty} \| x_n - x \| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded below is convergent.

Lemma 2.1. [14] *Every regular cone is normal.*

Definition 2.2. [9] *Let X be a non-empty set. An N-cone metric space on X is a function $N : X^3 \to E$, that satisfies the following conditions for all $x, y, z, a \in X$*

(1) $N(x, y, z) \geq 0$*;*

(2) $N(x, y, z) = 0$ *if and only if* $x = y = z$*;*

(3) $N(x, y, z) \leq N(x, x, a) + N(y, y, a) + N(z, z, a)$*.*

Then, the function N *is called an* N*-cone metric and the pair* (X, N) *is called an* N*-cone metric space.*

Proposition 2.3. [9] *If* (X, N) *is an* N*-cone metric space for all* $x, y, z \in X$*, we have* $N(x, x, y) = N(y, y, x)$*.*

Definition 2.4. [9] *Let* (X, N) *be an* N*-cone metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$*. If for every* $c \in E$ *with* $0 << c$ *there is* N *such that for all* $n > N$, $N(x_n, x_n, x) << c$*, then* $\{x_n\}$ *is said to be convergent.* $\{x_n\}$ *converges to* x *and* x *is the limit of* $\{x_n\}$*. We denote this by* $x_n \to x$ *as* $(n \to +\infty)$*.*

Lemma 2.5. [9] *Let* (X, N) *be an* N*-cone metric space and* P *be a normal cone with normal constant* k*. Let* $\{x_n\}$ *be a sequence in* X*. If* $\{x_n\}$ *converges to* x *and* $\{x_n\}$ *also converges to* y *then* $x = y$*. That is the limit of* $\{x_n\}$*, if exists, is unique.*

Definition 2.6. [9] *Let* (X, N) *be an* N*-cone metric space and* $\{x_n\}$ *be a sequence in* X*. If for any* $c \in E$ *with* $0 << c$ *there is* N *such that for* $m, n > N$, $N(x_n, x_n, x_m) << c$*, then* $\{x_n\}$ *is called a Cauchy sequence in* X*.*

Definition 2.7. [9] *Let* (X, N) *be an* N*-cone metric space. If every Cauchy sequence in* X *is convergent in* X*, then* X *is called a complete* N*-cone metric space.*

Lemma 2.8. [9] *Let* (X, N) *be an* N*-cone metric space and* $\{x_n\}$ *be a sequence in* X*. If* $\{x_n\}$ *converges to* x*, then* $\{x_n\}$ *is a Cauchy sequence.*

Definition 2.9. [9] *Let* (X, N) *and* (X', N') *be* N*-cone metric spaces. Then, a function* $f : X \to X'$ *is said to be continuous at a point* $x \in X$ *if it is sequentially continuous at* x*, that is, whenever* $\{x_n\}$ *is convergent to* x *we have* $\{fx_n\}$ *is convergent to* fx*.*

Lemma 2.10. [9] *Let* (X, N) *be an* N*-cone metric space and* P *be a normal cone with normal constant* k*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in* X *and suppose that* $x_n \to x, y_n \to y$ *as* $n \to +\infty$*. Then* $N(x_n, x_n, y_n) \to N(x, x, y)$ *as* $n \to +\infty$*.*

Remark 2.11. [9] *If* $x_n \to x$ *in an* N*-cone metric space* X*, then every subsequence of* $\{x_n\}$ *converges to* x*.*

Proposition 2.12. [9] *Let* (X, N) *be an* N*-cone metric space and* P *be a cone in a real Banach space* E*. If* $u \leq v, v << w$*, then* $u << w$*.*

Lemma 2.13. [9] *Let (X, N) be an N-cone metric space and P be an N-cone in a real Banach space E and $k_1, k_2, k_3, k_4, k > 0$. If $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ in X and*

$$\begin{aligned} ka &\leq k_1N(x_n, x_n, x) + k_2N(y_n, y_n, y) \\ &\quad +k_3N(z_n, z_n, z) + k_4N(p_n, p_n, p) \end{aligned}$$

then $a = 0$.

Definition 2.14. [4] *Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:*

(1) $0 \leq q(x, y)$ for all $x, y \in X$,

(2) $q(x, y) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(3) for each $x \in X$ and $n \geq 1$, if $q(x, x_n) \leq u$ for some $u = u_x \in P$, then $q(x, y) \leq u$ whenever $\{y_n\}$ is a sequence in X converging to point $y \in X$,

(4) for all $c \in E$ with $0 \ll c$, there exist $e \in E$ with $0 \ll e$ such that $q(z, x) \leq e$ and $q(z, y) \leq e$ imply $d(x, y) \ll c$.

Lemma 2.15. [4] *Let (X, d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to 0. Then the following hold:*

(1) If $q(x_n, y) \leq u_n$ and $q(x_n, z) \leq u_n$, then $y = z$.

(2) If $q(x_n, y_n) \leq u_n$ and $q(x_n, z) \leq u_n$, then $\{y_n\}$ converges to z.

(3) If $q(x_n, x_m) \leq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y, x_n) \leq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Definition 2.16. *An element $(x, y, z) \in X^3$ is to be a tripled point of a mapping $F : X^3 \to X$ if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.*

3. Fixed point theorems

In this section, we prove some tripled fixed point theorems under the concept of c-distance by using the idea of F-invariant in an N-cone metric space.

Definition 3.1. *Let (X, N) be an N-cone metric space and $F : X^3 \to X$ be a given* *mapping. Let M be a non-empty subset of X^6. We say that M is an F-invariant subset of X^6 if for $x, y, z, w, e, s \in X$, we have*

$$F_1(x, y, z, w, e, s) \in M \Leftrightarrow (s, e, w, z, y, x) \in M,$$
$$F_2(x, y, z, w, e, s) \in M \Rightarrow (F(x, y, z), F(y, z, x),$$
$$F(z, x, y), F(w, e, s), F(e, s, w), F(s, w, e)) \in M.$$

Definition 3.2. *Let (X, N) be an N-cone metric space endowed with a partial order $\leq$. Let $F : X^3 \to X$ be a mapping satisfying the mixed monotone property; that is, for all $x, y, z \in X$, we have*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$
$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Example 3.3. Let (X, N) be an N-cone metric space endowed with a partial order $\leq$. Let $F : X^3 \to X$ be a mapping satisfying the mixed monotone property; that is, for all $x, y, z \in X$, we have

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$
$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Define the subset $M \subseteq X^6$ by

$$M = \{(a, b, c, d, e, s) : d \leq a, b \leq e, s \leq c\}.$$

Then M is F-invariant of X^6.

Theorem 3.4. *Let (X, N) be a complete N-cone metric space and q be a c-distance on X. Let M be a nonempty subset of X^6 and $F : X^3 \to X$ be a function such that*

$$\begin{aligned} q(F(x, y, z), F(x^*, y^*, z^*)) + q(F(y, z, x), F(y^*), z^*, x^*) \\ +q(F(z, x, y), F(z^*), x^*, y^*) \leq k(q(x, x^*) + q(y, y^*) + q(z, z^*)) \end{aligned} \tag{3.1}$$

for some $k \in [0, 1)$ and all $x, y, z, x^*, y^*, z^* \in X$ with $(x, y, z, x^*, y^*, z^*) \in M$ or $(x^*, y^*, z^*, x, y, z) \in M$. If M is F invariant and there exist $x_0, y_0, z_0 \in X$ such that

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) \in M,$$

then F has a tripled fixed point (u, v, w). Furthermore, if $(u, v, w, u, v, w) \in M$, then $q(u, u) = 0, q(v, v) = 0$ and $q(w, w) = 0$.

Proof. Since $F(X \times X \times X) \subseteq X$, we can construct three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$x_n = F(x_{n-1}, y_{n-1}, z_{n-1}), y_n = F(y_{n-1}, z_{n-1}, x_{n-1})$$

$$z_n = F(z_{n-1}, x_{n-1}, y_{n-1}) \tag{3.2}$$

for all $n \in \mathbb{N}$. Since

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) = (x_1, y_1, z_1, x_0, y_0, z_0) \in M$$

and M is an F-invariant set, we get

$$(F(x_1, y_1, z_1), F(y_1, z_1, x_1), F(z_1, x_1, y_1), F(x_0, y_0, z_0),$$
$$F(y_0, z_0, x_0), F(z_0, x_0, y_0)) = (x_2, y_2, z_2, x_1, y_1, z_1) \in M.$$

Again, using the fact that M is an F-invariant set, we have

$$(F(x_2, y_2, z_2), F(y_2, z_2, x_2), F(z_2, x_2, y_2), F(x_1, y_1, z_1),$$
$$F(y_1, z_1, x_1), F(z_1, x_1, y_1)) = (x_3, y_3, z_3, x_2, y_2, z_2) \in M.$$

By repeating the argument similar to the above, we get

$$F(x_{n-1}, y_{n-1}, z_{n-1}), F(y_{n-1}, z_{n-1}, x_{n-1}), F(z_{n-1}, x_{n-1}, y_{n-1}),$$
$$F(x_{n-1}, y_{n-1}, z_{n-1}) = (x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \in M$$

for all $n \in \mathbb{N}$. From (3.1), we have

$$\begin{aligned} & q(x_n, x_{n+1}) + q(y_n, y_{n+1}) + q(z_n, z_{n+1}) \\ & = q(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ & + q(F(y_{n-1}, z_{n-1}, x_{n-1}, F(y_n, z_n, x_n)) + q(F(z_{n-1}, x_{n-1}, y_{n-1}, F(z_n, x_n, y_n)) \\ & \leq k(q(x_{n-1}, x_n) + q(y_{n-1}, y_n) + q(z_{n-1}, z_n)). \end{aligned} \tag{3.3}$$

We repeat the above process for n-times, we get

$$\begin{aligned} & q(x_n, x_{n+1}) + q(y_n, y_{n+1}) + q(z_n, z_{n+1}) \\ & \leq k^n(q(x_0, x_1) + q(y_0, y_1) + q(z_0, z_1)). \end{aligned} \tag{3.4}$$

Put $q_n = q(x_n, x_{n+1}) + q(y_n, y_{n+1}) + q(z_n, z_{n+1})$. Then from (3.4) we have

$$q_n \leq k^n q_0. \tag{3.5}$$

Let $m, n \in \mathbb{N}$ with $m > n$. Then we have

$$q(x_n, x_m) \leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m),$$

$$q(y_n, y_m) \leq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + \cdots + q(y_{m-1}, y_m),$$

and

$$q(z_n, z_m) \leq q(z_n, z_{n+1}) + q(z_{n+1}, z_{n+2}) + \cdots + q(z_{m-1}, z_m).$$

Then we have

$$\begin{aligned} & q(x_n, x_m) + q(y_n, y_m) + q(z_n, z_m) \\ & = q_n + q_{n+1} + \cdots + q_{m-1} \\ & \leq k^n q_0 + k^{n+1} q_0 + \cdots + k^{m-1} q_0 \\ & = (k^n + k^{n+1} + \cdots + k^{m-1}) q_0 \\ & = k^n (1 + k + k^2 + \cdots + k^{m-1-n}) q_0 \\ & \leq \frac{k^n}{1-k} q_0. \end{aligned} \tag{3.6}$$

From (3.6) we have

$$q(x_n, x_m) \leq \frac{k^n}{1-k} q_0 \to 0, n \to +\infty,$$

$$q(y_n, y_m) \leq \frac{k^n}{1-k} q_0 \to 0, n \to +\infty$$

and

$$q(z_n, z_m) \leq \frac{k^n}{1-k} q_0 \to 0, n \to +\infty.$$

Thus, by Lemma 2.15(3), $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X. Since X is complete, there exist u, v and $w \in X$ such that $x_n \to u, y_n \to v$ and $z_n \to w$ as $n \to \infty$. Then

$$q(x_n, u) \leq \frac{k^n}{1-k} q_0, \tag{3.7}$$

$$q(y_n, v) \leq \frac{k^n}{1-k} q_0 \tag{3.8}$$

and

$$q(z_n, w) \leq \frac{k^n}{1-k} q_0. \tag{3.9}$$

On the other hand, we can get

$$\begin{aligned}
&q(x_n, F(u, v, w)) + q(y_n, F(v, w, u)) + q(z_n, F(w, u, v)) \\
&= q(F(x_{n-1}, y_{n-1}, z_{n-1}), F(u, v, w) \\
&+ q(F(y_{n-1}, z_{n-1}, x_{n-1}, F(v, w, u)) \\
&+ q(F(z_{n-1}, x_{n-1}, y_{n-1}, F(w, u, v)) \\
&\leq k(q(x_{n-1}, u) + q(y_{n-1}, v) + q(z_{n-1}, w)) \\
&\leq k(\frac{k^{n-1}}{1-k} q_0 + \frac{k^{n-1}}{1-k} q_0 + \frac{k^{n-1}}{1-k} q_0) \\
&= \frac{3k^n}{1-k} q_0.
\end{aligned} \tag{3.10}$$

Therefore

$$q(x_n, F(u, v, w)) \leq \frac{3k^n}{1-k} q_0, \tag{3.11}$$

$$q(x_n, F(v, w, u)) \leq \frac{3k^n}{1-k} q_0 \tag{3.12}$$

and

$$q(x_n, F(w, u, v)) \leq \frac{3k^n}{1-k} q_0. \tag{3.13}$$

Also, from (3.7), we have

$$q(x_n u) \leq \frac{k^n}{1-k} q_0 \leq \frac{3k^n}{1-k} q_0. \tag{3.14}$$

Then by Lemma 2.15(1), (3.11) and (3.14), we have $u = F(u, v, w)$. Also, $v = F(v, w, u)$ and $w = F(w, u, v)$. Therefore, (u, v, w) is a tripled fixed point of F. Finally we assume that $(u, v, w, u, v, w) \in M$. We have

$$\begin{aligned}
& q(u, u) + q(v, v) + q(w, w)) \\
= \;& q(F(u, v, w), F(u, v, w) \\
& + q(v, w, u, F(v, w, u)) + q(w, u, v, F(w, u, v)) \\
\leq \;& k(q(u, u) + q(v, v) + q(w, w)).
\end{aligned}$$

Since $0 \leq k < 1$, by Lemma 2.15(1), we have $q(u, u) + q(v, v) + q(w, w) = 0$. But $q(u, u) \geq 0, q(v, v) \geq 0$ and $q(w, w) \geq 0$. Hence, $q(u, u) = 0, q(v, v) = 0$ and

$q(w,w) = 0$.

Theorem 3.5. *Let (X,N) be a complete N-cone metric space and q be a c-distance on X. Let M be a nonempty subset of X^6 and $F : X^3 \to X$ be a function such that*

$$q(F(x,y,z),F(x^*,y^*,z^*)) + q(F(y,z,x),F(y^*),z^*,x^*)$$
$$+q(F(z,x,y),F(z^*),x^*,y^*) \leq k(q(x,x^*) + q(y,y^*) + q(z,z^*))$$

for some $k \in [0,1)$ and all $x,y,z,x^*,y^*,z^* \in X$ with $(x,y,z,x^*,y^*,z^*) \in M$ or $(x^*,y^*,z^*,x,y,z) \in M$. Suppose that for any three elements x,y and $z \in X$, we have $(x,y,z,x,y,z) \in M$ or $(y,z,x,y,z,x) \in M$ or $(z,x,y,z,x,y) \in M$. Then the tripled fixed point has the form (u,u,u), where $u \in X$.

Proof. From theorem 3.4, there exists a tripled fixed point $(u,v,w) \in X^3$. Therefore

$$u = F(u,v,w), v = F(v,u,w), w = F(w,u,v).$$

Moreover, $q(u,u) = 0, q(v,v) = 0$ and $q(w,w) = 0$ if $(u,v,w,u,v,w) \in M$. From the additional hypothesis, we have $(u,v,w,u,v,w) \in M$ or $(v,w,u,v,w,u) \in M$ or $(w,u,v,w,u,v) \in M$. By (3.1), we get

$$\begin{aligned} & q(u,v) + q(w,u) + q(v,w) \\ & = q(F(u,v,w),F(v,w,u) \\ & + q(F(w,u,v),F(u,v,w)) \\ & + q(F(v,w,u),F(w,u,v)) \\ & \leq k(q(u,v) + q(v,w) + q(w,u)). \end{aligned} \tag{3.15}$$

Since M is an F-invariant set, $(w,v,u,w,v,u) \in M$. By applying the contractive condition, we have $(u,v,w,u,v,w) \in M$ or $(v,w,u,v,w,u) \in M$ or $(w,u,v,w,u,v) \in M$. By (3.1), we get

$$\begin{aligned} & q(w,v) + q(u,w) + q(v,u)) \\ = \; & q(F(w,u,v),F(v,w,u) \\ & +q(F(u,v,w),F(w,u,v)) \\ & +q(F(v,w,u),F(u,v,w)) \\ \leq \; & k(q(u,v) + q(v,w) + q(w,u)). \end{aligned}$$

Since $0 \leq k < 1$, we get that $q(u,v) + q(w,u) + q(v,w) = 0$. Therefore, $q(u,v) = 0, q(w,u) = 0$ and $q(v,w) = 0$. We also have $q(u,u) = 0, q(w,w) = 0$ and $q(v,v) = 0$. Let $u_n = 0$ and $x_n = u$. Then

$$q(x_n,u) \leq u_n$$

and

$$q(x_n, v) \le u_n.$$

Then by Lemma 2.15(1), we have $u = v$. Also, we have $u = w$ and $w = v$.By using the same way for the arrangement in (3.15), we have the same results. Therefore, the triple fixed point of F has the form (u, u, u).

Theorem 3.6. *Let (X, N) be a complete N-cone metric space and q be a c-distance on X. Let M be a nonempty subset of X^6 and $F : X^3 \to X$ be a function such that*

$$q(F(x, y, z), F(x^*, y^*, z^*)) + q(F(y, z, x), F(y^*), z^*, x^*)$$
$$+q(F(z, x, y), F(z^*), x^*, y^*) \le k(q(x, x^*) + q(y, y^*) + q(z, z^*))$$

for some $k \in [0, 1)$ and all $x, y, z, x^*, y^*, z^* \in X$ with $(x, y, z, x^*, y^*, z^*) \in M$ or $(x^*, y^*, z^*, x, y, z) \in M$.
Also, suppose that

(i) there exist $x_0, y_0, z_0 \in X$ such that

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) \in M,$$

(ii) if three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ with $(x_{n+1}, y_{n+1}, z_{n+1}, x_n, y_n, z_n) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x, y_n \to y$ and $z_n \to z$, then $(x, y, z, x_n, y_n, z_n) \in M$ for all $n \in \mathbb{N}$.

If M is an F invariant set, F has a tripled fixed point. Furthermore, if $(u, v, w, u, v, w) \in M$, then $q(u, u) = 0, q(v, v) = 0$ and $q(w, w) = 0$.

Proof. We can construct three Cauchy sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1} \in M$$

for all $n \in \mathbb{N}$. Moreover, we from the assumption $x_n \to u, y_n \to v$ and $z_n \to w$ where $u, v, w \in X$.Therefore, by the assumption, we have $(u, v, w, x_n, y_n, z_n) \in M$. Since F is continuous,taking $n \to +\infty$ in (3.2), we get

$$\lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} F(x_n, y_n, z_n) = F(\lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n, \lim_{n\to+\infty} z_n) = F(u, v, w),$$

$$\lim_{n\to+\infty} y_{n+1} = \lim_{n\to+\infty} F(y_n, z_n, x_n) = F(\lim_{n\to+\infty} y_n, \lim_{n\to+\infty} z_n, \lim_{n\to+\infty} x_n) = F(v, w, u),$$

and

$$\lim_{n\to+\infty} z_{n+1} = \lim_{n\to+\infty} F(z_n, x_n, y_n) = F(\lim_{n\to+\infty} z_n, \lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n) = F(w, u, v).$$

By the uniqueness of the limits, we have $u = F(u, v, w), v = F(v, w, u)$ and $w = F(w, u, v)$. Therefore, (u, v, w) is a tripled fixed point of F. We can also prove $q(u, u) = 0, q(v, v) = 0$ and $q(w, w) = 0$ as in case of Theorem 3.4.

Example 3.7. Consider a mapping $F : X^3 \to X$ by $F(x, y, z) = \frac{3x+2y+z}{12}$ for all $(x, y, z) \in X^3$. Let $M = X^6$. Then M is an F-invariant set. Assume that $x, y, z, x^*, y^*, z^* \in X$ with $(x, y, z, x^*, y^*, z^*) \in M$ or $(x^*, y^*, z^*, x, y, z) \in M$. Since M is F-invariant, there exist $x_0, y_0, z_0 \in X$ such that

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) \in M.$$

Now, applying the contractive condition we have

$$\begin{aligned}
&q(F(x,y,z), F(x^*,y^*,z^*)) + q(F(y,z,x), F(y^*), z^*, x^*) + q(F(z,x,y), F(z^*), x^*, y^*) \\
&= (F(x^*), y^*, z^*), F(x^*), y^*, z^*)) + (F(y^*), z^*, x^*), F(y^*), z^*, x^*)) \\
&+(F(z^*), x^*, y^*), F(z^*), x^*, y^*)) \\
&= (\frac{3x^* + 2y^* + z^*}{12}, \frac{3x^* + 2y^* + z^*}{12}) + (\frac{3y^* + 2z^* + x^*}{12}, \frac{3y^* + 2z^* + x^*}{12}) \\
&+(\frac{3z^* + 2x^* + y^*}{12}, \frac{3z^* + 2x^* + y^*}{12}) \\
&= \frac{1}{2}(x^*, x^*) + (y^*, y^*) + (z^*, z^*) \\
&\leq k(q(x, x^*) + q(y, y^*) + q(z, z^*)),
\end{aligned}$$

where $k = \frac{2}{3} \in [0, 1)$. Hence, all the conditions of Theorem 3.4 are satisfied. Therefore, F has a tripled fixed point. It is easy to say that $(0, 0, 0)$ is the tripled fixed point of F.

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