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COMMON FIXED POINTS OF SUZUKI TYPE $Z-$CONTRACTION OF TWO PAIRS OF SELFMAPS WITH A RATIONAL EXPRESSION VIA $\Psi-$SIMULATION FUNCTION

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Abstract: In this paper, we define Suzuki type $Z-$contraction with a rational expression via $\Psi-$simulation function and prove the existence and uniqueness of common fixed points of two pairs of selfmaps by using reciprocal continuity and weakly compatible property. An example is provided in support of our result.

Keywords and Phrases: Common fixed points, simulation function, compatible maps, weakly compatible maps, reciprocal continuity.

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1. Introduction

In 2015, Khojasteh, Shukla and Radenović [11] introduced the concept of simulation function ζ, and the notion of Z-contraction with respect to ζ, which generalizes the Banach contraction principle and unifies several known types of contractions in complete metric spaces. The technique of using a simulation function in establishing the existence of fixed points became famous by the works of Karapınar [9], Olgun, Bicer and Alyildiz [13], Dolícanin-Dekić [6], Karapınar and Agarwal [10], Alqahtani and Karapınar [1], Aydi, Karapınar and Rakočević [3], Roldán López de Hierro, Karapınar, Roldán López de Hierro and Martínez-Moreno [16].

In this paper, we denote $\mathbb{R}^+ = [0, +\infty)$ and $N=$ the set of all natural numbers. $\Psi = \{\psi \mid \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotonically increasing and $\psi(t) = 0$ if and only if $t = 0\}$.

Definition 1.1. *A function $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a $\Psi-$simulation function if there exists $\psi \in \Psi$ such that*

(η_1) $\eta(0,0) = 0$;

(η_2) $\eta(t,s) < \psi(s) - \psi(t)$ *for all* $s, t > 0$;

(η_3) *If* $\{t_n\}$, $\{s_n\}$ *are sequences in* $(0, +\infty)$ *such that* $\lim\limits_{n\to+\infty} t_n = \lim\limits_{n\to+\infty} s_n > 0$, *then* $\limsup\limits_{n\to+\infty} \eta(t_n, s_n) < 0$.

Here we note that if ψ is the identity map in (η_2), then we call η is a simulation function (see [9, Definition 2.1]).

Example 1.1. Define $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\eta(t,s) = \frac{1}{2}s^2 - t^2, s \geq 0, t \geq 0$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t^2, t \geq 0$. Then
$\eta(t,s) = \frac{1}{2}\psi(s) - \psi(t) < \psi(s) - \psi(t)$ so that (η_2) holds.
Further (η_1) and (η_3) hold trivially. Therefore η is a $\Psi-$simulation function.
But it is not a simulation function, since
$\eta(10,2) = \frac{10^2}{2} - 2^2 = 50 - 4 \not< 8$ and so (η_2) fails to hold when ψ is the identity map.

For more examples on simulation functions, we refer [11].

The following are examples of $\Psi-$simulation functions.

Example 1.2. [2, Example 2] Define $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

(i) $\eta(t,s) = \alpha\psi(s) - \psi(t)$, for all $s \geq 0, t \geq 0$, where $\alpha \in [0,1)$, for some $\psi \in \Psi$.

(ii) $\eta(t,s) = \phi(\psi(s)) - \psi(t)$, for all $s \geq 0, t \geq 0$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\phi(0) = 0, 0 < \phi(s) < s$ for each $s > 0$ and $\limsup\limits_{t\to s} \phi(t) < s$. (for instance, $\phi(s) = \alpha s$ where $0 \leq \alpha < 1$), for some $\psi \in \Psi$.

(iii) $\eta(t,s) = \phi(s)\psi(s) - \psi(t)$, for all $s \geq 0, t \geq 0$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\limsup\limits_{t\to s} \phi(t) < 1$ for each $s > 0$ and for some $\psi \in \Psi$.

(iv) $\eta(t,s) = \psi(s) - \phi(s) - \psi(t)$, for all $s \geq 0, t \geq 0$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\phi(0) = 0$ and $\liminf\limits_{t\to s} \phi(t) > 0$ for each $s > 0$ and for some $\psi \in \Psi$.

Notation. We denote Z_Ψ for the set of all $\Psi-$simulation functions.

Theorem 1.1. [2, Theorem 2] *Let (X,d) be a complete metric space and let $A, B : X \to X$ be two maps such that*

$$\frac{1}{2}\min\{d(x,Ax), d(y,By)\} \leq d(x,y) \text{ implies } \eta(d(Ax,By), m(x,y)) \geq 0, \quad (1)$$

where $\eta \in Z_\Psi$ *and*

$$m(x,y) = \begin{cases} \max\{d(x,y), \frac{d(x,Ax)d(y,By)}{d(x,y)}\} & if\ \ x \neq y \\ d(Ax,Bx) & if\ \ x = y \end{cases}$$

for all $x, y \in X$*. If* A *and* B *are continuous then* A *and* B *have a unique common fixed point in* X *(that is, there is a unique* $u \in X$ *such that* $Au = Bu = u$*).*

A pair of maps that satisfy (1) is known as Suzuki type Z-contraction pair of selfmaps. For more works in the direction of finding the existence of fixed points/common fixed points, we refer [4], [12] and [14].

Definition 1.2. (Jungck [7]) *A pair* (f, g) *of selfmaps of a metric space* (X, d) *is said to be compatible if* $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = z$ *for some* $z \in X$*.*

Definition 1.3. (Jungck and Rhoades [8]) *Let* f *and* g *be selfmaps of a metric space* (X, d)*. The pair* (f, g) *is said to be weakly compatible if they commute at their coincidence points i.e.,* $fgx = gfx$ *whenever* $fx = gx, x \in X$*.*

Every compatible pair of maps is weakly compatible, but its converse is not true (see [8, Example 5.1]).

Definition 1.4. (Pant [15]) *Two selfmaps* f *and* g *of a metric space* (X, d) *are called reciprocally continuous if* $\lim\limits_{n\to\infty} fgx_n = fz$ *and* $\lim\limits_{n\to\infty} gfx_n = gz$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = z$ *for some* z *in* X*.*

Motivated by the works of Alsubaie, Alqahtani, Karapınar, and Roldán López de Hierro [2], in Section 2 of this paper, we extend Theorem 1.1 to two pairs of selfmaps by using reciprocal continuity and weakly compatible property. For this purpose, we define a Suzuki type $Z-$contraction of two pairs of selfmaps with a rational expression via $\Psi-$simulation function and prove the existence and uniqueness of common fixed points. In Section 3, we draw some corollaries to our main result and give an example in support of our main result.

2. Main result

In the following, we introduce Suzuki type $Z-$contraction with a rational expression via $\Psi-$simulation function for two pairs of selfmaps.

Definition 2.1. *Let* (X, d) *be a metric space. Let* A, B, S *and* T *be selfmaps of* X *which satisfy the following condition: if there exists a* Ψ*-simulation function* η

such that

$$\begin{aligned}\frac{1}{2}\min\{d(Sx,Ax),d(Ty,By)\} \leq \max\{d(Sx,Ty),d(Ax,By)\}\\ \textit{implies } \eta(d(Ax,By),m(x,y)) \geq 0\end{aligned} \tag{2}$$

where

$$m(x,y) = \begin{cases} \max\{d(Sx,Ty), \frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)}\} & \text{if } Sx \neq Ty \\ d(Ax,By) & \text{if } Sx = Ty \end{cases} \tag{3}$$

for all $x, y \in X$. Then we say that the pairs (A,S) and (B,T) satisfy Suzuki type $Z-$contraction with a rational expression with respect to a $\Psi-$simulation function η.

Let A, B, S and T be maps from a metric space (X,d) into itself and satisfying

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X).$$

Thus, for any $x_0 \in X$, there exists $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. Similarly, for $x_1 \in X$, we choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on.
In general, we define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \tag{4}$$

for $n = 0, 1, 2, \ldots$.

The following lemma is useful to prove that the sequence $\{y_n\}$ is Cauchy in X.

Lemma 2.1. (Babu and Sailaja [5]) *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*
i) $\lim\limits_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \epsilon$ *ii)* $\lim\limits_{k\to+\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$
iii) $\lim\limits_{k\to+\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$ *iv)* $\lim\limits_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon.$

We initiate this paper with the following two propositions which we apply to prove our main result.

Proposition 2.1. *Let (X,d) be a metric space. Let A, B, S and T be selfmaps of X and the pairs (A,S) and (B,T) satisfy Suzuki type $Z-$contraction with a rational expression with respect to a $\Psi-$simulation function η. Then we have the following:*

(i) *If $A(X) \subseteq T(X)$ and the pair (B,T) is weakly compatible, and if z is a common fixed point of A and S then z is a common fixed point of A, B, S and T and it is unique.*

(ii) *If $B(X) \subseteq S(X)$ and the pair (A, S) is weakly compatible, and if z is a common fixed point of B and T then z is a common fixed point of A, B, S and T and it is unique.*

Proof. First we prove (i).
Let z be a common fixed point of A and S.
Then $Az = Sz = z$.
Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = Az$.
Therefore $Az = Sz = Tu = z$.
We now prove that $Az = Bu$.
Suppose that $Az \neq Bu$.
Now, since $Az = Sz$, we have
$\frac{1}{2}\min\{d(Sz, Az), d(Tu, Bu)\} = 0 \leq \max\{d(Tu, Sz), d(Az, Bu)\}$, and hence
$\eta(d(Az, Bu), m(z, u)) \geq 0$,
where $m(z, u) = d(Az, Bu)$, since $Sz = Tu = z$.
Therefore $0 \leq \eta(d(Az, Bu), d(Az, Bu))$
$< \psi(d(Az, Bu)) - \psi(d(Az, Bu)) = 0$,
a contradiction.
Therefore $Az = Bu$.
Hence $Bu = Az = Sz = Tu = z$.
Since the pair (B, T) is weakly compatible and $Tu = Bu$, we have
$BTu = TBu$, i.e., $Bz = Tz$.
We now prove that $Bz = z$.
Suppose that $Bz \neq z$. So $Bz \neq Az$. Since $Az = Sz$, clearly
$\frac{1}{2}\min\{d(Sz, Az), d(Tz, Bz)\} = 0 \leq \max\{d(Tz, Sz), d(Az, Bz)\}$, and hence
$\eta(d(Az, Bz), m(z, z)) \geq 0$,
where $m(z, z) = d(Az, Bz)$, since $Sz = Tz = z$.
Therefore $0 \leq \eta(d(Az, Bz), d(Az, Bz))$
$< \psi(d(Az, Bz)) - \psi(d(Az, Bz)) = 0$,
a contradiction.
Therefore $Bz = z$.
Hence $Bz = Tz = z$.
Therefore $Az = Bz = Sz = Tz = z$.
Therefore z is a common fixed point of A, B, S and T.
Suppose z' is also a common fixed point of A, B, S, and T, with $z \neq z'$, then
$Az' = Bz' = Sz' = Tz' = z'$ and $d(z, z') > 0$.
Now, we have
$\frac{1}{2}\min\{d(Sz, Az), d(Tz', Bz')\} = 0 \leq \max\{d(Tz', Sz), d(Az, Bz')\}$, and hence
$\eta(d(Az, Bz'), m(z, z')) \geq 0$, which implies that $\eta(d(z, z'), m(z, z')) \geq 0$

where $m(z, z') = \max\{d(Sz, Tz'), \frac{d(Sz,Az)d(Tz',Bz')}{d(Sz,Tz')}\}$
$= d(Sz, Tz') = d(z, z')$.
Therefore $0 \leq \eta(d(z, z'), d(z, z'))$
$< \psi(d(z, z')) - \psi(d(z, z')) = 0$,
a contradiction.
Therefore $z = z'$. Hence z is the unique common fixed point of A, B, S and T.
Let us now prove (ii).
Let z be a common fixed point of B and T.
Then $Bz = Tz = z$. Since $B(X) \subseteq S(X)$, there exists $v \in X$ such that $Sv = z$.
Therefore $Bz = Tz = Sv = z$.
We now prove that $Bz = Av$.
Suppose that $Bz \neq Av$.
Now, since $Tz = Bz$, we have
$\frac{1}{2}\min\{d(Sv, Av), d(Tz, Bz)\} = 0 \leq \max\{d(Tz, Sv), d(Av, Bz)\}$, and hence
$\eta(d(Av, Bz), m(v, z)) \geq 0$,
where $m(v, z) = d(Av, Bz)$, since $Sv = Tz = z$.
Therefore $\eta(d(Av, Bz), d(Av, Bz)) \geq 0$, i.e.,
$0 \leq \eta(d(Av, Bz), d(Av, Bz))$
$< \psi(d(Av, Bz)) - \psi(d(Av, Bz)) = 0$,
a contradiction.
Therefore $Av = Bz$.
Hence $Av = Bz = Sv = Tz = z$.
Since the pair (A, S) is weakly compatible and $Av = Sv$, we have
$ASv = SAv$, i.e., $Az = Sz$.
We now prove that $Az = z$.
Suppose that $Az \neq z$. Since $Az = Sz$, clearly
$\frac{1}{2}\min\{d(Sz, Az), d(Tz, Bz)\} = 0 \leq \max\{d(Tz, Sz), d(Az, Bz)\}$, and hence
$\eta(d(Az, Bz), m(z, z)) \geq 0$,
where $m(z, z) = d(Az, Bz)$, since $Sz = Tz = z$.
Therefore $0 \leq \eta(d(Az, Bz), d(Az, Bz))$
$< \psi(d(Az, Bz)) - \psi(d(Az, Bz)) = 0$,
a contradiction.
Therefore $Az = z$.
Hence $Az = Sz = z$.
Therefore $Az = Bz = Sz = Tz = z$.
Therefore z is a common fixed point of A, B, S and T.
Suppose z' is also a common fixed point of A, B, S, and T, with $z \neq z'$, then
$Az' = Bz' = Sz' = Tz' = z'$ and $d(z, z') > 0$.

Now, we have
$\frac{1}{2}\min\{d(Sz,Az),d(Tz',Bz')\}=0\leq \max\{d(Tz',Sz),d(Az,Bz')\}$, and hence
$\eta(d(Az,Bz'),m(z,z'))\geq 0$, which implies that $\eta(d(z,z'),m(z,z'))\geq 0$
where $m(z,z')=\max\{d(Sz,Tz'),\frac{d(Sz,Az)d(Tz',Bz')}{d(Sz,Tz')}\}$
$=d(Sz,Tz')=d(z,z')$.
Therefore $0\leq \eta(d(z,z'),d(z,z'))$
$<\psi(d(z,z'))-\psi(d(z,z'))=0$,
a contradiction.
Therefore $z=z'$.

Hence z is the unique common fixed point of A,B,S and T.

Proposition 2.2. *Let (X,d) be a metric space. Let A,B,S and T be selfmaps of X such that $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$. If the pairs (A,S) and (B,T) satisfy Suzuki type $Z-$contraction with a rational expression with respect to a $\Psi-$simulation function η, then for any $x_0\in X$, the sequence $\{y_n\}$ defined by (4) for $n=0,1,2,...$, is Cauchy in X.*

Proof. Let $x_0\in X$ be arbitrary. Let $\{y_n\}$ be a sequence defined by (4).
Assume that $y_n=y_{n+1}$ for some n.
First we suppose that n is even.
We write $n=2m$ for some $m\in\mathbb{N}$.
Clearly, we have
$\frac{1}{2}\min\{d(y_{2m+1},y_{2m+2}),d(y_{2m},y_{2m+1})\}=0\leq\max\{d(y_{2m+1},y_{2m}),d(y_{2m+2},y_{2m+1})\}$,
i.e., $\frac{1}{2}\min\{d(Sx_{2m+2},Ax_{2m+2}),d(Tx_{2m+1},Bx_{2m+1})\}$
$\leq\max\{d(Sx_{2m+2},Tx_{2m+1}),d(Ax_{2m+2},Bx_{2m+1})\}$.
Therefore $\eta(d(Ax_{2m+2},Bx_{2m+1}),m(x_{2m+2},x_{2m+1}))\geq 0$,
where $m(x_{2m+2},x_{2m+1})=\max\{d(Sx_{2m+2},Tx_{2m+1}),\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+1},Bx_{2m+1})}{d(Sx_{2m+2},Tx_{2m+1})}\}$
$=\max\{d(y_{2m+1},y_{2m}),\frac{d(y_{2m+1},y_{2m+2})d(y_{2m},y_{2m+1})}{d(y_{2m+1},y_{2m})}\}$
$=\max\{d(y_{2m+1},y_{2m}),d(y_{2m+1},y_{2m+2})\}$.
$=\max\{0,d(y_{2m+1},y_{2m+2})\}=d(y_{2m+1},y_{2m+2})$.
$0\leq\eta(d(y_{2m+2},y_{2m+1}),d(y_{2m+2},y_{2m+1}))$
$<\psi(d(y_{2m+2},y_{2m+1}))-\psi(d(y_{2m+2},y_{2m+1}))=0$ provided $d(y_{2m+2},y_{2m+1})>0$,
a contradiction.
Therefore $d(y_{2m+2},y_{2m+1})=0$ so that $y_{2m+2}=y_{2m+1}$, and hence

$$y_{2m+2}=y_{2m+1}=y_{2m}. \tag{5}$$

We now suppose that n is odd.
We write $n=2m+1$ for some $m\in\mathbb{N}$.
Clearly, we have

$\frac{1}{2}\min\{d(y_{2m+1},y_{2m+2}),d(y_{2m+2},y_{2m+3})\}=0\leq\max\{d(y_{2m+1},y_{2m+2}),d(y_{2m+2},y_{2m+3})\}$,
i.e., $\frac{1}{2}\min\{d(Sx_{2m+2},Ax_{2m+2}),d(Tx_{2m+3},Bx_{2m+3})\}$
$\leq\max\{d(Sx_{2m+2},Tx_{2m+3}),d(Ax_{2m+2},Bx_{2m+3})\}$.
Therefore $\eta(d(Ax_{2m+2},Bx_{2m+3}),m(x_{2m+2},x_{2m+3}))\geq 0$
where $m(x_{2m+2},x_{2m+3})=\max\{d(Sx_{2m+2},Tx_{2m+3}),\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+3},Bx_{2m+3})}{d(Sx_{2m+2},Tx_{2m+3})}\}$
$=\max\{d(y_{2m+1},y_{2m+2}),\frac{d(y_{2m+1},y_{2m+2})d(y_{2m+2},y_{2m+3})}{d(y_{2m+1},y_{2m+2})}\}$
$=\max\{d(y_{2m+1},y_{2m+2}),d(y_{2m+2},y_{2m+3})\}$.
If maximum of $\{d(y_{2m+1},y_{2m+2}),d(y_{2m+2},y_{2m+3})\}=d(y_{2m+2},y_{2m+3})$ then
$0\leq\eta(d(y_{2m+2},y_{2m+3}),d(y_{2m+2},y_{2m+3}))$
$<\psi(d(y_{2m+2},y_{2m+3}))-\psi(d(y_{2m+2},y_{2m+3}))=0$,
a contradiction.
Therefore $m(x_{2m+2},x_{2m+3})=d(y_{2m+1},y_{2m+2})$.
Hence $d(y_{2m+2},y_{2m+3})\leq d(y_{2m+1},y_{2m+2})=0$.
Therefore $d(y_{2m+2},y_{2m+3})=0$.
Hence $y_{2m+2}=y_{2m+3}$.
Therefore

$$y_{2m+3}=y_{2m+2}=y_{2m+1}. \tag{6}$$

Therefore from (5) and (6), we have $y_{n+k}=y_n$ for all $k=1,2,3,\ldots$.
Hence $\{y_n\}$ is a constant sequence so that $\{y_n\}$ is Cauchy.
We now show that $\{y_n\}$ is Cauchy when $y_n\neq y_{n+1}$ for all $n=1,2,\ldots$. Assume that $y_n\neq y_{n+1}$ for all $n\in\mathbb{N}$.

Case (i): n is odd.
We write $n=2m+1$ for some $m\in\mathbb{N}$.
Clearly, we have
$\frac{1}{2}\min\{d(y_{2m+1},y_{2m+2}),d(y_{2m},y_{2m+1})\}\leq\max\{d(y_{2m+1},y_{2m}),d(y_{2m+2},y_{2m+1})\}$,
i.e., $\frac{1}{2}\min\{d(Sx_{2m+2},Ax_{2m+2}),d(Tx_{2m+1},Bx_{2m+1})\}$
$\leq\max\{d(Sx_{2m+2},Tx_{2m+1}),d(Ax_{2m+2},Bx_{2m+1})\}$ and
hence $\eta(Ax_{2m+2},Bx_{2m+1}),m(x_{2m+2},x_{2m+1})\geq 0$
where $m(x_{2m+2},x_{2m+1})=\max\{d(Sx_{2m+2},Tx_{2m+1}),\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+1},Bx_{2m+1})}{d(Sx_{2m+2},Tx_{2m+1})}\}$
$=\max\{d(y_{2m+1},y_{2m}),\frac{d(y_{2m+1},y_{2m+2})d(y_{2m},y_{2m+1})}{d(y_{2m+1},y_{2m})}\}$
$=\max\{d(y_{2m+1},y_{2m}),d(y_{2m+1},y_{2m+2})\}$.
If maximum of $\{d(y_{2m+1},y_{2m}),d(y_{2m+1},y_{2m+2})\}=d(y_{2m+1},y_{2m+2})$, then
$0\leq\eta(d(y_{2m+2},y_{2m+1}),d(y_{2m+2},y_{2m+1}))$
$<\psi(d(y_{2m+2},y_{2m+1}))-\psi(d(y_{2m+2},y_{2m+1}))=0$,
a contradiction.
Therefore $d(y_{2m+1},y_{2m+2})\leq d(y_{2m},y_{2m+1})$.

Case (ii): n is even.
We write $n = 2m$ for $m \in \mathbb{N}$.
Clearly, we have
$\frac{1}{2}\min\{d(y_{2m-1}, y_{2m}), d(y_{2m-2}, y_{2m-1})\} \leq \max\{d(y_{2m-1}, y_{2m-2}), d(y_{2m}, y_{2m-1})\}$,
i.e., $\frac{1}{2}\min\{d(Sx_{2m}, Ax_{2m}), d(Tx_{2m-1}, Bx_{2m-1})\}$
$\leq \max\{d(Sx_{2m}, Tx_{2m-1}), d(Ax_{2m}, Bx_{2m-1})\}$
implies $\eta(d(Ax_{2m}, Bx_{2m-1}), m(x_{2m}, x_{2m-1})) \geq 0$
where $m(x_{2m}, x_{2m-1}) = \max\{d(Sx_{2m}, Tx_{2m-1}), \frac{d(Sx_{2m}, Ax_{2m})d(Tx_{2m-1}, Bx_{2m-1})}{d(Sx_{2m}, Tx_{2m-1})}\}$
$= \max\{d(y_{2m-1}, y_{2m-2}), \frac{d(y_{2m-1}, y_{2m})d(y_{2m-2}, y_{2m-1})}{d(y_{2m-1}, y_{2m-2})}\}$
$= \max\{d(y_{2m-1}, y_{2m-2}), d(y_{2m-1}, y_{2m})\}$.
If maximum of $\{d(y_{2m-1}, y_{2m-2}), d(y_{2m-1}, y_{2m})\} = d(y_{2m-1}, y_{2m})$, then
$0 \leq \eta(d(y_{2m}, y_{2m-1}), d(y_{2m}, y_{2m-1}))$
$< \psi(d(y_{2m}, y_{2m-1})) - \psi(d(y_{2m}, y_{2m-1})) = 0$,
a contradiction.
Therefore $m(x_{2m}, x_{2m-1}) = d(y_{2m-1}, y_{2m-2})$.
Therefore $d(y_{2m}, y_{2m-1}) \leq d(y_{2m-1}, y_{2m-2})$.
Therefore from Cases (i) and (ii), we have

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n) \text{ for all } n = 1, 2, \dots . \tag{7}$$

Therefore $\{d(y_n, y_{n+1})\}$ is a nonnegative monotone decreasing sequence of reals.
Let $\lim_{n\to+\infty} d(y_n, y_{n+1}) = r, r \geq 0$.
We now show that $r = 0$.
Suppose $r > 0$. Since $\lim_{n\to+\infty} d(y_{2n}, y_{2n+1}) = r$ and $\lim_{n\to+\infty} d(y_{2n-1}, y_{2n}) = r$.
We have
$\frac{1}{2}\min\{d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}) \leq \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}$,
and hence $\eta(d(y_{2n+2}, y_{2n+1}), m(x_{2n+2}, x_{2n+1}) \geq 0$
where $m(x_{2n+2}, x_{2n+1}) = \max\{d(y_{2n+1}, y_{2n}), \frac{d(y_{2n+1}, y_{2n+2})d(y_{2n}, y_{2n+1})}{d(y_{2n+1}, y_{2n})}\}$
$= \max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2})\}$.
If maximum of $\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2})\} = d(y_{2n+1}, y_{2n+2})$, then
$0 \leq \eta(d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}))$
$< \psi(d(y_{2n+2}, y_{2n+1})) - \psi(d(y_{2n+2}, y_{2n+1})) = 0$,
a contradiction.
Therefore $m(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n})$.
Therefore $0 \leq \eta(d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}))$.
On letting $\limsup_{n\to+\infty}$, and by using (η_3) we get
$0 \leq \limsup_{n\to+\infty} \eta(d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n})) < 0$,

a contradiction.
Therefore $r = 0$, i.e., $\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0$.
We now show that $\{y_n\}$ is Cauchy.
It is sufficient to show that $\{y_{2n}\}$ is Cauchy.

Suppose that $\{y_{2n}\}$ is not Cauchy. Then, by Lemma 2.1, it follows that there exist $\epsilon > 0$ and subsequences $\{y_{2n_k}\}$ and $\{y_{2m_k}\}$ such that $n_k > m_k > k$ and $d(y_{2n_k}, y_{2m_k}) \geq \epsilon$ and $d(y_{2m_k-1}, y_{2n_k}) < \epsilon$ and

(i) $\lim_{k\to+\infty} d(y_{2n_k}, y_{2m_k}) = \epsilon$ (ii) $\lim_{k\to+\infty} d(y_{2n_k}, y_{2m_k-1}) = \epsilon$

(iii) $\lim_{k\to+\infty} d(y_{2n_k+1}, y_{2m_k}) = \epsilon$ (iv) $\lim_{k\to+\infty} d(y_{2n_k-1}, y_{2m_k-2}) = \epsilon$.

Therefore, $m(x_{2n_k}, x_{2m_k-1}) = \max\{d(Sx_{2n_k}, Tx_{2m_k-1}), \frac{d(Sx_{2n_k},Ax_{2n_k})d(Tx_{2m_k-1},Bx_{2m_k-1})}{d(Sx_{2n_k},Tx_{2m_k-1})}\}$.
On letting $k \to +\infty$, we have

$$\begin{aligned}\lim_{k\to+\infty} m(x_{2n_k}, x_{2m_k-1}) &= \lim_{k\to+\infty} \max\{d(y_{2n_k-1}, y_{2m_k-2}), \tfrac{d(y_{2n_k-1},y_{2n_k})d(y_{2m_k-2},y_{2m_k-1})}{d(y_{2n_k-1},y_{2m_k-2})}\}\\ &= \max\{\epsilon, 0\} = \epsilon.\end{aligned}$$

Also $\lim_{k\to+\infty} d(y_{2n_k}, y_{2m_k-1}) = \epsilon$.
Therefore from (η_3), we have

$$\limsup_{n\to+\infty} \eta(d(y_{2n_k}, y_{2m_k-1}), m(x_{2n_k}, x_{2m_k-1})) < 0. \tag{8}$$

We now show that if $n_k > m_k > k$,

$$\begin{aligned}&\frac{1}{2}\min\{d(Sx_{2n_k}, Ax_{2n_k}), d(Tx_{2m_k-1}, Bx_{2m_k-1})\}\\ &\qquad\qquad \leq \max\{d(Sx_{2n_k}, Tx_{2m_k-1}), d(Ax_{2n_k}, Bx_{2m_k-1})\}.\end{aligned} \tag{9}$$

Since $n_k > m_k$ and $\{d(y_n, y_{n+1})\}$ is decreasing, we have

$$\begin{aligned}d(Sx_{2n_k}, Ax_{2n_k}) &= d(y_{2n_k-1}, y_{2n_k})\\ &\leq d(y_{2m_k}, y_{2m_k-1})\\ &\leq d(y_{2m_k-1}, y_{2m_k-2})\\ &= d(Bx_{2m_k-1}, Tx_{2m_k-1}).\end{aligned}$$

Therefore $\frac{1}{2}\min\{d(Sx_{2n_k}, Ax_{2n_k}), d(Tx_{2m_k-1}, Bx_{2m_k-1})\} \leq \frac{1}{2}d(Tx_{2m_k-1}, Bx_{2m_k-1})$.
Therefore for sufficiently large $k, n_k > m_k > k$, we show that
$\frac{1}{2}d(Tx_{2m_k-1}, Bx_{2m_k-1}) \leq \max\{d(Sx_{2n_k}, Tx_{2m_k-1}), d(Ax_{2n_k}, Bx_{2m_k-1})\}$.
Suppose $\max\{d(Sx_{2n_k}, Tx_{2m_k-1}), d(Ax_{2n_k}, Bx_{2m_k-1})\} = d(Sx_{2n_k}, Tx_{2m_k-1})$. Then $\frac{1}{2}d(Tx_{2m_k-1}, Bx_{2m_k-1}) \leq d(Sx_{2n_k}, Tx_{2m_k-1})$, i.e.,

$$\frac{1}{2}d(y_{2m_k-2}, y_{2m_k-1}) \leq d(y_{2n_k-1}, y_{2m_k-2}). \tag{10}$$

Let $\epsilon > 0$ be given. Since $\lim\limits_{n\to+\infty} d(y_{2n_k}, y_{2n_k+1}) = 0$, there exists $N_1 \in \mathbb{N}$ such that for $k \geq N_1, d(y_{2n_k-1}, y_{2n_k}) < \frac{\epsilon}{8}$.
Also there exists $N_2 \in \mathbb{N}$ such that for $k \geq N_2, d(y_{2m_k-2}, y_{2m_k-1}) < \frac{\epsilon}{8}$ and $d(y_{2m_k-1}, y_{2m_k}) < \frac{\epsilon}{8}$.
Therefore $k > \max\{N_1, N_2\}$ and for $n_k > m_k > k$,

$$\begin{aligned}\epsilon \leq d(y_{2n_k}, y_{2m_k}) &\leq d(y_{2n_k}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2m_k-1}) \\ &\quad + d(y_{2m_k-1}, y_{2m_k}) \\ &< \frac{\epsilon}{8} + d(y_{2n_k-1}, y_{2m_k-2}) + \frac{\epsilon}{8} + \frac{\epsilon}{8}, \text{ and hence}\end{aligned}$$

$\frac{5\epsilon}{8} < d(y_{2n_k-1}, y_{2m_k-2})$.
Therefore for $k > \max\{N_1, N_2\}$ and $n_k > m_k > k$,

$$\begin{aligned}d(y_{2m_k-2}, y_{2m_k-1}) < \frac{\epsilon}{8} \leq \frac{5\epsilon}{8} &< d(y_{2n_k-1}, y_{2m_k-2}) \\ &\leq \max\{d(y_{2n_k-1}, y_{2m_k-2}), d(y_{2n_k}, y_{2m_k-1})\} \\ &= \max\{d(Sx_{2n_k}, Tx_{2m_k-1}), d(Ax_{2n_k}, Bx_{2m_k-1})\}.\end{aligned}$$

Therefore (10) holds, which in turn (9) holds.
Consequently, for sufficiently large $k \in \mathbb{N}$ and $n_k > m_k > k$, we have
$\eta(d(Ax_{2n_k}, Bx_{2m_k-1}), m(x_{2n_k}, x_{2m_k-1})) \geq 0$.
i.e., $\eta(d(y_{2n_k}, y_{2m_k-1}), m(x_{2n_k}, x_{2m_k-1})) \geq 0$, and hence
$\limsup\limits_{k\to+\infty} \eta(d(y_{2n_k}, y_{2m_k-1}), m(x_{2n_k}, x_{2m_k-1})) \geq 0$,
a contradiction to (8).
Therefore $\{x_n\}$ is Cauchy.
The following theorem is the main result of this paper.

Theorem 2.1. *Let (X, d) be a complete metric space. Let A, B, S and T be selfmaps of X and satisfy $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and the pairs (A, S) and (B, T) satisfy Suzuki type $Z-$contraction with a rational expression with respect to a $\Psi-$simulation function η. Further, assume that either*
(i) *(A, S) is a reciprocally continuous and compatible pair of maps and (B, T) is a pair of weakly compatible maps (or)*
(ii) *(B, T) is a reciprocally continuous and compatible pair of maps and (A, S) is a pair of weakly compatible maps.*
Then A, B, S and T have a unique common fixed point in X.
Proof. Let $x_0 \in X$ be arbitrary. We define the sequence $\{y_n\}$ by (4) for $n = 0, 1, 2, \ldots$.
By Proposition 2.2, it follows that $\{y_n\}$ is Cauchy in X.
Since X is complete, there exists $z \in X$ such that $\lim\limits_{n\to+\infty} y_n = z$.
Consequently, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also convergent to $z \in X$,

we have

$$\lim_{n\to+\infty} y_{2n} = \lim_{n\to+\infty} Ax_{2n} = \lim_{n\to+\infty} Tx_{2n+1} = z, \tag{11}$$

and

$$\lim_{n\to+\infty} y_{2n+1} = \lim_{n\to+\infty} Bx_{2n+1} = \lim_{n\to+\infty} Sx_{2n+2} = z. \tag{12}$$

Case (i): First, we assume that (i) holds.
Since (A, S) is reciprocally continuous, by using (11) and (12) it follows that $\lim_{n\to+\infty} ASx_{2n+2} = Az$ and $\lim_{n\to+\infty} SAx_{2n+2} = Sz$.
Since (A, S) is compatible, we have
$\lim_{n\to+\infty} d(ASx_{2n+2}, SAx_{2n+2}) = 0$, which implies that $\lim_{n\to+\infty} d(Az, Sz) = 0$.
Therefore $Az = Sz$.
We now show that $Az = z$.
Since $Sz = Az$, if we suppose that $Az \neq z$, then
$\frac{1}{2}\min\{d(Sz, Az), d(Tx_{2n+1}, Bx_{2n+1})\} = 0 \leq \max\{d(Tx_{2n+1}, Sz), d(Az, Bx_{2n+1})\}$
and hence

$$\eta(d(Az, Bx_{2n+1}), m(z, x_{2n+1})) \geq 0 \tag{13}$$

where $m(z, x_{2n+1}) = \max\{d(Sz, Tx_{2n+1}), \frac{d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{d(Sz,Tx_{2n+1})}\}$.
On letting $n \to +\infty$, we get

$$\lim_{n\to+\infty} m(z, x_{2n+1}) = \max\{d(Sz, z), \frac{d(Sz,Az)d(z,z)}{d(Sz,z)}\}$$
$$= d(Sz, z) = d(Az, z) > 0.$$

Also, since $\lim_{n\to+\infty} d(Az, Bx_{2n+1}) = d(Az, z)$, by applying (η_3) to (13), we have
$0 \leq \limsup_{n\to+\infty} \eta(d(Az, Bx_{2n+1}), m(z, x_{2n+1})) < 0$,
a contradiction.
Therefore $Az = z$, so that z is a common fixed point of A and S.
Now, by Proposition 2.1, it follows that $Az = Bz = Sz = Tz = z$.

Case (ii): We now assume that (ii) holds.
Since (B, T) is reciprocally continuous, by using (11) and (12), it follows that $\lim_{n\to+\infty} BTx_{2n+1} = Bz$ and $\lim_{n\to+\infty} TBx_{2n+1} = Tz$.
Since (B, T) is compatible, we have
$\lim_{n\to+\infty} d(BTx_{2n+1}, TBx_{2n+1}) = 0$, which implies that $\lim_{n\to+\infty} d(Bz, Tz) = 0$.
Therefore $Bz = Tz$.
We now show that $Tz = z$.
Since $Tz = Bz$, if we suppose that $Tz \neq z$, then
$\frac{1}{2}\min\{d(Sx_{2n+2}, Ax_{2n+2}), d(Tz, Bz)\} = 0 \leq \max\{d(Sx_{2n+2}, Tz), d(Ax_{2n+2}, Bz)\}$

which implies that

$$\eta(d(Ax_{2n+2}, Bz), m(x_{2n+2}, z)) \geq 0 \tag{14}$$

where $m(x_{2n+2}, z) = \max\{d(Sx_{2n+2}, Tz), \frac{d(Sx_{2n+2}, Ax_{2n+2})d(Tz,Bz)}{d(Sx_{2n+2},Tz)}\}$.
On letting $n \to +\infty$, we get
$\lim\limits_{n\to+\infty} m(x_{2n+2}, z) = \max\{d(z, Tz), \frac{d(z,z)d(Tz,Bz)}{d(z,Tz)}\}$
$= d(z, Tz) > 0.$
Also, since $\lim\limits_{n\to+\infty} d(Ax_{2n+2}, Bz) = d(z, Bz)$, by applying (η_3), to (14) we have
$0 \leq \limsup\limits_{n\to+\infty} \eta(d(Ax_{2n+2}, Bz), m(x_{2n+2}, z)) < 0,$
a contradiction.
Therefore $Tz = z$, and hence z is a common fixed point of B and T.
Now by applying Proposition 2.1, it follows that $Az = Bz = Sz = Tz = z$.

Hence, it follows that z is a common fixed point of A, B, S and T.

3. Corollaries and an example

Corollary 3.1. *Suppose that $\{A_n\}_{n=1}^{+\infty}, S$ and T be selfmaps of a complete metric space (X, d) and satisfy $A_1(X) \subseteq S(X)$ and $A_1(X) \subseteq T(X)$ and the inequality*

$$\begin{gathered}\frac{1}{2}\min\{d(Sx,A_1x), d(Ty, A_jy) \leq \max d(Sx, Ty), d(A_1x, A_jy)\} \\ \textit{implies } \eta(d(A_1x, A_jy), m(x, y)) \geq 0\end{gathered} \tag{15}$$

where
$$m(x, y) = \begin{cases} \max\{d(Sx, Ty), \frac{d(Sx,A_1x)d(Ty,A_jy)}{d(Sx,Ty)}\} & \textit{if } \; Sx \neq Ty \\ d(A_1x, A_jy) & \textit{if } \; Sx = Ty \end{cases} \quad \textit{for all } x, y \in X.$$
Further, assume that either
(i) (A_1, S) *is a reciprocally continuous and compatible pair of maps and* (A_1, T) *is a pair of weakly compatible maps* (*or*)
(ii) (A_1, T) *is a reciprocally continuous and compatible pair of maps and* (A_1, S) *is a pair of weakly compatible maps.*

Then $\{A_n\}_{n=1}^{+\infty}, S$ and T have a unique common fixed point in X.

Proof. Under the assumptions of A_1, S and T, the existence of unique common fixed point z of A_1, S and T follows by choosing $A = B = A_1$ in Theorem 2.1.
Therefore $A_1z = Sz = Tz = z$.
Now, let $j \in \mathbb{N}$ with $j \neq 1$.
Since
$\frac{1}{2}\min\{d(Sz, A_1z), d(Tz, A_jz)\} = 0 \leq \max\{d(Sz, Tz), d(A_1z, A_jz)\}$,
we have

$\eta(d(A_1z, A_jz), m(z,z)) \geq 0$

where $m(z,z) = \begin{cases} \max\{d(Sz,Tz), \frac{d(Sz,A_1z)d(Tz,A_jz)}{d(Sz,Tz)}\} & \text{if } Sz \neq Tz \\ d(A_1z, A_jz) & \text{if } Sz = Tz. \end{cases}$

Since $Sz = Tz = z$, we have

$m(z,z) = d(A_1z, A_jz)$. Therefore

$0 \leq \eta(d(A_1z, A_jz), d(A_1z, A_jz))$

$< \psi(d(A_1z, A_jz)) - \psi(d(A_1z, A_jz)) = 0,$

a contradiction.

Therefore $A_1z = z = A_jz$ for $j = 1, 2, 3, ...$ and uniqueness of common fixed point follows from the inequality (15).

Hence $\{A_n\}_{n=1}^{+\infty}, S$ and T have a unique common fixed point in X.

Corollary 3.2. *Let (X, d) be a complete metric space and let A, B, S and T be selfmaps of X and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and for every $x, y \in X$,*

$$\frac{1}{2}\min\{d(Sx, Ax), d(Ty, By)\} \leq \max\{d(Sx, Ty), d(Ax, By)\} \tag{16}$$

which implies that

$$\psi(d(Ax, By)) \leq \psi(m(x,y)) - \phi(m(x,y)) \tag{17}$$

where $m(x,y)$ is as defined in Definition 2.1, $\psi \in \Psi$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\liminf\limits_{s\to t} \phi(s) > 0$ for each $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$. Further, assume that either (i) *or* (ii) *of Theorem 2.1 holds. Then A, B, S and T have a unique common fixed point in X.*

Proof. By defining $\eta(t,s) = \psi(s) - \phi(s) - \psi(t)$ for all $t, s \geq 0$ as in Example 1.2 (iv), we can easily see that η is a $\Psi-$simulation function.

Therefore the inequality (16) implies $\eta(d(Ax, By), m(x,y)) \geq 0$, so that (17) holds. Hence, by applying Theorem 2.1, the conclusion of the corollary follows.

Corollary 3.3. *Let (X, d) be a complete metric space and let A, B, S and T be selfmaps of X and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and for every $x, y \in X$,*

$$\frac{1}{2}\min\{d(Sx, Ax), d(Ty, By)\} \leq \max\{d(Sx, Ty), d(Ax, By)\}$$

which implies that

$$\psi(d(Ax, By)) \leq \psi(m(x,y)) - \phi(m(x,y))$$

where $\psi \in \Psi$ and ϕ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$, and $m(x,y)$ is as defined in Definition 2.1. Further, assume that either (i) *or* (ii) *of Theorem 2.1 holds. Then A, B, S and T have a unique common fixed point in X.*

Proof. Since ϕ is lower semi-continuous, if

$$\lim_{n\to+\infty} s_n = l > 0,$$

then, we have

$$\liminf_{n\to+\infty} \phi(s_n) \geq \phi(l) > 0.$$

Hence, by Corollary 3.2, the conclusion of this corollary follows.

Corollary 3.4. *Let (X, d) be a complete metric space and let A, B, S and T be selfmaps of X and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and for every $x, y \in X$,*

$$\frac{1}{2}\min\{d(Sx, Ax), d(Ty, By)\} \leq \max\{d(Sx, Ty), d(Ax, By)\}$$

which implies that

$$\psi(d(Ax, By)) \leq \rho(m(x, y))\psi(m(x, y))$$

where $\psi \in \Psi$, and $m(x, y)$ is as defined in Definition 2.1 and $\rho : \mathbb{R}^+ \to [0, 1)$ is a function such that $\rho(t) = 0$ if and only if $t = 0$ and $\limsup_{t\to s} \rho(t) < 1$ for each $s > 0$. Further, assume that either (i) *or* (ii) *of Theorem 2.1 holds. Then A, B, S and T have a unique common fixed point in X.*

Proof. By defining $\eta(t, s) = \rho(s)\psi(s) - \psi(t)$ for all $t, s \geq 0$, we can easily see that η is a $\Psi-$simulation function. Now by applying Theorem 2.1, the conclusion follows.

Corollary 3.5. *Let (X, d) be a complete metric space and let A, B, S and T be selfmaps of X and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and for every $x, y \in X$,*

$$\begin{gathered}\frac{1}{2}\min\{d(Sx, Ax), d(Ty, By)\} \leq \max\{d(Sx, Ty), d(Ax, By)\} \\ \textit{implies } \psi(d(Ax, By)) \leq \phi(\psi(m(x, y)))\end{gathered} \tag{18}$$

where $\psi \in \Psi$ and $m(x, y)$ is as defined in Definition 2.1 and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that for each $t > 0$ and $\phi(t) < t$, and $\limsup_{s\to t} \phi(s) < t$ and $\phi(t) = 0$ if and only if $t = 0$. Further, assume that either (i) *or* (ii) *of Theorem 2.1 holds. Then A, B, S and T have a unique common fixed point in X.*

Proof. Define $\eta(t,s) = \phi(\psi(s)) - \psi(t)$ for all $t, s \geq 0$ as in Example 1.2 (ii). Then η is a $\Psi-$simulation function. Therefore, from the inequality (18) we have

$$\frac{1}{2}\min\{d(Sx,Ax), d(Ty,By)\} \leq \max\{d(Sx,Ty), d(Ax,By)\}$$

implies

$$\eta(d(Ax,By), m(x,y)) = \phi(\psi(m(x,y))) - \psi(d(Ax,By)) \geq 0 \text{ for all } x, y \in X.$$

Therefore A, B, S and T satisfy the inequality (2). Hence, by Theorem 2.1, the conclusion holds.

Remark 3.1. *Suppose that $\psi \in \Psi$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semi-continuous function such that $\phi(t) < t$ for each $t > 0$ and $\phi(t) = 0$ if and only $t = 0$. Then for any sequence $\{s_n\}$ in $(0, +\infty)$ with $\lim\limits_{n\to+\infty} s_n = l > 0$, one can obtain that $\limsup\limits_{n\to+\infty} \phi(\psi(s_n)) < \psi(l)$.*

Corollary 3.6. *Let (X,d) be a complete metric space and let A, B, S and T be selfmaps of X and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and for every $x, y \in X$,*

$$\frac{1}{2}\min\{d(Sx,Ax), d(Ty,By)\} \leq \max\{d(Sx,Ty), d(Ax,By)\}$$

which implies that

$$\psi(d(Ax,By)) \leq \phi(\psi(m(x,y)))$$

where $\psi \in \Psi$ and $m(x,y)$ is as defined in Definition 2.1 and ϕ is defined as in Remark 3.1. Further, assume that either (i) *or* (ii) *of Theorem 2.1 holds. Then A, B, S and T have a unique common fixed point in X.*

Proof. By choosing $\eta(t,s)$ as in Corollary 3.5, and by applying Theorem 2.1, the conclusion follows.

Example 3.1. Let $X = [0,1]$ with the usual metric. We define selfmaps A, B, S and

T on X by $A(x) = \begin{cases} \frac{x^2}{4} \text{ if } x \in [0, \frac{1}{2}) \\ 0 \text{ if } x \in [\frac{1}{2}, 1] \end{cases}$, $B(x) = \begin{cases} \frac{x^2}{6} \text{ if } x \in [0, \frac{1}{2}) \\ 0 \text{ if } x \in [\frac{1}{2}, 1] \end{cases}$

$S(x) = \begin{cases} x^2 \text{ if } x \in [0, \frac{1}{2}) \\ 1 \text{ if } x \in [\frac{1}{2}, 1] \end{cases}$ and $T(x) = \begin{cases} \frac{x^2}{2} \text{ if } x \in [0, \frac{1}{2}) \\ 0 \text{ if } x \in [\frac{1}{2}, 1] \end{cases}$.

Then $A(X) = [0, \frac{1}{16})$, $B(X) = [0, \frac{1}{24})$, $S(X) = [0, \frac{1}{4}) \cup \{1\}$, $T(X) = [0, \frac{1}{8})$ so that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. We define $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\eta(t,s) = \frac{8}{9}s^2 - t^2$. We define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t^2, t \geq 0$. Then clearly

$\psi \in \Psi$ and $\eta(t,s) = \frac{8}{9}\psi(s) - \psi(t) = \alpha\psi(s) - \psi(t) < \psi(s) - \psi(t)$ where $\alpha = \frac{8}{9}$.
In the following we show that the pairs (A,S) and (B,T) satisfy the hypotheses of Theorem 2.1.
Since $\lim\limits_{n\to+\infty} Ax_n = 0$ and $\lim\limits_{n\to+\infty} Sx_n = 0$, we have
$\lim\limits_{n\to+\infty} ASx_n = \lim\limits_{n\to+\infty} Ax_n^2 = \lim\limits_{n\to+\infty} \frac{x_n^4}{4} = 0 = A0$ and
$\lim\limits_{n\to+\infty} SAx_n = \lim\limits_{n\to+\infty} S\frac{x_n^2}{4} = \lim\limits_{n\to+\infty} \frac{x_n^4}{16} = 0 = S0.$
Therefore $\lim\limits_{n\to+\infty} ASx_n = \lim\limits_{n\to+\infty} SAx_n$.
Therefore (A,S) is reciprocally continuous and compatible.
Since $Bx = Tx = 0$ for $x \in [\frac{1}{2}, 1]$, we have $TBx = T0 = 0 = B0 = BTx$.
Hence the pair (B,S) is weakly compatible.
We now verify the inequality (2).

Case (i): First, we suppose that $x \geq y$.

Subcase (i): Let $x, y \in [0, \frac{1}{2})$.
$d(Ax,Sx) = |\frac{x^2}{4} - x^2| = \frac{3}{4}x^2$, $d(Ax,By) = |\frac{x^2}{4} - \frac{y^2}{6}|$
$d(Ty,By) = |\frac{y^2}{6} - \frac{y^2}{2}| = \frac{1}{3}y^2$, $d(Sx,Ty) = |x^2 - \frac{y^2}{2}|$
$\frac{1}{2}\min\{d(Sx,Ax), d(Ty,By)\} \leq \max\{d(Sx,Ty), d(Ax,By)\}$, i.e.,
$\frac{1}{2}\min\{\frac{3}{4}x^2, \frac{1}{3}y^2\} \leq \max\{|x^2 - \frac{y^2}{2}|, |\frac{x^2}{4} - \frac{y^2}{6}|\}$.
If $\frac{1}{2}$ minimum of $\{\frac{3}{4}x^2, \frac{1}{3}y^2\}$ is $\frac{3}{8}x^2$, then
$\frac{3}{8}x^2 \leq \frac{1}{6}y^2$ implies $\frac{9}{4}x^2 \leq y^2$ which fails to hold because
$\frac{3}{8}x^2 \leq |\frac{x^2}{4} - \frac{y^2}{6}|$ implies $\frac{3}{8}x^2 \leq \frac{x^2}{4} - \frac{y^2}{6}$ implies $\frac{y^2}{6} \leq \frac{-3}{8}x^2 + \frac{x^2}{4} = \frac{-x^2}{8} < 0$.
Therefore $\frac{1}{2}$ minimum of $\{\frac{3}{4}x^2, \frac{1}{3}y^2\}$ is $\frac{1}{6}y^2$, and $\frac{1}{6}y^2 \leq |\frac{x^2}{4} - \frac{y^2}{6}| = \frac{x^2}{4} - \frac{y^2}{6}$
so that

$$y^2 \leq \frac{3}{4}x^2. \tag{19}$$

Therefore $\frac{1}{2}\min\{\frac{1}{3}y^2, \frac{3}{4}x^2\} \leq |\frac{x^2}{4} - \frac{y^2}{6}| \leq \max\{|\frac{x^2}{4} - \frac{y^2}{6}|, |x^2 - \frac{y^2}{2}|\}$.
Then

$$\begin{aligned}
\eta(d(Ax,By), m(x,y)) &= \tfrac{8}{9}m(x,y)^2 - d(Ax,By)^2 \\
&= \tfrac{8}{9}|x^2 - \tfrac{y^2}{2}|^2 - |\tfrac{x^2}{4} - \tfrac{y^2}{6}|^2 \\
&= \tfrac{8}{9}(x^2 - \tfrac{y^2}{2})^2 - (\tfrac{x^2}{4} - \tfrac{y^2}{6})^2 \\
&= \tfrac{8}{9}(x^4 + \tfrac{y^4}{4} - x^2y^2) - (\tfrac{x^4}{16} + \tfrac{y^4}{36} - \tfrac{x^2y^2}{12}) \\
&= (\tfrac{8}{9} - \tfrac{1}{16})x^4 + (\tfrac{8}{36} - \tfrac{1}{36})y^4 - (\tfrac{8}{9} + \tfrac{1}{12})x^2y^2 \\
&= \tfrac{117}{144}x^4 + \tfrac{7}{36}y^4 - \tfrac{35}{36}x^2y^2 \\
&\geq \tfrac{117}{144}x^4 + \tfrac{7}{36}y^4 - \tfrac{35}{36} \times \tfrac{3}{4}x^4 \qquad \text{(from (19))} \\
&= \tfrac{117}{144}x^4 - \tfrac{35}{48}x^4 + \tfrac{7}{36}y^4 \\
&= \tfrac{1}{12}x^4 + \tfrac{7}{36}y^4 \geq 0.
\end{aligned}$$

Subcase (ii): $x, y \in [\frac{1}{2}, 1]$.

$d(Sx, Ax) = 1, d(Ax, By) = 0, d(Ty, By) = 0, d(Sx, Ty) = 1.$

$$\begin{aligned}\tfrac{1}{2}\min\{d(Ax, Sx), d(Ty, By)\} &= \tfrac{1}{2}\min\{1, 0\}\\ &= 0 \leq \max\{0, 1\} = \max\{d(Sx, Ty), d(Ax, By)\}\end{aligned}$$

and hence $\eta(d(Ax, By), m(x, y)) = \frac{8}{9}m(x, y)^2 - d(Ax, By)^2 = \frac{8}{9}m(x, y)^2 \geq 0.$

Subcase (iii): Let $x \in [0, \frac{1}{2}), y \in [\frac{1}{2}, 1]$.

$d(Sx, Ax) = |x^2 - \frac{x^2}{4}| = \frac{3}{4}x^2, d(Ty, By) = 0,\ d(Ax, By) = \frac{x^2}{4}, d(Sx, Ty) = x^2.$

$$\begin{aligned}\tfrac{1}{2}\min\{d(Sx, Ax), d(Ty, By)\} = \tfrac{1}{2}\{\tfrac{3}{4}x^2, 0\} &= 0 \leq \max\{x^2, \tfrac{x^2}{4}\}\\ &= \max\{d(Sx, Ty), d(Ax, By)\}\end{aligned}$$

and so
$$\begin{aligned}\eta(d(Ax, By), m(x, y)) = \tfrac{8}{9}m(x, y)^2 &\geq d(Ax, By)^2\\ &= \tfrac{8}{9}(x^2)^2 - (\tfrac{x^2}{4})^2\\ &= (\tfrac{8}{9} - \tfrac{1}{16})x^4 = \tfrac{99}{144}x^4 \geq 0.\end{aligned}$$

Subcase (iv): Let $x \in [\frac{1}{2}, 1], y \in [0, \frac{1}{2})$.

$d(Sx, Ax) = 1, d(Ty, By) = |\frac{y^2}{2} - \frac{y^2}{6}| = \frac{1}{3}y^2,\ d(Sx, Ty) = |1 - \frac{y^2}{2}|,$

$d(Ax, By) = |\frac{y^2}{6}| = \frac{y^2}{6}.$

$$\begin{aligned}\tfrac{1}{2}\min\{d(Sx, Ax), d(Ty, By) = \tfrac{1}{2}\min\{1, \tfrac{1}{3}y^2\}\} = \tfrac{1}{6}y^2 &\leq 1 - \tfrac{y^2}{2} = d(Sx, Ty)\\ &\leq \max\{d(Sx, Ty), d(Ax, By)\}\end{aligned}$$

and hence
$$\begin{aligned}\eta(d(Ax, By), m(x, y)) &= \tfrac{8}{9}m(x, y)^2 - d(Ax, By)^2\\ &= \tfrac{8}{9}(1 - \tfrac{y^2}{2})^2 - (\tfrac{1}{6}y^2)^2\\ &= \tfrac{8}{9}(1 + \tfrac{y^4}{4} - y^2) - \tfrac{1}{36}y^4\\ &= \tfrac{8}{9} + \tfrac{8}{36}y^4 - \tfrac{8}{9}y^2 - \tfrac{1}{36}y^4\\ &= \tfrac{8}{9} + \tfrac{7}{36}y^4 - \tfrac{8}{9}y^2 \geq \tfrac{7}{36}y^4 \geq 0.\end{aligned}$$

Therefore in this case the inequality (2) holds.

Similarly, we can see that A, B, S and T satisfy the inequality (2) for the case $x \leq y$.

Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.1, and '0' is the unique common fixed point of A, B, S and T.

Remark 3.2. *Theorem 1.1 follows as a corollary to Theorem 2.1 by choosing $S = I$ and $T = I$ in Theorem 2.1.*

4. Conclusion

We defined Suzuki type $Z-$contraction with a rational expression via $\Psi-$simulation function and proved the existence and uniqueness of common fixed points of two pairs of selfmaps in complete metric spaces by using reciprocal continuity and weakly compatible property (Theorem 2.1). Here we note that the class of all $\Psi-$simulation functions is more general than the class of all simulation functions

(Example 1.1). Our result (Theorem 2.1) extends Theorem 1.1 [2, Theorem 2] to two pairs of selfmaps. We provided an example in support of our main result.

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