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CHARACTERIZATIONS OF t^2-REVERSIBLE RINGS

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Abstract: This article aims to investigate the ring theoretic structures of (strongly) t^2-reversible ring using the concept of non-zero tripotent elements. A ring R is said to be t^2-reversible if $ab = 0$ implies $bat^2 = 0$ for all $a, b \in R$ and t is a non-zero tripotent element of R. It is proved that R is a t^2-reversible ring if and only if t^2 is left semicentral and t^2Rt^2 is a reversible ring. We also introduce and establish several characteristics of strongly t^2-reversible rings. It is proved that every strongly t^2-reversible ring is also a t^2-reversible ring but the converse need not be true. Moreover we call, R is a right (left) t^2-reduced ring if $N(R)t^2 = 0$ ($t^2N(R) = 0$), where $N(R)$ stands for the set of all nilpotent elements of R and we have established some of its properties.

Keywords and Phrases: t^2-reversible rings, strongly t^2- reversible rings, t^2-reduced rings, tripotent elements.

2020 Mathematics Subject Classification: 16A30, 16A50, 16E50, 16D30.

1. Introduction

All rings are associative with identity throughout this paper. Assuming that R is a ring, we denote its centre as $Z(R)$ and its set of all nilpotent elements as $N(R)$ respectively. Additionally, the $n \times n$ upper triangular matrix ring over R is denoted by the symbol $M_n(R)$. For a ring R, an element t is said to be tripotent if $t^3 = t$, the set of all non-zero tripotent elements is denoted by $T(R)$. It is obvious that all idempotents are tripotents but the converse is not true. For example let, $R =$

$M_2(\mathbb{R})$ then $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a tripotent element in R, as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ but not idempotent, as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \neq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Following [5], an idempotent element $e \in R$ is called left (resp. right)-semicentral if $(1-e)Re = 0$ (resp. $eR(1-e) = 0$).

A ring is usually called reduced if it has zero as the only nilpotent element. Accordind to Lambek [12], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. In [1], used the term ZC_3 for symmetric. Clearly, commutative rings are symmetric. Also every reduced rings are symmetric by [1, Theorem 1.3]. The notion of reversible ring was first introduced by Cohn [4] in 1999. A ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for any $a, b \in R$. Anderson and Camillo [1] used the term ZC_2 for reversibility. After that many researchers had studied the notion of reversible rings and extended in many different ways (refer to [13], [9], [10], [17], [7]).

Kose et al. [11] introduced the right (left) e-reversible rings and they defined a ring R to be right e-reversible (left e-reversible) if for any $a, b \in R$, $ab = 0$ implies $bae = 0$ ($eba = 0$), where e is an idempotent element in R. Also they established various properties of right e-reversibility in a ring. Later on Sabah et al. [16] introduced a strong condition on the Kose's notion and they defined a ring R is to be e-strongly reversible if $ab = 0$ implies $bea = 0$ for any $a, b \in R$. In recent time Chaturvedi and Verma [3] also studied the e-reversible rings and some associated ring extensions.

In this article, the results appeared in Sabah et al. [16], Kose et al. [11] and Chaturvedi and Verma [3] are extended and generalized using the concept of non-zero tripotent elements in a ring. With the aid of non-zero tripotent elements, the objective is to investigate and define a new type of ring known as a (strongly) t^2-reversible ring. Moreover, we introduce and study the notion of t^2-reduced rings and some associated concepts.

2. t^2-Reversible and Strongly t^2-reversible Rings

In this section we introduce t^2-reversible and strongly t^2-reversible rings and some examples are presented to illustrate the concepts. We begin with the following definitions.

Definition 2.1. *Let R be a ring and $t \in T(R)$. Then,*

(1) R is called t^2-reversible if $ab = 0$ implies $bat^2 = 0$, for all $a, b \in R$.

(2) R is called strongly t^2-reversible if $ab = 0$ implies $bt^2a = 0$, for all $a, b \in R$.

Remark 2.1. *In the above Definition 2.1, we have observed that, whenever $t \in T(R)$, then t^2 is always idempotent, as $(t^2)^2 = (t^3)t = t^2$. But t need not be an idempotent element. For $t = -1$ then $(-1)^3 = -1$ so $t \in T(R)$ and $((-1)^2)^2 = (-1)^2$ so t^2 is idempotent. But $(-1)^2 \neq -1$. So, t is not an idempotent element.*

Example 2.1. Every reversible ring is t^2-reversible for any non-zero tripotent element t in R, but the converse need not be true.
Let us consider, $R = M_2(\mathbb{Z}_3)$, where $\mathbb{Z}_3 = \{-1, 0, 1\}$ is the field. Then $T = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \in T(R)$. Let $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in R$, where a and b are non-zero elements of $\mathbb{Z}_3$. Then $AB = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies that $BAT^2 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This shows that R is a t^2-reversible ring. But R is not a reversible ring, since

$$BA = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ as } ba \neq 0.$$

Example 2.2. Every strongly t^2-reversible ring is also a t^2-reversible for any non-zero tripotent element t of the ring (it follows from Theorem 2.1 and 2.2). But the converse need not be true.
In Example 2.1, R is a t^2-reversible ring but not strongly t^2-reversible because

$$BT^2A = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ as } ba \neq 0.$$

Remark 2.2.

(1) From the Definition 2.1, it is clear that for $t = -1, 1$; we have R is a reversible ring if and only if R is a strongly t^2-reversible ring if and only if R is a t^2-reversible ring, as both -1 and 1 are in $T(R)$.

(2) Let R be a ring and e be an idempotent element in R. Since every idempotent is also a tripotent but tripotent need not be an idempotent element. So, $e \in T(R)$ then e^2 must be an idempotent. So by [3], every e-reversible ring is also e^2-reversible. But the converse need not be true.
Because in Example 2.1, we have R is a E^2-reversible ring for $E = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ but R is not E-reversible, as $E = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is not an idempotent element.

2.1. Some properties

In this section we discuss some basic properties of (strongly) t^2-reversible rings using the concept of non-zero tripotent elements in R.

Theorem 2.1. *Let R be a ring such that $t \in T(R)$. Then the subsequent conditions are equivalent.*

(1) R is a t^2-reversible ring;

(2) t^2Rt^2 is a reversible ring and t^2 is left semicentral.

Proof. (1) $\implies$ (2). Let R be a t^2-reversible ring. Since t is tripotent, so t^2 is always an idempotent element in R. Thus, $(t^2)^2 = t^2 \implies t^2(1-t^2) = 0$. For each $x \in R$, $t^2(1-t^2)x = 0$. Since R is a t^2-reversible ring so, $(1-t^2)xt^2t^2 = 0 \implies (1-t^2)xt^2 = 0 \implies xt^2 = t^2xt^2$. Thus, t^2 is left semicentral. Secondly, let $a, b \in t^2Rt^2$ such that $ab = 0$. Since t^2Rt^2 is a subring of R and R is a t^2-reversible ring. So we get, $bat^2 = 0 \implies ba = 0$, as $at^2 = a$. Thus t^2Rt^2 is a reversible ring.

(2) $\implies$ (1). Let us assume that condition (2) is true. Let $a, b \in R$ such that $ab = 0$. Then the elements $t^2at^2, t^2bt^2 \in t^2Rt^2$ and t^2Rt^2 is a reversible ring. So we get, $t^2at^2t^2bt^2 = t^2at^2bt^2 = 0$ implies $t^2bt^2t^2at^2 = t^2bt^2at^2 = 0$, as $t^3 = t$ so, $t^4 = t^2$. Since t^2 is a left semi central, so $t^2bt^2at^2 = 0 \implies bt^2at^2 = 0 \implies bat^2 = 0$. Thus we get, $ab = 0$ implies $bat^2 = 0$ for $a, b \in R$. This shows that R is a t^2-reversible ring.

Theorem 2.2. *Let R be a ring such that $t \in T(R)$. Then the subsequent conditions are equivalent.*

(1) R is a strongly t^2-reversible ring;

(2) t^2Rt^2 is a reversible ring and $t^2 \in Z(R)$.

Proof. (1) $\implies$ (2). Let us assume that condition (1) is true. Since $t \in T(R)$, so t^2 is idempotent in R. Thus for each $x \in R$, $x(1-t^2)t^2 = 0$. Since R is a strongly t^2-reversible ring, so we have $t^2t^2x(1-t^2) = 0 \implies t^2x(1-t^2) = 0 \implies t^2x = t^2xt^2$. Again R is a strongly t^2-reversible ring, so R is a t^2-reversible ring. This implies that t^2 is left semicentral by Theorem 2.1. Thus $t^2x = xt^2$ and so $t^2 \in Z(R)$. Again by Theorem 2.1, we get t^2Rt^2 is a reversible ring.

(2) $\implies$ (1). Suppose condition (2) holds. Let $a, b \in R$ such that $ab = 0$. Since t^2Rt^2 is a reversible ring, thus from the second part of Theorem 2.1 we have, $t^2bt^2at^2 = 0$. Again since $t^2 \in Z(R)$ so, $t^2a = at^2$ and $t^2b = bt^2$ for all $a, b \in R$. This implies that $bt^2a = 0$. Thus R is a strongly t^2-reversible ring.

We have the following corollary as a consequence of Theorem 2.1 and Theorem 2.2.

Corollary 2.2.1. *Let R be a ring and $t \in T(R)$. Then R is a strongly t^2-reversible ring if and only if R is a t^2-reversible ring and $t^2 \in Z(R)$.*

Lemma 2.3. *Let R be a ring and $t \in T(R)$. Then the subsequent conditions are equivalent.*

(1) R is a reversible ring;

(2) R is both t^2-reversible and $(1-t^2)$-reversible ring.

Proof. (1) $\implies$ (2). It is obvious.
(2) $\implies$ (1). Suppose condition (2) holds. Let $a, b \in R$ such that $ab = 0$. Then $ba(1-t^2)^2 = ba(1-2t^2+t^4) = ba(1-t^2) = 0$, as R is a $(1-t^2)$-reversible ring and $t^3 = t$. This implies that $ba = bat^2$. Again R is a t^2-reversible ring. So, we have $bat^2 = 0$. Hence $ba = 0$. This shows that R is a reversible ring.

Corollary 2.3.1. *Let R be a ring and $t \in T(R)$. Then the subsequent conditions are equivalent.*

(1) R is a reversible ring;

(2) R is a strongly t^2-reversible and $(1-t^2)R(1-t^2)$ is a reversible ring.

Proof. (1) $\implies$ (2). Let us assume that condition (1) is true. Since R is a reversible ring, so R is a t^2-reversible and an Abelian ring. This implies $t^2 \in Z(R)$ and hence R is a strongly t^2-reversible ring. Again $(1-t^2)R(1-t^2)$ is a subring of R. Thus $(1-t^2)R(1-t^2)$ is also a reversible ring.

(2) $\implies$ (1). Suppose condition (2) holds. By Theorem 2.2, we have R is a strongly $(1-t^2)$-reversible ring, as $(1-t^2) \in Z(R)$ and $(1-t^2)R(1-t^2)$ is a reversible ring. This implies that R is both t^2-reversible and $(1-t^2)$-reversible ring. Thus by Lemma 2.3, R is a reversible ring.

Generalising the notion defined by F. Meng et. al [15], the following concept is defined using tripotent elements. A ring R is called left t^2-reflexive if $xRt^2 = 0 \implies t^2Rx = 0$ for any $x \in R$ and $t \in T(R)$.

Theorem 2.4. *For a ring R and $t \in T(R)$, the following conditions are equivalent.*

(1) R is a strongly t^2-reversible ring;

(2) R is a t^2-reversible and left t^2-reflexive.

Proof. Suppose that R is a strongly t^2-reversible ring. Then by Corollary 2.2.1, we get t^2 is central and R is a t^2-reversible ring. Let $x \in R$ such that $xRt^2 = 0$. This implies $xt^2 = 0$ and since t^2 is central so, $t^2Rx = Rxt^2 = 0$. This implies that R is left t^2-reflexive.

Conversely let, condition (2) holds. Since R is t^2-reversible, so by Theorem 2.1 we get t^2 is left semicentral and hence $(1-t^2)Rt^2 = 0$ this implies that $t^2R(1-t^2) = 0$, as R is left t^2-reflexive. Thus $t^2 \in Z(R)$ and hence R is a strongly t^2-reversible ring by Corollary 2.2.1.

Following [16], a ring R and an R-bimodule ${}_RM_R$, the trivial extension of R by M is the ring $U(R, M) = R \oplus M$ under the operations $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2+m_1r_2)$, where $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Then $U(R, M)$ is isomorphic to the ring of all matrices of the form $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$ where $r \in R$ and $m \in M$ under usual matrix operations.

Theorem 2.5. *Let R be a reduced ring and $t \in T(R)$. Then $U(R, R)$ is a T^2-reversible ring if and only if R is a t^2-reversible ring, where $T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$.*

Proof. We assume that $U(R, R)$ is a T^2-reversible ring. Let $a, b, c, d \in R$ such that $ab = 0$. Let $A = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & d \\ 0 & b \end{pmatrix} \in U(R, R)$, so $AB = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix}\begin{pmatrix} b & d \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & ad+cb \\ 0 & ab \end{pmatrix} = \begin{pmatrix} 0 & ad+cb \\ 0 & 0 \end{pmatrix}$, as $ab = 0$. Now $(ba)^2 = baba = 0 \implies ba = 0$, as R is reduced. Again $ad + cb = 0 \implies ada + cba = 0 \implies ada = 0 \implies adad = 0 = (ad)^2$. Since R is reduced so, $ad = 0$ this gives us $cb = 0$. Hence $AB = 0$. Since $U(R, R)$ is T^2-reversible so we get $BAT^2 = 0$. This implies that $\begin{pmatrix} b & d \\ 0 & b \end{pmatrix}\begin{pmatrix} a & c \\ 0 & a \end{pmatrix}\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} bat^2 & bct^2+dat^2 \\ 0 & bat^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, $bat^2 = 0$, $bct^2 = 0$, $dat^2 = 0$. Thus $ab = 0$ implies that $bat^2 = 0$ for $a, b \in R$ and $t \in T(R)$. Hence R is a t^2-reversible ring.

Conversely let R be a t^2-reversible ring. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in U(R, R)$ such that $AB = 0$. This implies $ac = 0$ and $ad + bc = 0$. Now $(ca)^2 = caca = 0$, since R is reduced so we have $ca = 0$ and hence $cat^2 = 0$. Also $ad+bc = 0 \implies ada + bca = 0 \implies ada = 0 \implies adad = 0 \implies (ad)^2 = 0$. Since R is reduced so $ad = 0$ this gives $bc = 0$. Again, R is a t^2-reversible ring so we get $dat^2 = 0$ and $cbt^2 = 0$. Thus $BAT^2 = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} =$

$\begin{pmatrix} cat^2 & cbt^2+dat^2 \\ 0 & cat^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. This shows that $U(R,R)$ is a T^2-reversible ring, where $T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ and for each $t = t^3$.

Theorem 2.6. *Let R be a ring and $T = \begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix} \in T(M_2(R))$ for each $r \in R$. Then $M_2(R)$ is a T^2-reversible ring if and only if R is a reversible ring.*

Proof. We assume that, $M_2(R)$ is a T^2-reversible ring. Let $a, b \in R$ such that $ab = 0$. So we have, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $M_2(R)$ is T^2-reversible, so we get, $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} ba & -bar \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus we get, $ba = 0$. This implies R is a reversible ring.

Conversely let, R be a reversible ring and $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in M_2(R)$ such that $AB = 0$. This implies $a_1a_2 = 0 = c_1c_2$. Since R is reversible so $a_2a_1 = 0$ and $c_2c_1 = 0$. Now, $BAT^2 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2$ $= \begin{pmatrix} a_2a_1 & -a_2a_1r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. Thus, $M_2(R)$ is a T^2-reversible ring.

Theorem 2.7. *Let R be a ring, $t \in T(R)$ and $T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in T(M_2(R))$. Then $M_2(R)$ is a T^2-reversible ring if and only if R is a t^2-reversible ring.*

Proof. Let us assume that, $M_2(R)$ is a T^2-reversible ring. Let $a, b \in R$ such that $ab = 0$. So we have, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $M_2(R)$ is T^2-reversible, so we get, $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} bat^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus we get, $bat^2 = 0$. This implies R is a t^2 reversible ring.

Conversely let, R be a reversible ring and $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in M_2(R)$ such that $AB = 0$. This implies $a_1a_2 = 0 = c_1c_2$. Since R is t^2 reversible so $a_2a_1t^2 = 0$ and $c_2c_1t^2 = 0$. Now $BAT^2 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}^2 =$ $\begin{pmatrix} a_2a_1t^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. Thus, $M_2(R)$ is a T^2-reversible ring.

Theorem 2.8. *Let R be a ring, $t \in T(R)$ and $T = \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix} \in T(M_2(R))$. Then $M_2(R)$ is a T^2-reversible ring if and only if R is a t^2-reversible ring.*

Proof. The proof is similar to the proof of Theorem 2.7.

3. t^2-Reduced Rings

In this section we define right (left) t^2-reduced rings and study some basic properties of it.

Definition 3.1. *Let R be a ring and $t \in T(R)$. Then,*

(1) R is called right t^2-reduced if $N(R)t^2 = 0$.

(2) R is called left t^2-reduced if $t^2N(R) = 0$.

Example 3.1. Let $R = M_3(F)$ and F is a field. Then $N(R) = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$, as

$$\begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is nilpotent.}$$

Let $t = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T(R)$. Then $N(R)t^2 = N(R) \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $= 0$. Thus, R is a right t^2-reduced ring. But R is not a left t^2-reduced ring as,

$$t^2N(R) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 N(R) = \begin{pmatrix} 0 & F & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Remark 3.1. *We can also construct a left t^2-reduced ring which is not right t^2-reduced. In Example 3.1, if we consider $R = M_3(F)$ as a 3×3 lower triangular matrix ring over the field F.*

Then clearly, $N(R) = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & F & 0 \end{pmatrix}$.

Let $t = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T(R)$. Then $t^2N(R) = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 N(R) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $= 0$. Thus R is a left t^2-reduced ring. But R is not a right t^2-reduced ring as,

$$N(R)t^2 = N(R)\begin{pmatrix}-1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}^2 = \begin{pmatrix}0 & 0 & 0\\ F & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \neq \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = 0.$$

Theorem 3.1. *Let R be a ring and $t \in T(R)$. Then the subsequent conditions are equivalent.*

(1) R is right t^2-reduced;

(2) t^2 is left semicentral in R and t^2Rt^2 is reduced.

Proof. (1) $\implies$ (2). Let $t \in T(R)$ then t^2 must be idempotent so, $t^2 = (t^2)^2$. This implies that $(1 - t^2)t^2 = 0$. So for any $x \in R$ we have, $(1 - t^2)xt^2 \in N(R)$ and $(1 - t^2)xt^2 \in N(R)t^2 = 0$, as R is a right t^2-reduced ring. This implies that $(1 - t^2)xt^2 = 0 \implies t^2xt^2 = xt^2$. Thus t^2 is left semicentral in R. Again $N(t^2Rt^2) \subseteq N(R)t^2 = 0 \implies t^2Rt^2$ is a reduced ring, by (1).

(2) $\implies$ (1). Let t^2 is left semicentral in R and t^2Rt^2 is reduced, then $N(R)t^2 = t^2Rt^2 = N(t^2Rt^2) = 0$. This implies that R is a t^2-reduced ring.

The following theorem is related to Theorem 3.1.

Theorem 3.2. *Let R be a ring and $t \in T(R)$. Then the subsequent conditions are equivalent.*

(1) R is left t^2-reduced;

(2) t^2 is right semicentral in R and t^2Rt^2 is reduced.

Following [6], a ring R is called (strongly) t^2-symmetric if $abc = 0$ implies ($act^2b = 0$) $acbt^2 = 0$, for all $a, b \in R$ and $t \in T(R)$.

Corollary 3.2.1. *Right t^2-reduced rings are t^2-symmetric rings.*

Proof. Let R be a right t^2-reduced ring. Then by Theorem 3.1, we get, t^2 is left semicentral in R and t^2Rt^2 is reduced. Since reduced rings are symmetric by [[1], Theorem 1.3]. So, t^2Rt^2 is a symmetric ring and t^2 is left semicentral in R. Thus by [[6], Theorem 2.1] we get R is a t^2-symmetric ring.

Remark 3.2. *By Example 3.1 and [[6], Theorem 2.2], we have observed that right t^2-reduced rings need not be strongly t^2-symmetric.*

Theorem 3.3. *Every t^2-symmetric ring is t^2-reversible.*

Proof. Let R be a t^2-symmetric ring and $t \in T(R)$. Let $x, y \in R$ such that $xy = 0$. Since, R is t^2-symmetric, so we have $1xy = 0$, (since $1 \in R$) implies that $1yxt^2 = 0 \implies yxt^2 = 0$. This shows that R is a t^2-reversible ring.

Remark 3.3. *In the above Theorem 3.3, we have observed that the t^2-reversible ring need not be t^2-symmetric by the following example.*

Example 3.2. Let us consider $R = M_2(\mathbb{R})$ where $\mathbb{R}$ is the field of all real numbers. Then $T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in T(R)$. Let $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \in R$, where a and b are non zero elements in $\mathbb{R}$. Then $AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ implies that $BAT^2 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. This shows that R is a T^2-reversible ring.
Now for $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$ we have, $ABC = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$
but $ACBT^2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, as $ab \neq 0$. Thus R is not a T^2-symmetric ring.

Remark 3.4. *In the above Example 3.2, it is seen that* $AB = 0$ *implies that* $BA = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. *Thus in Example 3.2, R is also a reversible ring.*

The following theorem is related to Theorem 3.3.

Theorem 3.4. *Every strongly* t^2*-symmetric ring is strongly* t^2*-reversible.*
Proof. Let R be a strongly t^2-symmetric ring and $t \in T(R)$. Let $x, y \in R$ such that $xy = 0$. Since R is strongly t^2-symmetric, so we have $1xy = 0$ implies that $1yt^2x = 0 \implies yt^2x = 0$. Thus R is a t^2-reversible ring.

Remark 3.5. *The converse of the above Theorem 3.4, need not be true in general. As in Example 3.2, it is observed that* $AB = 0$ *implies* $BT^2A = 0$, *so R is a strongly* T^2*-reversible ring. But for* $ABC = 0$ *which implies that* $ACT^2B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, *as* $ab \neq 0$. *So, R is not a strongly* T^2*-symmetric ring.*

Remark 3.6. *From Examples 2.2; 3.2 and Remark 3.5, we have concluded that, every strongly* t^2 *reversible ring is a* t^2*-reversible but* t^2*-reversible rings may or may not be strongly* t^2*-reversible rings.*

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