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BEST PROXIMITY AND FIXED POINT OUTCOMES IN METRIC SPACES FOR THE PROXIMAL CONTRACTION OF $\alpha_0 - (\psi_0, g_0)$

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Abstract: In this paper, we introduce new notions of $\alpha_0 - (\psi_0, g_0)$ - proximal contraction of Type-I and Type-II and modified $\alpha_0 - (\psi_0, g_0)$ - proximal contraction. In the setting of these notions, we prove certain fixed point theorems in metric space. Additionally, a few applications are provided to show how the results can be used.

Keywords and Phrases: Best proximity point, metric space, fixed point, proximal generalized contraction.

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1. Introduction

A broad range of issues appearing in various branches of pure and applied mathematics, including discrete and continuous dynamic systems, differential equations, and variational analysis. Fixed point theory is essential for solving equations of the mentioned type, the solutions to which are the fixed points of the mapping $F : \mathfrak{X} \to \mathfrak{X}$, where $\mathfrak{X}$ is a non-empty set. Potential applications of this theory include the study of equilibrium points in physics, economics, and engineering. However, if F is a non self-mapping, the above fixed point equation might not have any solutions. In this case, it would be of some interest to find a rough solution, p, that is optimal in the sense that the distance between p and Fp is as small as possible. When examining this kind of element, the best proximity point theory is a helpful tool. Given M and N two non-empty subsets of a metric space and $F : M \to N$ a non self-mapping, the equation $Fp = p$ does not necessarily have a solution, which is known as a fixed point of the mapping F. In this context, best proximity point theory is an useful tool in studying such kind of element.

Definition 1.1. *"Let M and N be two non-empty subsets of a metric space $(\mathfrak{X}, d_{\mathfrak{X}})$ and a non self-mapping $F : M \to N$. An element $p \in M$ such that $d_{\mathfrak{X}}(p, Fp) = d_{\mathfrak{X}}(M, N)$ is a best proximity point of the non self-mapping F. Clearly, a fixed point, defined as $p = Fp$, is the best proximity point if F is a self-mapping."*

Numerous authors have researched the best proximity point theory of non self-mappings since its inception; see the seminal papers of Fan [4] and Kirk et al. [10]. Numerous prerequisites for the existence of the optimal proximity point are examined in ([1, 2, 3, 5, 6, 12, 13, 15, 18, 19]). For multi valued mappings, some noteworthy best proximity point results are reported in [7] also see the references therein.

2. Preliminaries

In this section, we will discuss some definitions and results from fixed point theory. These definitions and results will help to make new theorems in best proximity point.

The term α_0 - admissible mapping was defined as follows by Samet et al. [17] in their study.

Definition 2.1. ([17]) *Let $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$ be a function. We say that a self - mapping $F : \mathfrak{X} \to \mathfrak{X}$ is α_0 - admissible if*
$p, q \in \mathfrak{X}, \quad \alpha_0(p, q) \geq 1 \quad \implies \alpha_0(Fp, Fq) \geq 1.$

They established several fixed point findings by applying this idea.

Theorem 2.2. ([17]) *Let $(\mathfrak{X}, d_{\mathfrak{X}})$ be a complete metric space and $F : \mathfrak{X} \to \mathfrak{X}$ be an α_0 - admissible mapping. Assume that the following conditions hold:*

1. *for all $p, q \in \mathfrak{X}$ we have*

$$\alpha_0(p, q) d_{\mathfrak{X}}(Fp, Fq) \leq \psi_0(d_{\mathfrak{X}}(p, q)), \tag{2.1}$$

where $\psi_0 : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\sum_{n=1}^{+\infty} \psi_0^n(t) < +\infty$ for each $t > 0$,

2. *there exists $p_0 \in \mathfrak{X}$ such that $\alpha_0(p_0, Fp_0) \geq 1$,*

3. *either F is continuous or for any sequence $\{p_n\}$ in $\mathfrak{X}$ with $\alpha_0(p_n, p_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $p_n \to p$ as $n \to +\infty$, then $\alpha_0(p_n, p) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*
   Then F has a fixed point.

Last but not least, let's remember how Karapinar et al. [9] first developed the idea of triangular α_0 - admissible mapping.

Definition 2.3. ([9]) *Let $\alpha_0 : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ be a function. We say that a self-mapping $F : \mathfrak{X} \to \mathfrak{X}$ is triangular α_0 - admissible if*
(i) $p, q \in \mathfrak{X}, \quad \alpha_0(p, q) \geq 1 \quad \Longrightarrow \alpha_0(Fp, Fq) \geq 1$,
(ii) $p, q, r \in \mathfrak{X}$,

$$\begin{cases} \alpha_0(p, r) \geq 1, \\ \alpha_0(r, q) \geq 1 \end{cases} \Longrightarrow \alpha_0(p, q) \geq 1.$$

Ran and Reurings [14] have initiated the study of weaker contraction in recent years by representing self-map in partially ordered metric space. Some of the latest findings of Mongkolkeha et al. [11] and Sadiq Basha et al. [16].

Theorem 2.4. ([16]) *Suppose that M, N be two closed members of a partially ordered complete metric space $(\mathfrak{X}, d_{\mathfrak{X}}, \preceq)$, M_0 is non-empty and the pair (M, N) has the V - property. Presume the following conditions are met by $F : M \to N$:*

1. *F is ordered immediately-holding $F(M_0) \subseteq N_0$ in such a way that,*

2. *there exist elements $p_0, p_1 \in M_0$ such that*
   $d_{\mathfrak{X}}(g_0 p_1, F p_0) = d_{\mathfrak{X}}(M, N)$ and $p_0 \preceq p_1$,

3. *for all $p, q, m, n \in M$,*

$$\begin{cases} g_0 p \preceq g_0 q, \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \Longrightarrow d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0 p, n) + d_{\mathfrak{X}}(g_0 q, m)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad -\psi_0(d_{\mathfrak{X}}(g_0 p, n), d_{\mathfrak{X}}(g_0 q, m)). \\ d_{\mathfrak{X}}(g_0 q, Fq) = d_{\mathfrak{X}}(M, N) \end{cases}$$

4. *if $\{p_m\}$ is an increasing sequence in M converging to $p \in M$, $\forall\, p \in \mathbb{N}$. Then F has a best proximity point.*

3. Main results

Let M and N be two nonempty subsets of a metric space $(\mathfrak{X}, d_{\mathfrak{X}})$. Following the notation, we get

$$M_0 := \{p \in M : d_{\mathfrak{X}}(p,q) = d_{\mathfrak{X}}(M,N),\ \textit{for some } q \in N\},$$

$$N_0 := \{q \in N : d_{\mathfrak{X}}(p,q) = d_{\mathfrak{X}}(M,N),\ \textit{for some } p \in M\}.$$

If $M \cap N \neq \emptyset$, then M_0 and N_0 are non-empty. Further, it is interesting to notice that M_0 and N_0 are contained in the boundaries of M and N, respectively, provided M and N are closed subsets of a normal linear space such that $d(M,N) > 0$ (see [8]).

Definition 3.1. *If there is a non-negative integer $\alpha_0 < 1$, then the mapping $F : M \to N$ is the proximal contraction, so for all m_1, m_2, p_1, p_2 in M,*
$d_{\mathfrak{X}}(m_1, Fp_1) = d_{\mathfrak{X}}(M,N) = d_{\mathfrak{X}}(m_2, Fp_2) \quad \Rightarrow d_{\mathfrak{X}}(m_1, m_2) \leq \alpha_0(d_{\mathfrak{X}}(p_1, p_2)).$

Definition 3.2. *Let M and N be two nonempty subsets of a metric space $(\mathfrak{X}, d_{\mathfrak{X}})$ and α_0 be a function. We can say a non self-map $F : M \to N$ is triangular α_0 - proximal admissible if, for all $p, q, r, p_1, p_2, m_1, m_2 \in M$,*

1.
$$\begin{cases} \alpha_0(p_1, p_2) \geq 1, \\ d_{\mathfrak{X}}(m_1, Fp_1) = d_{\mathfrak{X}}(M,N), \\ d_{\mathfrak{X}}(m_2, Fp_2) = d_{\mathfrak{X}}(M,N) \end{cases} \Longrightarrow \alpha_0(m_1, m_2) \geq 1,$$

2.
$$\begin{cases} \alpha_0(p, r) \geq 1, \\ d_{\mathfrak{X}}(r, q) \geq 1 \end{cases} \Longrightarrow \alpha_0(p, q) \geq 1.$$

As motivated by this paper, we introduce new notions of modified $\alpha_0 - (\psi_0, g_0)$ - proximal contraction, $\alpha_0 - (\psi_0, g_0)$ - proximal contraction of Type-I and Type-II in this paper. We also prove some fixed point theorems in the setting of metric spaces.

Now, we introduce the new class of proximal contractions.

Definition 3.3. *Let $F : M \to N$, $g_0 : M \to M$ be two maps. Let $\psi_0 : [0, \infty) \to [0, \infty)$ satisfy*
$\psi_0(0) = 0, \quad \psi_0(\mathfrak{t}) < \mathfrak{t}, \quad$ *and* $\quad lim_{\mathfrak{s} \to \mathfrak{t}^+} sup\, \psi_0(\mathfrak{s}) < \mathfrak{t}$ *for each* $\mathfrak{t} > 0$.
Then, F is said to be a (ψ_0, g_0) - proximal contraction if
$d_{\mathfrak{X}}(m_1, Fp_1) = d_{\mathfrak{X}}(M,N) = d_{\mathfrak{X}}(m_2, Fp_2) \quad \Rightarrow d_{\mathfrak{X}}(m_1, m_2) \leq \psi_0(d_{\mathfrak{X}}(gp_1, gp_2))$

for all m_1, m_2, p_1, p_2 *in* M.

Definition 3.4. *Presume* M, N *be two nonempty elements of a metric space* $(\mathfrak{X}, d_{\mathfrak{X}})$ *and* $\alpha_0 : M \times M \to [0, +\infty)$ *be a function. We're suggesting that a non self-mapping* $F : M \to N$ *is*

1. *a improved* $\alpha_0 - (\psi_0, g_0)$ *- proximal contraction if, for all* $m, n, p, q \in M$,

$$\begin{cases} \alpha_0(g_0p, g_0r) \geq 1, \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fr) = d_{\mathfrak{X}}(M, N) \end{cases}$$

$$\Longrightarrow d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, n) + d_{\mathfrak{X}}(g_0q, m)) - \psi_0(d_{\mathfrak{X}}(g_0p, n), d_{\mathfrak{X}}(g_0q, m)), \tag{3.1}$$

2. *an* $\alpha_0 - (\psi_0, g_0)$ *- proximal contraction of Type-I if, for all* $m, n, p, q \in M$,

$$\begin{cases} d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N) \end{cases}$$

$$\Longrightarrow \alpha_0(p, q) d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, n) + d_{\mathfrak{X}}(g_0q, m)) - \psi_0(d_{\mathfrak{X}}(g_0p, n), d_{\mathfrak{X}}(g_0q, m)),$$

where $0 \leq \alpha_0(g_0p, g_0q) \leq 1$ *for all* $g_0p, g_0q \in M$

3. *an* $\alpha_0 - (\psi_0, g_0)$ *- proximal contraction of Type-II if,* $\forall m, n, p, q \in M$,

$$\begin{cases} d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N) \end{cases}$$

$$\Longrightarrow (\alpha_0(g_0p, g_0q) + l)^{d_{\mathfrak{X}}(m,n)} \leq (l+1)^{\frac{1}{2} d_{\mathfrak{X}}(g_0p,n) + d_{\mathfrak{X}}(g_0q,m)) - \psi_0(d_{\mathfrak{X}}(g_0p,n), d_{\mathfrak{X}}(g_0q,m))}.$$

Theorem 3.5. *Let us suppose* M, N *be two non-empty members of a metric space* $(\mathfrak{X}, d_{\mathfrak{X}})$ *so* M *is complete and* M_0 *is nonempty. Let* $F : M \to N$ *is a continuous modified* $\alpha_0 - (\psi_0, g_0)$ *- proximal contraction* $g_0 : M \to M$ *satisfy the following conditions:*

1. F *is a triangular* $\alpha_0 - (\psi_0, g_0)$ *- proximal admissible mapping and* $F(M_0) \subseteq N_0$,

2. $\exists\, p_0, p_1 \in M_0$ *s.t.*

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_0, g_0p_1) \geq 1$

Then there is a best proximity point for F. *Moreover, the best proximity point is unique, if, for each* $p, q \in M$ *s.t.*

$d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$, *we have* $\alpha_0(g_0p, g_0q) \geq 1$.

Proof. By (2), there exists $p_0, p_1 \in M_0$ such that

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_0, g_0p_1) \geq 1$.

On the other hand, since $F(M_0) \subseteq N_0$, then there exists $p_2 \in M_0$ such that

$$d_{\mathfrak{X}}(g_0p_2, Fp_1) = d_{\mathfrak{X}}(M, N).$$

Since F is the allowable near end of the triangle α_0, we have $\alpha_0(g_0p_1, g_0p_2) \geq 1$. Thus

$$d_{\mathfrak{X}}(g_0p_2, Fp_1) = d_{\mathfrak{X}}(M, N).$$

and $\alpha_0(g_0p_1, g_0p_2) \geq 1$.

Since $F(M_0) \subseteq N_0$, then $\exists\, p_3 \in M_0$ s.t.

$$d_{\mathfrak{X}}(g_0p_3, Fp_2) = d_{\mathfrak{X}}(M, N).$$

Next, F is a triangular $\alpha_0-(\psi_0, g_0)$ - proximal admissible, it becomes $\alpha_0(g_0p_2, g_0p_3) \geq 1$ and hence

$$d_{\mathfrak{X}}(g_0p_3, Fp_2) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_2, g_0p_3) \geq 1$.

In this step, we create a $\{p_a\}$ sequence in such a way that

$$\begin{cases} \alpha_0(g_0p_{a-1}, g_0p_a) \geq 1 \\ d_{\mathfrak{X}}(g_0p, Fp_{a-1} = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(g_0p_{a+1}, Fp_a) = d_{\mathfrak{X}}(M, N), \end{cases} \tag{3.2}$$

for all $a \in \mathbb{N}$. Now, from (3.1) with $m = g_0p_a$, $n = g_0p_{a-1}$ and $g_0p = gp_a$, we get

$$\begin{aligned} d_{\mathfrak{X}}(g_0p_a, g_0p_{a+1}) &\leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) + d_{\mathfrak{X}}(g_0p_a, g_0p_a)) \\ &- \psi_0(d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}), d_{\mathfrak{X}}(g_0p_a, g_0p_a)) \\ &= \frac{1}{2}d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) - \psi_0(d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}, 0)) \\ &\leq \frac{1}{2}d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) \\ &\leq (d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_a) + d_{\mathfrak{X}}(g_0p_a, g_0p_{a+1})), \end{aligned} \tag{3.3}$$

which implies that $d_{\mathfrak{X}}(g_0p_a, g_0p_{a+1}) \leq d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_a)$. It follows that the sequence $\{\delta_a\}$, where $\delta_a = \delta(g_0p_a, g_0p_{a+1})$ is decreasing and so $\exists\ \delta \geq 0$ s.t. $\delta_a \to \delta$ while $a \to \infty$. Later, we will take limit $a \to +\infty$ in (3.3), it become

$$\delta \leq \frac{1}{2}\delta(g_0p_{a-1}, g_0p_{a+1}) \leq \frac{1}{2}(\delta + \delta) = 2\delta,$$

that is,

$$lim_{a\to+\infty} d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) = 2\delta. \tag{3.4}$$

Again, taking the limit as $a \to +\infty$ in (3.3) and (3.4) and the continuity of ψ_0, we get

$$\delta \leq \delta - \psi_0(2\delta, 0).$$

and so $\psi_0(2\delta, 0) = 0$. Therefore, by the property of ψ_0, we get $\delta = 0$, that is,

$$lim_{a\to+\infty} d_{\mathfrak{X}}(g_0p_{a+1}, g_0p_a) = 0. \tag{3.5}$$

Next, we show g_0p_a is a Cauchy sequence. Then there is an $\epsilon > 0$ and two subsequences $\{u(\bar{l})\}$ and $\{v(\bar{l})\}$ s.t. for all positive integer $\bar{l}$,

$v(\bar{l}) > u(\bar{l}) > \bar{l}$, $\quad d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) \geq \epsilon$, $\quad d_{\mathfrak{X}}(g_0p_{v(\bar{l})-1}, g_0p_{v(\bar{l})}) < \epsilon$.

The smallest number reaches $u(\bar{l})$ go for $v(\bar{l})$.

This means that we get $\bar{l} \in \mathbb{N}$ for all of them.

$$\begin{aligned} \varepsilon \leq d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) &\leq d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{v(\bar{l})-1}) + d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})+1}) \\ &\leq d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{v(\bar{l})-1}) + \epsilon. \end{aligned}$$

Making limit as $\bar{l} \to +\infty$, we obtain and using (3.5), we get

$$lim_{\bar{l}\to+\infty} d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) = \epsilon. \tag{3.6}$$

Again, from

$$d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) \leq d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+1}) + d_{\mathfrak{X}}(g_0p_{u(\bar{l})+1}, g_0p_{v(\bar{l})+1}) + d_{\mathfrak{X}}(g_0p_{v(\bar{l})+1}, g_0p_{v(\bar{l})})$$

and

$$d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})+1}) \leq d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+1}) + d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})}) + d_{\mathfrak{X}}(g_0p_{v(\bar{l})+1}, g_0p_{v(\bar{l})}),$$

Proceeding limit as $\bar{l} \to +\infty$, by (3.5) and (3.6), we deduce

$$lim_{\bar{l}\to+\infty} d_{\mathfrak{X}}(g_0p_{v(\bar{l})+1}, g_0p_{u(\bar{l})+1}) = \epsilon. \tag{3.7}$$

Similarly,

$$lim_{\bar{l}\to+\infty} d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_u(\bar{l}) + 1) = \epsilon \tag{3.8}$$

and

$$lim_{\bar{l}\to+\infty} d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})+1}) = \epsilon. \tag{3.9}$$

We're going to explain that

$$\alpha_0(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})}) \geq 1, \ where\ v(\bar{l}) > u(\bar{l}) > \bar{l}. \tag{3.10}$$

F is a triangular $\alpha_0 - (\psi_0, g_0)$ - proximal admissible mapping and

$$\begin{cases} \alpha_0(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+1}) \geq 1, \\ \alpha_0(g_0p_{u(\bar{l})+1}, g_0p_{u(\bar{l})+2}) \geq 1. \end{cases}$$

With condition (2) of Definition (3.2), we have

$$\alpha_0(g_0p_{u(\bar{l})+1}, g_0p_{u(\bar{l})+2}) \geq 1.$$

Again, F is $\alpha_o - (\psi_0, g_0)$ - triangular proximal map,

$$\begin{cases} \alpha_0(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+2}) \geq 1, \\ \alpha_0(g_0p_{u(\bar{l})+2}, g_0p_{u(\bar{l})+3}) \geq 1. \end{cases}$$

With condition (2) of Definition (3.2), we have

$$\alpha_0(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+3}) \geq 1.$$

Therefore, we get (3.10) through this process.

On the second side, we do know that

$$\begin{cases} \alpha_0(g_0p_{u(\bar{l})+1}, Fp_{v(\bar{l})}) = d_{\mathfrak{X}}(M,N), \\ \alpha_0(g_0p_{v(\bar{l})+1}, Fp_{u(\bar{l})}) = d_{\mathfrak{X}}(M,N). \end{cases}$$

Therefore, we have

$$\begin{aligned} d_{\mathfrak{X}}(g_0p_{u(\bar{l})+1}, g_0p_{v(\bar{l})+1}) \leq \frac{1}{2}(&d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})+1}) + d_{\mathfrak{X}}(g_0pv(\bar{l}), g_0p_{u(\bar{l})+1})) \\ &-\psi_0(d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})+1}), d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})+1})). \end{aligned}$$

Picking limit as $\bar{l} \to +\infty$ and using (3.7), (3.8), (3.9), the continuity of ψ_0, one become

$$\begin{aligned} \epsilon &\leq \frac{1}{2}(\epsilon + \epsilon) - \psi_0(\epsilon, \epsilon) \\ \epsilon &\leq \epsilon - \psi_0(\epsilon, \epsilon) \end{aligned}$$

and hence $\psi_0(\epsilon, \epsilon) = 0$, which leads to the contradiction $\epsilon = 0$. Thus, $\{p_a\}$ is a CS. Ahead M has been completed, there is $z \in M$ so $p_a \to r$. Hereinafter, $d_{\mathfrak{X}}(g_0p_{a+1}, Fp_a) = d_{\mathfrak{X}}(M,N) \quad \text{for} \quad \text{all} \quad a \in \mathbb{N} \cup \{0\}$.

Selecting limit as $a \to +\infty$, we gather $d_{\mathfrak{X}}(r, Fr) = d_{\mathfrak{X}}(M,N)$, owing to the f consistency.

Lastly, we demonstrate the uniqueness of point $p \in F$ s.t. $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M,N)$. Suppose, in fact, that there is $p, q \in M$ which are best proximity points, viz. $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M,N) = d_{\mathfrak{X}}(g_0q, Fq)$.

Since $\alpha_0(g_0p, g_0q) \geq 1$, we have

$$\begin{aligned} d_{\mathfrak{X}}(g_0p, g_0q) &\leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, g_0q) + d_{\mathfrak{X}}(g_0q, g_0p) - \psi_0(d_{\mathfrak{X}}(g_0p, g_0q), d_{\mathfrak{X}}(g_0q, g_0p)) \\ &= d_{\mathfrak{X}}(g_0p, g_0q) - \psi_0(d_{\mathfrak{X}}(g_0p, g_0q), d_{\mathfrak{X}}(g_0q, g_0p)), \end{aligned}$$

which implies $d_{\mathfrak{X}}(g_0p, g_0q) = 0, \quad \text{that is} \quad g_0p = g_0q$.

Corollary 3.6. *Let M,N be non-empty subsets of a metric space $(\mathfrak{X}, d_{\mathfrak{X}})$ to this extent M is complete and M_0 is non-empty. Presume $F : M \to N$ and $g_0 : M \to M$ are continuous $\alpha_0 - (\psi_0, g_0)$ - proximal contraction of Type-I or a continuous $\alpha_0 - (\psi_0, g_0)$ - proximal contraction mapping of the Type-II s.t. the following requirements satisfied:*

1. *F is a triangular $\alpha_0 - (\psi, g_0)$ - proximal admissible mapping and $F(M_0) \subseteq N_0$.*

2. *there exists* $p_0, p_1 \in M_0$ *such that*

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_0, g_0p_1) \geq 1$.

Then the F *will have a best proximity point. Furthermore, if, for every* $p, q \in M$, $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$, *we have* $\alpha_0(g_0p, g_0q) \geq 1$, *the best proximity point is unique.*

Definition 3.7. *Let* M, N *are two nonempty subsets of a metric space* $(\mathfrak{X}, d_{\mathfrak{X}})$. *The* (M, N) *pair is said to have the* V*-property if, for each* $\{q_n\}$ *sequence of* N *that satisfies the* $d_{\mathfrak{X}}(p, q_n) \to d_{\mathfrak{X}}(p, N)$ *condition, for each* $p \in M$ *sequence,* $q \in N$ *is given such that* $d_{\mathfrak{X}}(p, q) = d_{\mathfrak{X}}(p, N)$.

Theorem 3.8. *Suppose* M, N *be two non-void elements of a metric space* $(\mathfrak{X}, d_{\mathfrak{X}})$ *s.t.* M *is complete, the pair* (M, N) *has the* V *- property and* M_0 *is complete. Assume that* $F : M \to N$ *and* $g_0 : M \to M$ *are modified* $\alpha_0 - (\psi_0, g_0)$ *- proximal contraction in such a way that the following criteria hold:*

1. F *is a triangular map of* $\alpha_0 - (\psi_0, g_0)$ *and* $F(M_0) \subseteq N_0$.

2. $p_0, p_1 \in M_0$ *occurs to such a degree that*

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_0, g_0p_1) \geq 1$,

3. *if* $\{g_0p_n\}$ *is a sequence in* M *such that* $\alpha_0(g_0p_n, g_0p_{n+1}) \geq 1$ *and* $g_0p_n \to g_0p$ *as* $n \to \infty$, *then* $\alpha_0(g_0p_n, g_0p) \geq 1$ $\forall n \in \mathbb{N} \cup \{0\}$.

Then, there is a best proximity point for F. *Furthermore, the best proximity point is unique if we have* $\alpha_0(g_0p, g_0q) \geq 1$ *for every* $p, q \in M$, *so that* $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$.

Proof. After the Theorem (3.5) is proved, there are Cauchy sequences $\{g_0p_a\} \subseteq M$ and $r \in M$ such that (3.2) keep $g_0p_a \to z$ as $a \to +\infty$. On next side, $\forall a \in \mathbb{N}$, write down

$$\begin{aligned} d_{\mathfrak{X}}(r, N) &\leq d_{\mathfrak{X}}(r, Fp_a) \\ &\leq d_{\mathfrak{X}}(r, g_0p_{a+1}) + d_{\mathfrak{X}}(g_0p_{a+1}, Fp_a) \\ &= d_{\mathfrak{X}}(r, g_0p_{a+1}) + d_{\mathfrak{X}}(M, N). \end{aligned}$$

Selecting this limit $p \to +\infty$, we take

$$lim_{a \to +\infty} d_{\mathfrak{X}}(r, Fp_a) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(M, N). \tag{3.11}$$

Since (M, N) has the V-attribute, there is $c \in N$, so $d_{\mathfrak{X}}(r, c) = d_{\mathfrak{X}}(M, N)$. Therefore $r \in M_0$. Moreover, since $F(M_0) \subseteq N_0$, then there is $n \in M$ such that

$$d_{\mathfrak{X}}(n, Fr) = d_{\mathfrak{X}}(M, N).$$

Now, by (3) and (3.2), we have $\alpha_0(g_0 p_a, r) \geq 1$ and $d_{\mathfrak{X}}(g_0 p_{a+1}, Fp_a) = d_{\mathfrak{X}}(M, N)$ for all $a \in \mathbb{N} \cup \{0\}$. Also, since F is a modified $\alpha_0 - (\psi_0, g_0)$ - proximal contraction, we get

$$d_{\mathfrak{X}}(g_0 p_{a+1}, v) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0 p_a, n) + d_{\mathfrak{X}}(r, g_0 p_{a+1})) - \psi_0(d_{\mathfrak{X}}(g_0 p_a, n), d_{\mathfrak{X}}(r, g_0 p_{a+1})$$

. Taking this as $a \to +\infty$in the equation, we have

$$d_{\mathfrak{X}}(r, n) \leq \frac{1}{2} d_{\mathfrak{X}}(r, n) - \psi_0(d_{\mathfrak{X}}(r, n), 0)$$

. This means that $d_{\mathfrak{X}}(r, n) = 0$, that is, $n = r$. Therefore, r is the best proximity point for F. The uniqueness of the best neighbour can easily follow the process in the theorem (3.5).

Corollary 3.9. *Let M and N be two non-empty members of a complete metric space $(\mathfrak{X}, d_{\mathfrak{X}})$ s.t. M is complete, the pair (M, N) has the V - property and M_0 is non-empty. Let $F : M \to N$ and $g_0 : M \to M$ are continuous $\alpha_0 - (\psi_0, g_0)$ - proximal contraction map of Type-I or a continuous $\alpha_0 - (\psi_0, g_0)$ - proximal contraction map of Type-II in such a way that the following terms and conditions hold:*

1. *F is a triangle $\alpha_0 - (\psi_0, g_0)$ - allowable near-end mapping and $F(M_0) \subseteq N_0$),*
2. *there exists elements $p_0, p_1 \in M_0$ such that*

   $$d_{\mathfrak{X}}(g_0 p_1, F p_0) = d_{\mathfrak{X}}(M, N)$$

   and $\alpha_0(g_0 p_0, g_0 p_1) \geq 1$,
3. *if $\{g_0 p_a\}$ is a sequence in M such that $\alpha_0(g_0 p_a, g_0 p_{a+1}) \geq 1$ and $g_0 p_a \to g_0 p$ as $a \to +\infty$, then $\alpha_0(g_0 p_a, g_0 p) \geq 1$ for all $a \in \mathbb{N} \cup \{0\}$.*

Then the F will have a best proximity point. Furthermore, for every $p, q \in M$ s.t. $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$, we have $\alpha_0(g_0p, g_0q) \geq 1$.

4. Some results in metric spaces endowed with a graph

Consistent with Jachymski [8], let $(\mathfrak{X}, d_{\mathfrak{X}})$ be a metric space and Δ denotes the diagonal of the cartesian product $\mathfrak{X} \times \mathfrak{X}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with $\mathfrak{X}$, and the set $\mathfrak{E}(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. The first result in this direction was given by Jachymski [16].

Definition 4.1. *Suppose that $(\mathfrak{X}, d\mathfrak{X})$ is a metric space containing a G graph. We say that a self-mapping $F : \mathfrak{X} \to \mathfrak{X}$ is a contraction of Banach G, or simply a contraction of G, if F retains the contour of G, i.e.*

$\forall p, q \in \mathfrak{X},\ (p, q) \in \mathfrak{E}(G) \Longrightarrow (Fp, Fq) \in \mathfrak{E}(G)$

And F reduces the weight of the G edges as follows:

$\exists\ \alpha_0 \in (0, 1),\ \forall\ p, q \in \mathfrak{X},\ (p, q) \in \mathfrak{E}(G) \Longrightarrow d_{\mathfrak{X}}(Fp, Fq) \leq \alpha_0 d_{\mathfrak{X}}(p, q)$.

Definition 4.2. *let M and N be two non-empty closed subsets of a metric space $(\mathfrak{X}, d_{\mathfrak{X}})$ own graph G. We are suggesting that $F : M \to N$ is a non-self map and $g_0 : M \to M$ are $G - (\psi_0, g_0)$ - proximal contraction, if, $m, n, p, q \in M$*

$$\begin{cases} (g_0p, g_0q) \in \mathfrak{E}(G), \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(N, Fq) = d_{\mathfrak{X}}(M, N). \end{cases}$$

$$\Longrightarrow d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, n) + d_{\mathfrak{X}}(g_0q, m)) - \psi_0(d_{\mathfrak{X}}(g_0p, n), d_{\mathfrak{X}}(g_0q, m))$$

and

$$\begin{cases} (g_0p, g_0q) \in \mathfrak{E}(G), \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N). \end{cases} \Longrightarrow (m, n) \in \mathfrak{E}(G),$$

Theorem 4.3. *Let M and N be two non-empty closed subsets of a complete metric space $(\mathfrak{X}, d_{\mathfrak{X}})$ endowed with a graph G. Assume that M is complete. and M_0 is non-empty and $F : M \to N$ and $g_0 : M \to M$ are continuous $G - (\psi_0, g_0)$ - proximal contraction map in such a way that the given terms and conditions hold:*

1. *$F(M_0) \subseteq N_0$,*

2. *then there exists elements $p_0, p_1 \in M_0$ such that*

$$d_{\mathfrak{X}}(g_0p_0, g_0p_0) = d_{\mathfrak{X}}(M, N)$$

and $(g_0p_0, g_0p_1) \in \mathfrak{E}(G)$,

3. *for all* $(g_0p, g_0q) \in \mathfrak{E}(G)$ *and* $(g_0q, g_0r) \in \mathfrak{E}(G)$, *we have* $(g_0p, g_0r) \in \mathfrak{E}(G)$.

Next, F *has a best proximity point. Additionally, the best proximity point is unique if, for every* $p, q \in M$ *such that* $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$, *we have* $(g_0p, g_0q) \in \mathfrak{E}(G)$.

Proof. Define $\alpha_0 : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$ by

$$\alpha_0(g_0p, g_0q) = \begin{cases} 1, & if\ (g_0p, g_0q) \in \mathfrak{E}(G), \\ 0, & otherwise. \end{cases}$$

First, we prove that F is a triangle $\alpha_0 - (\psi_0, G)$-near-end allowable map.

$$\begin{cases} \alpha_0(g_0p, g_0q) \geq 1, \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N). \end{cases}$$

Therefore, we obtain

$$\begin{cases} (g_0p, g_0q) \in \mathfrak{E}(G), \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N). \end{cases}$$

Since F is a $G - (\psi_0, g_0)$ - proximal contraction map, we get $(m, n) \in \mathfrak{E}(G)$, that is $\alpha_0(g_0m, g_0n) \geq 1$ and

$$d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, n) + d_{\mathfrak{X}}(g_0q, m)) - \psi_0(d_{\mathfrak{X}}(g_0p, n) d_{\mathfrak{X}}(g_0q, m)).$$

Also, let $\alpha_0(g_0p, r) \geq 1$ and $\alpha_0(r, g_0q) \geq 1$, then $\alpha_0(r, g_0q) \geq 1$, then $(r, g_0q) \geq 1$, then $(g_0p, r) \in \mathfrak{E}(G)$ and $(r, g_0q) \in \mathfrak{E}(G)$. As a result, we deduce from (3) that $(g_0p, g_0q) \in \mathfrak{E}(G)$ is $\alpha_0(g_0p, g_0q) \geq 1$.

Thus, F be $\alpha_0 - (\psi_0, g_0)$ - triangular proximal admissible mapping with $F(M_0) \subseteq N_0$. In addition, F is continuously modified $\alpha_0 - (\psi_0, g_0)$ - proximal contraction. From (2), there is $p_0, p_1 \in M_0$ s.t. $d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$ and $(g_0p_0, g_0p_1) \in \mathfrak{E}(G)$, that is, $d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$ and $\alpha_0(g_0p_0, g_0p_1) \geq 1$. All the conditions of Theorem (3.5) are thus fulfilled, and F has a single fixed point.

In the same way, we prove the following theorem by using the Theorem (3.8).

Theorem 4.4. *Presume* M *and* N *are non-empty closed members of a metric* *space* $(\mathfrak{X}, d_{\mathfrak{X}})$ *provided with a graph* G. *Assume that,* M *is complete, the pair* (M, N) *has the* V *- property and* M_0 *is non-empty. Also, suppose that* $F : M \to N$ *and* $g_0 : M \to M$ *are* $G - (\psi_0, g_0)$ *- proximal contraction map in a way that the following criteria hold:*

1. $F(M_0) \subseteq N_0$,
2. *there exists elements* $p_0, q_1 \in M_0$ *such that*
$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$
and $(g_0p_0, g_0p_1) \in \mathfrak{E}(G)$,
3. $\forall\ (p, q) \in \mathfrak{E}(G)$ *and* $(q, r) \in \mathfrak{E}(G)$, *we get* $(p, r) \in \mathfrak{E}(G)$
4. *if* $\{p_a\}$ *is a sequence in* $\mathfrak{X}$ *such that* $(p_a, p_{a+1}) \in \mathfrak{E}(G)$ *for all* $a \in \mathbb{N} \cup \{0\}$ *and* $p_a \to p$ *as* $a \to +\infty$, *so* $(p_a, p) \in \mathfrak{E}(G)\ \forall\ a \in \mathbb{N} \cup \{0\}$.

Then, F *has a best proximity point. Further, the best proximity point is unique if, for each* $p, q \in M$ *just like that* $d_{\mathfrak{X}}(p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(q, Fq)$, *we get* $(p, q) \in \mathfrak{E}(G)$.

We collect multiple fixed point theorems in this chapter, which are consequences of the results mentioned in the important area.

Theorem 4.5. *Let* $(\mathfrak{X}, d_{\mathfrak{X}})$ *be a complete metric space. Assume that* $F : X \to X$ *and* $g_0 : M \to M$ *be a continuous self-map fulfills the below requirements:*

1. (a) F *is triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*
   (b) *there is* p_0 *in* $\mathfrak{X}$ *so* $\alpha_0(g_0p_0, Fp_0) \geq 1$,
   (c) *for all* $p, q \in \mathfrak{X}$, $\alpha_0(g_0p, g_0q)d_{\mathfrak{X}}(Fp, Fq) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, Fq) + d_{\mathfrak{X}}(g_0q, Fp)) - \psi_0(d_{\mathfrak{X}}(g_0p, Fq), d_{\mathfrak{X}}(g_0q, Fp))$. *Then there's a fixed point of* F.
2. (a) F *be* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*
   (b) $\exists\ p_0$ *in* $\mathfrak{X}$ *s.t.* $\alpha_0(g_0p_0, Fp_0) \geq 1$,
   (c) $\forall\ p, q \in \mathfrak{X}$,
$$(\alpha_0(g_0p, g_0q)+l)^{d_{\mathfrak{X}}(Fp,Fq)} \leq (u+1)^{\frac{1}{2}(d_{\mathfrak{X}}(g_0p,Fq)+d_{\mathfrak{X}}(g_0q,Fp))-\psi_0(d_{\mathfrak{X}}(g_0p,Fq),d_{\mathfrak{X}}(g_0q,Fp)}.$$
   Then F *has a fixed point.*
3. (a) F *is triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*

(b) *there is* p_0 *in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_0, Fp_0) \geq 1$,

(c) *absolutely* $p, q \in \mathfrak{X}$, $\alpha_0(g_0p, g_0q)d_{\mathfrak{X}}(Fp, Fq) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, Fq)+d_{\mathfrak{X}}(g_0q, Fp)) - \psi_0(d_{\mathfrak{X}}(g_0p, Fq), d_{\mathfrak{X}}(g_0q, Fp))$. *Then* F *has a fixed point.*

(d) *if* $\{g_0p_a\}$ *is a sequence in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_a, g_0p_{a+1}) \geq 1$ *and* $p_a \to p$ *as* $a \to +\infty$, *then* $\alpha_0(g_0p_a, g_0p) \geq 1$ $\forall$ $a \in \mathbb{N}$. *Then there is a fixed point at* F.

4. (a) F *be triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*

(b) *there is* p_0 *in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_0, Fp_0) \geq 1$,

(c) $\forall$ $p, q \in \mathfrak{X}$,

$$(\alpha_0(g_0p, g_0q)+1)^{d_{\mathfrak{X}}(Fp,Fq)} \leq 2^{[\frac{1}{2}(d_{\mathfrak{X}}(g_0p,Fq)+d_{\mathfrak{X}}(g_0q,Fp))-\psi_0(d_{\mathfrak{X}}(g_0p,Fq),d_{\mathfrak{X}}(g_0q,Fp))]}.$$

Then F *has a fixed point.*

(d) *if a sequence* $\{g_0p_a\}$ *in* M *such that* $\alpha_0(g_0p_a, g_0p_{a+1}) \geq 1$ *and* $p_a \to p$ *as* $a \to +\infty$, *then* $\alpha_0(g_0p_a, g_0p) \geq 1$ $\forall$ $a \in \mathbb{N}$. *Then there is a fixed point of* F.

5. Application to fixed point theorems

Theorem 5.1. *Let* $(\mathfrak{X}, d_{\mathfrak{X}})$ *be a complete metric space. Assume that* $F : \mathfrak{X} \to \mathfrak{X}$ *and* $g_0 : M \to M$ *are continuous self-mapping satisfying the following conditions:*
(i) F *is triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*
(ii) there exists p_0 *in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_0, Fp_0) \geq 1$,
(iii) for all $p, q \in \mathfrak{X}$,

$$\alpha_0(g_0p, g_0q)d_{\mathfrak{X}}(Fp, Fq) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, Fq)+d_{\mathfrak{X}}(g_0q, Fp))-\psi_0(d_{\mathfrak{X}}(g_0p, Fq), d_{\mathfrak{X}}(g_0q, Fp)).$$

Then T *has a fixed point.*

Theorem 5.2. *Let* $(\mathfrak{X}, d_{\mathfrak{X}})$ *be a complete metric space. Assume that* $F : \mathfrak{X} \to \mathfrak{X}$ *and* $g_0 : M \to M$ *are continuous self-mapping satisfying the following conditions:*

1. F *is triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*

2. *there exists* p_0 *in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_0, Fp_0) \geq 1$,

3. *for all* $p, q \in \mathfrak{X}$,

$$(\alpha_0(g_0p, g_0q) + l)^{d_{\mathfrak{X}}(Fp,Fq)} \leq (l+1)^{\frac{1}{2}(d_{\mathfrak{X}}(g_0p,Fq)+d_{\mathfrak{X}}(g_0q,Fp))-\psi_0(d_{\mathfrak{X}}(g_0p,Fq),d_{\mathfrak{X}}(g_0q,Fp)}.$$

Then T *has a fixed point.*

Theorem 5.3. *Let $(\mathfrak{X}, d_{\mathfrak{X}})$ be a complete metric space. Assume that $F : \mathfrak{X} \to \mathfrak{X}$ and $g_0 : M \to M$ are continuous self-mapping satisfying the following conditions:*

1. *F is triangular $\alpha_0 - (\psi_0, g_0)$ - admissible,*
2. *there exists p_0 in $\mathfrak{X}$ such that $\alpha_0(g_0p_0, Fp_0) \geq 1$,*
3. *for all $p, q \in \mathfrak{X}$, $\alpha_0(g_0p, g_0q)d_{\mathfrak{X}}(Fp, Fq) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, Fq) + d_{\mathfrak{X}}(g_0q, Fp)) - \psi_0(d_{\mathfrak{X}}(g_0p, Fq), d_{\mathfrak{X}}(g_0q, Fp))$. Then F has a fixed point.*
4. *if $\{g_0p_n\}$ is a sequence in $\mathfrak{X}$ such that $\alpha_0(g_0p_n, g_0p_{n+1}) \geq 1$ and $p_n \to p$ as $n \to +\infty$, then $\alpha_0(g_0p_n, g_0p) \geq 1$ for all $n \in \mathbb{N}$. Then F has a fixed point.*

Theorem 5.4. *Let $(\mathfrak{X}, d_{\mathfrak{X}})$ be a complete metric space. Assume that $F : \mathfrak{X} \to \mathfrak{X}$ and $g_0 : M \to M$ are continuous self-mapping satisfying the following conditions:*

1. *F is triangular $\alpha_0 - (\psi_0, g_0)$ - admissible,*
2. *there exists p_0 in $\mathfrak{X}$ such that $\alpha_0(g_0p_0, Fp_0) \geq 1$,*
3. *for all $p, q \in \mathfrak{X}$,*

$$(\alpha_0(g_0p, g_0q) + 1)^{d_{\mathfrak{X}}(Fp, Fq)} \leq 2^{[\frac{1}{2}(d_{\mathfrak{X}}(g_0p, Fq) + d_{\mathfrak{X}}(g_0q, Fp)) - \psi_0(d_{\mathfrak{X}}(g_0p, Fq), d_{\mathfrak{X}}(g_0q, Fp))]}.$$

   Then F has a fixed point.

4. *if $\{g_0p_n\}$ is a sequence in M such that $\alpha_0(g_0p_n, g_0p_{n+1}) \geq 1$ and $p_n \to p$ as $n \to +\infty$, then $\alpha_0(g_0p_n, g_0p) \geq 1$ for all $n \in \mathbb{N}$. Then F has a fixed point.*

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