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ROUGH IDEAL CONVERGENT SEQUENCE SPACES OF BOUNDED LINEAR OPERATORS

Shivani Sharma and Sanjay Mishra

Department of Mathematics,
Lovely Professional University,
Phagwara - 144411, Punjab, INDIA

E-mail : shivani.saggi@gmail.com, drsanjaymishra1@gmail.com

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Abstract: In this paper, using the concept of rough ideal convergence in normed linear spaces, we introduce rough ideal convergence for bounded linear operators. We also introduce some rough ideal convergent sequence spaces of bounded linear operators and further investigate and study some inclusion relations of these spaces, decomposition theorem and algebraic properties.

Keywords and Phrases: Ideal, Filter, Rough Ideal Convergence, Bounded Linear Operator.

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1. Introduction and Preliminaries

Convergence of sequences has always remained a subject of interest to the researchers. Several new types of convergence of sequences were introduced and studied by the researchers and named as usual convergence, uniform convergence, strong convergence, weak convergence, statistical convergence, ideal convergence etc. In 2001, the notion of rough convergence was first introduced by Phu [22] for finite dimensional normed linear spaces. It is a new type of convergence which involves extending the radius of convergence of a non-convergent but bounded sequence.

Let (x_i) be a sequence in some normed linear space $(X, \|\cdot\|)$ and r be any non-negative real number. Then (x_i) is said to be r-convergent to x_*, denoted by

$x_i \xrightarrow{r} x_*$, if there exist $i_\varepsilon \in \mathbb{N}$ such that

$$i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon, \forall \varepsilon > 0.$$

where r and x_* are called the roughness degree and the r-limit point of the sequence (x_i) respectively. It is easy to see that r-limit is not unique and for $r = 0$ we get the classical convergence. The set of r-limit points of the sequence (x_i) is denoted by $LIM^r x_i = \{x_* \colon x_i \xrightarrow{r} x_*\}$. $LIM^r x_i$ is bounded, closed and convex.

Phu also introduced the notion of rough Cauchy sequence and investigated the dependence of $LIM^r x_i$ on the roughness degree r. He also defined rough limit points and degree of roughness and studied basic results for finite dimensional normed linear spaces. Since then it has attracted a lot of interest from various scholars. This notion has been extended to various spaces like infinite dimensional normed linear spaces [24], metric spaces, S-metric spaces, cone metric spaces in [19], [5] and [6].

The statistical version of rough convergence was introduced by Aytar in [2]. In [4] he defined the concepts of rough statistical cluster point and rough statistical limit point of a sequence in a finite dimensional normed space and applied these definitions to sequences of functions. Recently, in [7] and [8] Demrici et al. introduced and studied rough $\mathcal{I}_2$ statistical convergence and rough statistical φ-convergence.

The idea of ideal convergence was given by Kostyrko et al. in [15] which is a generalization of statistical convergence. Since then, various authors have investigated its various generalizations and applications in several spaces [9], [25], [26], [28], [20] etc. Rough ideal convergence was introduced by Pal et al. in [21] and Dundar et al. [10] gave the idea of rough I-convergence in normed linear spaces independently. Furthermore, various other related results were introduced and studied in [2], [17], [4] and [18]. Various attempts have been made to study the algebraic, topological and geometrical properties of $LIM^r x_i$ of rough convergent sequences in various spaces.

Several applications of statistical convergence and ideal convergence have been discussed in summability theory and approximation theory in [9], [11], [12] and [27]. Sequences of bounded linear operators are a common occurrence, often associated with problems such as Fourier series convergence, interpolation polynomial sequences, and numerical integration techniques, as seen in [1], [16] and [13]. In these scenarios, the focus typically revolves around the convergence of operator sequences, the boundedness of associated norm sequences, or similar characteristics. Several sequences which are non-convergent in usual sense are rough ideal convergent. Thus, rough ideal convergence can be studied for the sequences of bounded linear operators. In this paper, we have extended the notion of rough ideal convergence to bounded linear operators using the results in [21]. We have defined some rough ideal convergent sequence spaces of operators. We have also studied some properties of these spaces when topologized through a norm and investigated inclusion relations, equivalent conditions, decomposition theorem and algebraic properties of such spaces.

We now recall some definitions and results from [21] and [14] that will be used in the next section of this paper.

Definition 1.1. (Ideal) *A non-empty collection $\mathcal{I}$ of subsets of non-empty set X is called an ideal on X if*

1. $\emptyset \in \mathcal{I}$,
2. *If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ and*
3. *If $A \in \mathcal{I}$, and $B \subseteq A$, then $B \in \mathcal{I}$.*

If for each $x \in X$, $\{x\} \in \mathcal{I}$, then $\mathcal{I}$ is called admissible. If $\mathcal{I} \neq \emptyset$ and $\mathcal{I} \neq \mathbb{P}(X)$, then $\mathcal{I}$ is called non-trivial.

Definition 1.2. (Filter) *A non-empty collection $\mathcal{F}$ of subsets of non-empty set X is called a filter on X if*

1. $\emptyset \notin \mathcal{F}$,
2. *If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ and*
3. *If $A \in \mathcal{F}$, and $A \subseteq B$, then $B \in \mathcal{F}$.*

Definition 1.3. (Filter associated with Ideal) *Let $\mathcal{I}$ be a non-trivial ideal of X. The family $\mathcal{F}(\mathcal{I}) = \{M \subset X : \exists\, A \in \mathcal{I}, M = X \backslash A\}$ is called filter associated with ideal.*

Definition 1.4. (Ideal Convergence) *A sequence (x_i) in a normed linear space $(X, \|\cdot\|)$ is said to be $\mathcal{I}$-convergent to L, if for every $\varepsilon > 0$, the set*

$$\{i \in \mathbb{N} : \|x_i - L\| \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim x_i = L$.

Definition 1.5. (Rough Ideal Convergence) *Let (x_i) be a sequence in normed linear space $(X, \|\cdot\|)$ and r be any non negative real number. The sequence (x_i) is said to be $r\mathcal{I}$-convergent to x, denoted by $x_i \xrightarrow{r\mathcal{I}} x$, if*

$$\{i \in \mathbb{N} : \|x_i - x\| \geq r + \varepsilon\} \in \mathcal{I}, \forall \varepsilon > 0,$$

where $\mathcal{I}$ is non-trivial admissible ideal on $\mathbb{N}$.

Theorem 1.6. *Let r be a non-negative real number. Then a sequence $x = (x_i)$ is $r\mathcal{I}$-convergent to x_* if and only if there exists a sequence $y = (y_i)$ such that $\mathcal{I} - \lim y = x_*$ and $\|x_i - y_i\| \leq r$, for $i \in \mathbb{N}$.*

Let X and Y be any two normed linear spaces. We denote the set of all linear operators from X to Y by

$$\mathcal{L}(T) = \{T = (T_k) \colon T_k \colon X \to Y, \text{ is linear for each } k \in \mathbb{N}\}.$$

Definition 1.7. (Bounded Linear Operator) *A linear operator $T \colon X \to Y$ is said to be bounded, if there exists a real $M > 0$ such that*

$$\|Tx\| \leq M\|x\|, \forall x \in X.$$

Let $\mathcal{B}_\infty(T)$ be the normed space of sequences of all bounded linear operators from a normed spaces X to Y with norm defined by

$$\|T\| = \sup_k \|T_k(x)\|.$$

The space $\mathcal{B}_\infty(T)$ is a Banach, if Y is a Banach space. Throughout, O and I represent zero and identity operators, respectively.

Definition 1.8. (Ideal Convergence of Sequence of Operators) *Let $\mathcal{I}$ be an ideal. The a sequence $U = (U_k) \in \mathcal{B}_\infty(T)$ is said to be $\mathcal{I}$-convergent to an operator T, if for every $\varepsilon > 0$, the set*

$$\{k \in \mathbb{N} \colon \|T_k(x) - T(x)\| \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim T_k = T$.

Definition 1.9. *Let X and Y be two normed linear spaces. A sequence (U_k) of operators $U_k \in \mathcal{B}_\infty(T)$ is said to be*

1. *Uniformly convergent, if (U_k) converges in the norm on $\mathcal{B}_\infty(T)$ i.e. $\|U_k - T\| \to 0$.*

2. *Strongly convergent, if $(U_k x)$ converges strongly in Y for every $x \in X$ i.e. $\|U_k(x) - T(x)\| \to 0$, for every $x \in X$.*

3. *Weakly convergent, if $(U_k(x))$ converges weakly in Y for every $x \in X$ i.e. $|h(U_k x) - h(Tx)| \to 0$, for every $x \in X$ and $h \in Y'$.*

Definition 1.10. (Sequence Space) *Any subspace λ of a linear space of sequences Λ is called sequence space.*

Definition 1.11. (Solid Sequence Space) *A sequence space P of operators is said to be solid (or normal), if $(\alpha_k U_k) \in P$ whenever $U_k \in P$ and for any sequence (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$ and is said to be monotone, if it contains the canonical preimages of its step space.*

Remark 1.12. *Every solid space is monotone.*

Definition 1.13. (Lipschitz Function) *Let X be any non-empty space. A function $f\colon X \to \mathbb{R}$ is said to be a Lipschitz if it satisfies Lipschitz condition,*

$$|f(x) - f(y)| \leq K|x - y|,$$

where K is known as the Lipschitz constant.

Throughout this paper, let $\mathcal{I}$ be a non-trivial admissible ideal on $\mathbb{N}$ and r be a non-negative real number. We consider the operators U_k, for each $k \in \mathbb{N}$ from the normed spaces $X = \mathbb{R}$ to $Y = \mathbb{R}$ over the field $\mathbb{R}$.

Motivated by the ideal convergence of sequence of operators, we define rough ideal convergence of sequence of operators.

Definition 1.14. (Rough Ideal Convergence of Sequence of Operators) *Let $\mathcal{I}$ be an ideal. A sequence $U = (U_k) \in \mathcal{B}_\infty(T) \subset \mathcal{L}(T)$ is said to be $r\mathcal{I}$-convergent to an operator T, if for $r > 0$ and $\varepsilon > 0$, the set*

$$\{k \in \mathbb{N}\colon \|U_k(x) - T(x)\| \geq r + \varepsilon\} \in \mathcal{I}.$$

In this case, we write $r\mathcal{I} - \lim U_k = T$.

Now we introduce the following rough ideal convergent classes of sequences of operators.

$$\mathcal{C}^{R\mathcal{I}}(T) = \{U = (U_k) \in \mathcal{B}_\infty(T)\colon \{k \in \mathbb{N}\colon \|U_k(x) - L(x)\| \geq r + \varepsilon\} \in \mathcal{I}, \text{ for some } L \in \mathcal{L}(T), r > 0\}, \text{ and}$$
$$\mathcal{C}_0^{R\mathcal{I}}(T) = \{U = (U_k) \in \mathcal{B}_\infty(T)\colon \{k \in \mathbb{N}\colon \|U_k(x)\| \geq r + \varepsilon\} \in \mathcal{I}, r > 0\}.$$

We also denote

$$\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T) = \mathcal{B}_\infty(T) \cap \mathcal{C}^{R\mathcal{I}}(T),\ \mathcal{G}_{\mathcal{C}_0}^{R\mathcal{I}}(T) = \mathcal{B}_\infty(T) \cap \mathcal{C}_0^{R\mathcal{I}}(T), \text{and}$$
$$\mathcal{C}^{\mathcal{I}}(T) = \{U = (U_k) \in \mathcal{B}_\infty(T)\colon \{k \in \mathbb{N}\colon \|U_k(x) - L(x)\| \geq \varepsilon\} \in \mathcal{I}, \text{ for some } L \in \mathcal{L}(T)\}$$

2. Main Results

This section is devoted to the study of some algebraic properties of rough ideal convergent classes of sequences of bounded linear operators, which we introduced in the previous section. We have also explored Cauchy like criteria for rough ideal convergent sequences of bounded linear operators and proved some equivalent conditions.

We first show that the classes of rough ideal convergent sequences of bounded linear operators are vector spaces over the field of $\mathbb{R}$.

Theorem 2.1. *The classes $\mathcal{C}^{R\mathcal{I}}(T), \mathcal{C}_0^{R\mathcal{I}}(T), \mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathcal{I}}(T)$ are linear spaces over $\mathbb{R}$.*

Proof. First we prove the space $\mathcal{C}^{R\mathcal{I}}(T)$ is linear. Let $U = (U_k), V = (V_k) \in \mathcal{C}^{R\mathcal{I}}(T)$ and α, β be scalars, then for some $r_1, r_2 > 0$ and for given $\varepsilon > 0$, there exist some $T_1, T_2 \in \mathcal{L}(T)$ such that

$$\{k \in \mathbb{N} \colon \|U_k(x) - T_1(x)\| \geq r_1 + \tfrac{\varepsilon}{2}\}, \{k \in \mathbb{N} \colon \|V_k(x) - T_2(x)\| \geq r_2 + \tfrac{\varepsilon}{2}\} \in \mathcal{I}.$$

Let $r = \max\{r_1, r_2\}$, then

$$\{k \in \mathbb{N} \colon \|U_k(x) - T_1(x)\| \geq r + \tfrac{\varepsilon}{2}\}, \{k \in \mathbb{N} \colon \|V_k(x) - T_2(x)\| \geq r + \tfrac{\varepsilon}{2}\} \in \mathcal{I}.$$

Let

$$P_1 = \{k \in \mathbb{N} \colon \|U_k(x) - T_1(x)\| < r + \tfrac{\varepsilon}{2}\}, P_2 = \{k \in \mathbb{N} \colon \|V_k(x) - T_2(x)\| < r + \tfrac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I}),$$

are such that ${P_1}^c, {P_2}^c \in \mathcal{I}$. Then

$$P_3 = \{k \in \mathbb{N} \colon \|(\alpha U_k)(x) + (\beta V_k)(x) - (\alpha T_1 - \beta T_2)\| \leq 2r + \varepsilon\} \supseteq (P_1 \cap P_2) \in \mathcal{F}(\mathcal{I})$$

Thus, $\alpha U_k + \beta V_k$ is rough ideal convergent, for all scalars α, β and $(U_k), (V_k) \in \mathcal{C}^{R\mathcal{I}}(T)$. Therefore, $\mathcal{C}^{R\mathcal{I}}(T)$ is linear. Similarly, we can prove that spaces $\mathcal{C}_0^{R\mathcal{I}}(T)$, $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathcal{I}}(T)$ are linear.

Let us now equip the linear spaces $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathcal{I}}(T)$ with a norm.

Theorem 2.2. *The spaces $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathcal{I}}(T)$ are normed linear spaces normed by*

$$\|T\|_* = \sup_k \|T_k(x)\|.$$

Proof. Proof omitted.

We now establish a relationship between rough ideal convergent sequences of bounded linear operators and rough Cauchy criteria.

Theorem 2.3. *A sequence $U = (U_k) \in \mathcal{B}_\infty(T)$ $r\mathcal{I}$-converges if and only if for $r > 0$ and $\varepsilon > 0$, there exists a set $N_{r\varepsilon} \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} \colon \|U_k(x) - U_{N_{r\varepsilon}}(x)\| < 2r + \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Proof. Let $U = (U_k) \in \mathcal{B}_\infty(T)$ be $r\mathcal{I}$- convergent for some $r > 0$ and let T be the $r\mathcal{I}$-limit of (U_k). Then the set,

$$B_{r\varepsilon} = \left\{k \in \mathbb{N} \colon \|U_k(x) - T(x)\| < r + \tfrac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}),\ \forall \varepsilon > 0.$$

Fix an $N_{r\varepsilon} \in B_{r\varepsilon}$. Then we have

$$\|U_k(x) - U_{N_{r\varepsilon}}(x)\| \leq \|U_k(x) - T(x)\| + \|U_{N_{r\varepsilon}}(x) - T(x)\| < r + \tfrac{\varepsilon}{2} + r + \tfrac{\varepsilon}{2} = 2r + \varepsilon$$

which holds for all $k \in B_{r\varepsilon}$. Thus, $\{k \in \mathbb{N} \colon \|U_k(x) - U_{N_{r\varepsilon}}(x)\| < 2r + \varepsilon\} \in \mathcal{F}(\mathcal{I})$.

Conversely, suppose that

$$\{k \in \mathbb{N} : \|U_k(x) - U_{N_{r\varepsilon}}(x)\| < 2r + \varepsilon\} \in \mathcal{F}(\mathcal{I}),\ \forall \varepsilon > 0.$$

Clearly, the set

$$K_{r\varepsilon} = \{k \in \mathbb{N} \colon U_k(x) \in [U_{N_{r\varepsilon}}(x) - (2r - \varepsilon), U_{N_{r\varepsilon}}(x) - (2r + \varepsilon)]\} \in \mathcal{F}(\mathcal{I}),\ \forall \varepsilon > 0.$$

Let $I_{r\varepsilon} = [U_{N_{r\varepsilon}}(x) - (2r - \varepsilon), U_{N_{r\varepsilon}}(x) - (2r + \varepsilon)]$. If we can fix $r > 0$ and $\varepsilon > 0$ then we have $K_{r\varepsilon}, K_{\frac{r\varepsilon}{2}} \in \mathcal{F}(\mathcal{I})$, implies $K_{r\varepsilon} \cap K_{\frac{r\varepsilon}{2}} \in \mathcal{F}(\mathcal{I})$. This further implies that $I_\varepsilon = I_{r\varepsilon} \cap I_{\frac{r\varepsilon}{2}} \neq \emptyset$. That is $\{k \in \mathbb{N} \colon U_k(x) \in I_\varepsilon\} \in \mathcal{F}(\mathcal{I})$. Let $diam\, I_\varepsilon$ be the length of the interval I_ε. Clearly, $diam\, I_\varepsilon \leq diam\, I_{r\varepsilon}$.

Continuing in this way, we can obtain a sequence of closed intervals

$$I_{r\varepsilon} = A_0 \supset A_1 \supset \ldots \supset A_k \supset \ldots$$

with the property that $diam\, A_k \leq \frac{1}{2}\, diam A_{k-1}, k = 2, 3, \ldots$ and $\{k \in \mathbb{N} \colon U_k(x) \in A_k\} \in \mathcal{F}(\mathcal{I})$ for $k = 1, 2, \ldots$. Then there exists some $\zeta \in \cap A_k, k \in \mathbb{N}$ such that $\zeta = r\mathcal{I} - \lim U_k(x)$. Thus, $U = (U_k) \in \mathcal{B}_\infty(T)$ $r\mathcal{I}$-convergent.

In the following theorem, we explore the relationship between ideal convergence and rough ideal convergence and give some equivalent conditions.

Theorem 2.4. *Let $\mathcal{I}$ be an admissible ideal. Then the following are equivalent:*

1. $(U_k) \in \mathcal{C}^{R\mathcal{I}}(T)$,

2. *For all $k \in \mathcal{I}$ there exists $(V_k) \in \mathcal{C}^{\mathcal{I}}(T)$ such that $\|U_k(x) - V_k(x)\| \leq r, r > 0$,*

3. *For all* $k \in \mathcal{I}$ *there exists* $(V_k) \in \mathcal{C}^{\mathcal{I}}(T)$ *and* $(W_k) \in {\mathcal{C}_0}^{R\mathcal{I}}(T)$ *such that* $U_k = V_k + W_k$,

4. *There exists a subset* $K = \{k_1, k_2, \ldots\}$ *of* $\mathbb{N}$ *such that* $K \in \mathcal{F}(\mathcal{I})$ *and* $\lim_{n\to\infty} \|U_{k_n}(x) - T(x)\| < r$, *where* T *is the* $r\mathcal{I}$*-limit of* (U_k).

Proof.

(1) $\Rightarrow$ **(2)** Let $(U_k) \in \mathcal{C}^{R\mathcal{I}}(T)$. Then for some $r > 0$ and $\varepsilon > 0$, there exists some T such that the set

$$\{k \in \mathbb{N}\colon \|U_k(x) - T(x)\| \geq r + \varepsilon\} \in \mathcal{I}.$$

Then by Theorem 1.6, there exists a sequence (V_k) as

$$V_k = \begin{cases} T, & \|U_k(x) - T(x)\| \leq r, \\ U_k + r\frac{T-U_k}{\|U_k - T\|}, & \text{otherwise.} \end{cases}$$

Clearly, $(V_k) \in \mathcal{C}^{\mathcal{I}}(T)$ and $\|U_k(x) - V_k(x)\| \leq r$, for all $k \in \mathbb{N}$.

(2) $\Rightarrow$ **(3)** We are given that for $(U_k) \in \mathcal{C}^{R\mathcal{I}}(T)$, then there exists $(V_k) \in \mathcal{C}^{\mathcal{I}}(T)$ such that for all $k \in \mathcal{I}$, $\|U_k(x) - V_k(x)\| \leq r$, where $r > 0$. Let $K = \{k \in \mathbb{N}\colon \|U_k - V_k\| > r\}$, then $K \in \mathcal{I}$. Define a sequence

$$W_k = \begin{cases} U_k - V_k, & k \in K, \\ O, & \text{otherwise.} \end{cases}$$

Then, $(W_k) \in \mathcal{C}_0^{R\mathcal{I}}(T)$.

(3) $\Rightarrow$ **(4)** Let $A = \{k \in \mathbb{N}\colon \|W_k(x)\| > r + \frac{\varepsilon}{2}\}$. Then $A^c \in \mathcal{F}(\mathcal{I})$. Let $A^c = K = \{k_1, k_2, k_3 \ldots\}$. Then,

$$\|W_{k_n}(x)\| < r + \tfrac{\varepsilon}{2}.$$

This implies that

$$\|U_{k_n}(x) - V_{k_n}(x)\| < r + \tfrac{\varepsilon}{2}.$$

Now,

$$\|U_{k_n}(x) - T(x)\| \leq \|U_{k_n}(x) - V_{k_n}(x)\| + \|V_{k_n}(x) - T(x)\| < r + \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = r + \varepsilon$$

Therefore, $\lim_{n\to\infty} \|U_{k_n}(x) - T(x)\| < r$.

(4) ⇒ (1) For $\varepsilon > 0$, we have

$$\{k \in \mathbb{N}\colon \|U_k(x) - T(x)\| \geq r + \varepsilon\} \subseteq K^c \cup \{k \in K\colon \|U_k(x) - T(x)\| \geq r + \varepsilon\}.$$

Hence, $(U_k) \in \mathcal{C}^{R\mathcal{I}}(T)$.

Theorem 2.5. *The inclusions* ${\mathcal{C}_0}^{R\mathcal{I}}(T) \subset \mathcal{C}^{R\mathcal{I}}(T) \subset \mathcal{B}_\infty(T)$ *hold.*
Proof. The inclusion ${\mathcal{C}_0}^{R\mathcal{I}}(T) \subset \mathcal{C}^{R\mathcal{I}}(T)$ is obvious. To show $\mathcal{C}^{R\mathcal{I}}(T) \subset \mathcal{B}_\infty(T)$, let $(U_k) \in \mathcal{C}^{R\mathcal{I}}(T)$. Then there exists some $T \in \mathcal{B}_\infty(T)$ such that the set

$$\{k \in \mathbb{N}\colon \|U_k(x) - T(x)\| \geq r + \varepsilon\} \in \mathcal{I}.$$

Now,

$$\|U_k(x)\| = \|U_k(x) - T(x) + T(x)\| \leq \|U_k(x) - T(x)\| + \|T(x)\|.$$

Taking supremum over k from both sides in the above inequality, we obtain, $(U_k) \in \mathcal{B}_\infty(T)$.

We now construct a Lipschitz function with the help of rough ideal convergent sequence space of bounded linear operators.

Theorem 2.6. *The function* $\mathcal{L}\colon \mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T) \to \mathbb{R}$ *defined by* $\mathcal{L}(U) = \|r\mathcal{I} - \lim U\|$, *for a fixed* $r > 0$ *is a Lipschitz function and therefore uniformity continuous.*
Proof. For a fixed $r > 0$, we first show that the function $\mathcal{L}$ is well defined. Let, $U, V \in \mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ be such that

$$U = V \Rightarrow r\mathcal{I} - \lim U = r\mathcal{I} - \lim V \Rightarrow \|r\mathcal{I} - \lim U\| = \|r\mathcal{I} - \lim V\|.$$

Thus, $\mathcal{L}$ is well defined. Then the sets

$$K_U = \{k \in \mathbb{N}\colon \|U_k(x) - \mathcal{L}(U)\| \geq r + \|U - V\|\} \in \mathcal{I}, \text{ and}$$
$$K_V = \{k \in \mathbb{N}\colon \|V_k(x) - \mathcal{L}(V)\| \geq r + \|U - V\|\} \in \mathcal{I},$$

where $U = (U_k), V = (V_k)$, and $\|U - V\| = \sup_k \|(U_k - V_k)(x)\|$. Clearly,

$${K_U}^c = \{k \in \mathbb{N}\colon \|U_k(x) - \mathcal{L}(U)\| < r + \|U - V\|\} \in \mathcal{F}(\mathcal{I}), \text{ and}$$
$${K_V}^c = \{k \in \mathbb{N}\colon \|V_k(x) - \mathcal{L}(V)\| < r + \|U - V\|\} \in \mathcal{F}(\mathcal{I}).$$

Hence, $K = {K_U}^c \cap {K_V}^c \in \mathcal{F}(\mathcal{I})$ is nonempty. For, $k \in K$

$$\begin{aligned}\|\mathcal{L}(U) - \mathcal{L}(V)\| &\leq \|\mathcal{L}(U) - U_k(x)\| + \|U_k(x) - V_k(x)\| + \|V_k(x) - \mathcal{L}(V)\| \\ &< (2r + 1)\|U - V\|.\end{aligned}$$

In the consequent theorems, we investigate some algebraic properties of the rough ideal convergent sequence spaces of bounded linear operators.

Theorem 2.7. *If $U = (U_k), V = (V_k) \in \mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ with $U_kV_k(x) = U_k(x) \cdot V_k(x)$, then $(U \cdot V) \in \mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ but $\mathcal{L}(U \cdot V) \neq \mathcal{L}(U)\mathcal{L}(V)$, where $\mathcal{L}(U) = \|r\mathcal{I} - \lim U\|$, for a fixed $r > 0$.*

Proof. For $\varepsilon = \|U - V\| = \sup_k \|(U_k - V_k)(x)\|$, we have

$$K_1 = \{k \in \mathbb{N} \colon \|U_k(x) - \mathcal{L}(U)\| < r + \varepsilon\} \in \mathcal{F}(\mathcal{I}), \text{ and}$$
$$K_2 = \{k \in \mathbb{N} \colon \|V_k(x) - \mathcal{L}(V)\| < r + \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Now,

$$\begin{aligned}\|U_kV_k(x) - \mathcal{L}(U)\mathcal{L}(V)\| &= \|U_k(x) \cdot V_k(x) - U_k(x)\mathcal{L}(V) + U_k(x)\mathcal{L}(V) - \mathcal{L}(U)\mathcal{L}(V)\| \\ &\leq \|U_k(x)\|\|V_k(x) - \mathcal{L}(V)\| + \|\mathcal{L}(V)\|\|U_k(x) - \mathcal{L}(U)\|.\end{aligned}$$

Since $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T) \subset \mathcal{B}_\infty(T)$, then there exists some $M \in \mathbb{R}$ such that $\|U_k(x)\| \leq M$. Therefore, we have

$$\begin{aligned}\|U_kV_k(x) - \mathcal{L}(U)\mathcal{L}(V)\| &\leq M(r + \varepsilon) + \|\mathcal{L}(V)\|(r + \varepsilon) \\ &= r(M + \|\mathcal{L}(V)\|) + \varepsilon(M + \|\mathcal{L}(V)\|) \\ &= r_* + \varepsilon_*, \ \forall k \in K_1 \cap K_2.\end{aligned}$$

Theorem 2.8. *The spaces $\mathcal{C}_0^{R\mathcal{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathcal{I}}(T)$ are solid and monotone.*

Proof. We shall prove the result for $\mathcal{C}_0^{R\mathcal{I}}(T)$ and the result for $\mathcal{G}_{\mathcal{C}_0}^{R\mathcal{I}}(T)$ can be obtained similarly. Let $(U_k) \in \mathcal{C}_0^{R\mathcal{I}}(T)$. Then

$$\{k \in \mathbb{N} \colon \|U_k(x)\| \geq r + \varepsilon\} \in \mathcal{I}.$$

Let (α_k) be a sequence of scalars with $\|\alpha_k\| \leq 1$ for all $k \in \mathbb{N}$. Then,

$$\|\alpha_k U_k(x)\| = \|\alpha_k\|\|U_k(x)\| \leq \|U_k(x)\|, \ \forall k \in \mathbb{N}.$$

Therefore,

$$\{k \in \mathbb{N} \colon \|\alpha_k U_k(x)\| \geq r + \varepsilon\} \in \mathcal{I}.$$

Thus, $\mathcal{C}_0^{R\mathcal{I}}(T)$ is solid and since every solid space is monotone, it follows that $\mathcal{C}_0^{R\mathcal{I}}(T)$ is solid and monotone.

Theorem 2.9. *The set $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ is a closed subspace of $\mathcal{B}_\infty(T)$.*

Proof. Let $U^n = ({U_k}^{(n)})$ be a Cauchy sequence in $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ such that ${U_k}^{(n)} \to U$. Since ${U_k}^{(n)} \in \mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$, there exists A_n such that for some $r > 0$

$$\{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - A_n\| \geq r + \varepsilon\} \in \mathcal{I}.$$

We need to show that

(a) (A_n) converges to A.

(b) If $V = \{k \in \mathbb{N} \colon \|U_k(x) - A\| < r + \varepsilon\}$, then $V^c \in \mathcal{I}$.

(a) Since $({U_k}^{(n)})$ is a Cauchy sequence in $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ implies that for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_k \|{U_k}^{(n)}(x) - {U_k}^{(m)}(x)\| < \tfrac{\varepsilon}{3}, \forall n, m \geq n_0.$$

For a given $\varepsilon > 0$, and some $r > 0$ consider

$$\begin{aligned} P_{nm} &= \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - {U_k}^{(m)}(x)\| < \tfrac{r+\varepsilon}{3}\}, \\ P_m &= \{k \in \mathbb{N} \colon \|{U_k}^{(m)}(x) - A_m\| < \tfrac{r+\varepsilon}{3}\}, \\ P_n &= \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - A_n\| < \tfrac{r+\varepsilon}{3}\}. \end{aligned}$$

Then, ${P_{nm}}^c, {P_n}^c, {P_m}^c \in \mathcal{I}$. Let $P^c = {P_{nm}}^c \cup {P_m}^c \cup {P_n}^c$, where

$$P = \{k \in \mathbb{N} \colon \|A_m - A_n\| < r + \varepsilon\}.$$

Then, $P^c \in \mathcal{I}$. We choose $n_0 \in P^c$, then for each $n, m \geq n_0$, we have

$$\begin{aligned} \{k \in \mathbb{N} \colon \|A_m - A_n\| < r + \varepsilon\} \supseteq \{&\{k \in \mathbb{N} \colon \|A_m - {U_k}^{(m)}(x)\| < \tfrac{r+\varepsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - {U_k}^{(m)}(x)\| < \tfrac{r+\varepsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - A_n\| < \tfrac{r+\varepsilon}{3}\}\}. \end{aligned}$$

Then (A_n) is a ρ-Cauchy sequence in $\mathbb{R}$ and since $\mathbb{R}$ is r-complete, so there exists some A in $\mathbb{R}$ such that A_n is r-convergent to A for some $r > 2^{-1}J(\mathbb{R})\rho$, where J is Jung's constant [24].

(b) Let $\varepsilon > 0$ be given. Since ${U_k}^{(n)} \to U$, then there exists some $n_0 \in \mathbb{N}$ and for some $r > 0$ such that

$$B = \{k \in \mathbb{N} \colon \|{U_k}^{(n_0)}(x) - U_k(x)\| < \tfrac{r+\varepsilon}{3}\} \qquad (*)$$

implies $B^c \in \mathcal{I}$. The number n_0 can be chosen in such a way such that together with $(*)$, we have

$$C = \{k \in \mathbb{N} \colon \|A_{n_0} - A\| < \tfrac{r+\varepsilon}{3}\},$$

such that $C^c \in \mathcal{I}$. Let $D^c = \{k \in \mathbb{N} \colon \|{U_k}^{(n_0)}(x) - A_{n_0}\| \geq \tfrac{r+\varepsilon}{3}\}$ then $D^c \in \mathcal{I}$.

Let $V^c = B^c \cup C^c \cup D^c$, where $V = \{k \in \mathbb{N}\colon \|U_k(x) - A\| < r + \varepsilon\}$. Therefore for each $k \in V^c$, we have

$$\{k \in \mathbb{N}\colon \|U_k(x) - A\| < r + \varepsilon\} \supseteq \{\{k \in \mathbb{N}\colon \|U_k(x) - {U_k}^{(n_0)}(x)\| < \tfrac{r+\varepsilon}{3}\} \cap \{k \in \mathbb{N}\colon \|{U_k}^{(n_0)}(x) - A_{n_0}\| < \tfrac{r+\varepsilon}{3}\} \cap \{k \in \mathbb{N}\colon \|A_{n_0} - A\| < \tfrac{r+\varepsilon}{3}\}\}.$$

Thus $V^c \in \mathcal{I}$.

Therefore, $\mathcal{G}_{\mathcal{C}}^{R\mathcal{I}}(T)$ is a closed subspace of $\mathcal{B}_\infty(T)$.

3. Conclusion

This work is the extension of idea of rough ideal convergence to the sequences of bounded linear operators. Theorem 2.4 establishes a relationship between rough ideal convergent sequences and ideal convergent sequences. In theorem 2.6, we have constructed a Lipschitz function with the help of rough ideal convergent sequence space of bounded linear operators. Delving into sequences of unbounded linear operators in future research promises to be intriguing. Furthermore, extending the concept of rough ideal convergence to sequences of linear operators in spaces of analytic functions could lead to the discovery of new and intriguing results.

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