South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 2 (2023), pp. 163-178

DOI: 10.56827/SEAJMMS.2023.1902.12 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

PATHWAY OPERATOR AND PATHWAY TRANSFORM OF GENERALIZED BESSEL FUNCTION

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(**Received:** *Feb. 26, 2023* **Accepted:** *Aug. 19, 2023* **Published:** *Aug. 30, 2023)*

Abstract: The current study aims to describe the generalized Bessel function's pathway fractional integral operator. The following noteworthy special cases have been brought forward for consideration. The pathway transform of the generalized Bessel function has also been developed.

Keywords and Phrases: Bessel function, Bessel-Maitland function, Wright generalized hypergeometric function.

2020 Mathematics Subject Classification: 33C90, 33B15, 26A33, 33C05.

1. Introduction and Preliminaries

Several authors discovered a number of intriguing integral operators comprising multiple kinds of special functions that could be useful in many diverse fields of physics, engineering, and applied sciences. Jain et al. [7], Singh et al. [22], Ghayasuddin et al. [6], Khan et al. [11], and we may refer to [3], [5] & [10], etc. Recently, Bessel function theory has become inextricably linked to the theory of certain forms of differential equations. The reader might refer to Watson [24] for a full overview of Bessel function applications. In this section, we are conscious of some work that was done earlier and is more well-known. The Bessel-Maitland function is described in the following way, according to [14]:

$$J_{\mu}^{\lambda}(z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{\Gamma(\lambda m + \mu + 1)m!}, \quad (\lambda > 0; z \in C). \tag{1}$$

Jain et al. [7] set out the specified function $J_{\mu,\sigma}^{\lambda}(z)$ as:

$$J_{\mu,\sigma}^{\lambda}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\mu+2\sigma+2m}}{\Gamma(\lambda m + \sigma + \mu + 1)\Gamma(m + \sigma + 1)m!}, \ (\lambda > 0, \mu, \sigma \in C; z \in C(-\infty, 0). \tag{2}$$

Singh et al. [22] implement the specified function $J_{\mu,q}^{\lambda}(z)$ as:

$$J_{\mu,q}^{\lambda,\varpi}(z) = \sum_{m=0}^{\infty} \frac{(-z)^m (\varpi)_{qm}}{\Gamma(\lambda m + \mu + 1)m!}, \tag{3}$$

where $\mu, \varpi, \lambda \in C, such\ that\ \Re(\mu) \geq -1, \Re(\lambda) > 0, \Re(\varpi) > 0; q \in (0,1) \cup N$, $(\varpi)_0 = 1, and\ (\varpi)_{qm} = \frac{\Gamma(\varpi+qm)}{\Gamma(\varpi)}$.

The following are the details of the new Bessel-Maitland function extension that Ghayasuddin et al. [6] investigate:

$$J_{\mu,q,\varrho}^{\lambda,\varpi,\zeta}(z) = \sum_{m=0}^{\infty} \frac{(-z)^m (\varpi)_{qm}}{\Gamma(\lambda m + \mu + 1)(\zeta)_{\varrho m}}, \tag{4}$$

where $\mu, \varpi, \lambda, \zeta \in C, such\ that\ \Re(\mu) \geq -1, \Re(\lambda) > 0, \Re(\varpi) > 0, \Re(\zeta) > 0, \varrho, q \geq 0$ and $q < \Re(\lambda) + \varrho$.

Khan et al. [11] have explored a novel generalized Bessel-Maitland function, which is defined as:

$$J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(z) = \sum_{m=0}^{\infty} \frac{(-z)^m (\varpi)_{qm} (\tau)_{\rho m}}{\Gamma(\lambda m + \mu + 1)(\zeta)_{\varrho m} (\xi)_{\sigma m}}, \tag{5}$$

where $\mu, \varpi, \lambda, \zeta, \tau, \sigma.\rho \in C, such\ that\ \Re(\mu) \geq -1, \Re(\lambda) > 0, \Re(\tau) > 0, \Re(\varpi) > 0, \Re(\zeta) > 0, \Re(\xi) > 0, \varrho, \sigma, q, \rho \geq 0$ and $q < \Re(\lambda) + \varrho$.

The classical beta function given by [21]:

$$B(\eta, \Theta) = \int_0^1 z^{\eta-1}(1-z)^{\Theta-1} dz = \frac{\Gamma(\eta)\Gamma(\Theta)}{\Gamma(\eta+\Theta)}, \Re(\eta) > 0, \Re(\Theta) > 0. \tag{6}$$

[23] provides the Wright generalized hypergeometric function defined as:

$$ {}_P\Psi_q\left[\begin{matrix}(e_1,f_1),(e_2,f_2),....,(e_p,f_p)\\(g_1,h_1),(g_2,h_2),....,(g_q,h_q)\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{\Pi_{i=1}^{p}(e_i+f_im)}{\Pi_{J=1}^{q}(g_j+h_jm)}\frac{z^m}{m!}. \tag{7}$$

Provided that $p,q\in\mathbb{N}=\mathbb{N}_0\cup\ \{0\},e_i,g_j\in C$ and $f_i,h_j\in\mathbb{R},(f_i,h_j\neq 0;i=1,2,...,p;j=1,2,...,q)$.

We also discuss the Hadamard product of two analytical functions because it is pertinent to the current focus of our investigation. This will assist us in abating the function that has emerged as a result of the product of two known functions.

$$h(z)=\sum_{r=0}^{\infty}a_rz^r,t(z)=\sum_{r=0}^{\infty}b_rz^r \tag{8}$$

be two power series with radii of convergence R_h and R_t respectively. Then, their Hadamard product [20] is the power series demarcate by

$$(h*t)(z)=\sum_{r=0}^{\infty}a_rb_rz^r. \tag{9}$$

Hadamard product series $(h*t)(z)$ has radius of convergence R if and only if $R_h.R_t\leq R$. If one of the power series defines a function exactly, then the Hadamard product series will do the same.

The H-function [18] can be described here by the mean of a Mellin-Barnes type integral:

$$H_{p,q}^{m,n}=H_{p,q}^{m,n}\left[\begin{matrix}(f_1,F_1),(f_2,F_2),....,(f_p,F_p)\\(g_1,G_1),(g_2,G_2),....,(g_q,G_q)\end{matrix};z\right]=\frac{1}{2\pi i}\int_L h(s)z^{-s}ds, \tag{10}$$

where

$$h(s)=\frac{\Pi_{i=1}^{m}\Gamma(g_i+G_is)\Pi_{i=1}^{n}\Gamma(1-f_i-F_is)}{\Pi_{i=m+1}^{q}\Gamma(1-g_i-G_is)\Pi_{i=n+1}^{p}\Gamma(f_i+F_is)}, \tag{11}$$

and $z^{-s}=exp[-s\{|z|+iargz\}],z\neq 0,i=\sqrt{-1}$, here $|z|$ is the natural logarithm of $|z|$ and arg z is not assuredly the principal value and m,n,p,q are integers s.t. $1\leq m\leq q,0\leq n\leq p,F_i,G_j\in\Re^+,f_i,g_j\in C,(i=1,2,...,p;j=1,2,...,q)$.An empty product in (11) is always interpreted as unity.The contour L in (10) splits the poles of the gamma function $\Gamma(g_i+G_is),i=1,2,...,m$ from the Gamma functions $\Gamma(1-f_i-F_is),i=1,2,...,n$.

2. Pathway integral representation of $J^{\lambda,\varpi,\zeta,\rho}_{\mu,q,\varrho,\sigma,\tau,\xi}(x)$

Pathway fractional integral (PFI) operator is a concept that was first proposed by Nair [19].This concept is connected to the pathway model that was established by Mathai [15] and Mathai & Haubold [16]. Due to the fact that it has such widespread use in the scientific and technological domains, the pathway fractional integral operator and pathway transform have been utilized by a great number of researchers. Ranjan et al. [8], recently made use of this pathway operator in their research by making use of the fractional integral operator, they were able to determine certain fractional integral features of the generalized Bessel function. The generalized extension of the K-Bessel Maitland function incorporating the pathway integral operator was established by Ahamad et al. [2] in their paper. They constructed the composition formulae for the pathway fractional integral operator associated with the extended Wright-Bessel function. The formulas were developed by Nirmal et al. [9]. Developing formulas for the pathway fractional integral operator that is connected with the Extended Bessel function is another goal for this sequel that we want to accomplish.
This operator can be considered of as a generalization of the traditional Riemann-Liouville fractional integral operator, which can be characterized as follows:

$$(P^{\phi,a,\Omega}_{0+}f)(x) = x^{\phi}\int_0^{\frac{x}{a(1-\Omega)}}\left(1-\frac{a(1-\Omega)\varphi}{x}\right)^{\frac{\phi}{1-\Omega}} f(\varphi)d\varphi. \tag{12}$$

Where Lebesgue measurable function$f \in L(a,b)$ for real or complex term valued function,
$\phi \in C, \Re(\phi) > 0, a > 0$ and $\Omega < 1$ (Ω is a pathway parameter).

The probability density function, also known as the p.d.f., is a model that illustrates a pathway for a real scalar Ω and scalar random variables in the following way:

$$f(x) = \frac{c}{|x|^{1-v}}[1-a(1-\Omega)|x|^{\delta}]^{\frac{\beta}{1-\Omega}}, \tag{13}$$

where $x \in (-\infty,\infty); \delta > 0; \beta \geq 0; 1-a(1-\Omega)|x|^{\delta} > 0; v > 0$ and c represent the normalizing constant for the pathway parameters respectively.

Furthermore, the normalising constants for $\Omega \in R$ are given as follows:

$$
= \begin{cases}
\frac{1}{2}\frac{\delta[a(1-\Omega)]^{\frac{v}{\delta}}\Gamma\left(\frac{v}{\delta}+\frac{\beta}{1-\Omega}+1\right)}{\Gamma(\frac{v}{\delta})\Gamma\left(\frac{\beta}{1-\Omega}+1\right)} & (\Omega < 1), \\
\frac{1}{2}\frac{\delta[a(1-\Omega)]^{\frac{v}{\delta}}\Gamma\left(\frac{\beta}{\Omega-1}\right)}{\Gamma(\frac{v}{\delta})\Gamma\left(\frac{\beta}{\Omega-1}-\frac{v}{\delta}\right)} & \left(\frac{1}{\Omega-1}-\frac{v}{\delta} > 0, \Omega > 1\right), \\
\frac{1}{2}\frac{\delta[a\beta]^{\frac{v}{\delta}}}{\Gamma(\frac{v}{\delta})}, & (\Omega \to 1).
\end{cases} \tag{14}
$$

It has been demonstrated that if $\Omega < 1$, finite range density with $1-a(1-\Omega)|x|^{\delta} > 0$, and (13) are members of the extended generalized type-1 beta family. Furthermore, the triangle density, uniform density, extended type-1 beta density, and a number of other probability density functions are all precise special instances of the pathway density function established in (13) for $\Omega < 1$. For instance, if $\Omega > 1$ and by setting$(1-\Omega) = -(\Omega-1)$ in (12) yield

$$
(P_{0+}^{\phi,\Omega} f)(x) = x^{\phi} \int_0^{\frac{-x}{a(\Omega-1)}} \left(1 + \frac{a(\Omega-1)\varphi}{x}\right)^{\frac{\phi}{-(\Omega-1)}} f(\varphi) d\varphi \tag{15}
$$

and

$$
f(x) = \frac{c}{|x|^{1-v}} [1 + a(\Omega-1)|x|^{\delta}]^{\frac{\beta}{-(\Omega-1)}}, \tag{16}
$$

assuming that $x \in (-\infty, \infty); \delta > 0; \beta \geq 0; \Omega > 1$ characterises the extended generalized type-2 beta model for real x. The individual cases of the density function (16) hold the type-2 beta density function, the p density function, and the density function. For $\Omega \to 1$, (12) diminishes to the Laplace integral transform. In a similar manner, if $\Omega = 0, a = 1$ and ϕ takes the place of $\phi-1$, then (12) diminishes to the familiar Riemann-Liouville (R-L) fractional integral operator $I_{0+}^{\phi}(f)$.

$$
(P_{0+}^{\phi-1,0} f)(x) = \Gamma(\phi)\left(I_{0+}^{\phi}(f)\right)(x), (\Re(\phi) > 1). \tag{17}
$$

Numerous fascinating demonstrations emerge as a consequence of the PFI operator (12), such as the fractional calculus connected to probability density functions and the significance of these functions in statistical theory. Many researchers are working on PFI formulae linked with a wide variety of special functions (see [1,4]).

Main Results

The following theorems provide evidence for the existence of the pathway integral operator of the generalized Bessel-Maitland function:

Theorem 2.1. *Let* $\mu, \varpi, \lambda, \zeta, \tau, \sigma.\rho \in C$, *be such that* $\Re(\mu) \geq -1, \Re(\lambda) > 0$, $\Re(\tau) > 0$, $\Re(\varpi) > 0$, $\Re(\zeta) > 0$, $\Re(\xi) > 0$, ϱ, σ, q, $\rho \geq 0$ *and* $q < \Re(\lambda) + \varrho, \Omega < 1$, $\Re\left(\frac{\phi}{1-\Omega}\right) > -1$. *Then there holds the following formula.*

$$P_{0+}^{\phi,\Omega}[t^{\psi-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(t)](x) = \frac{x^{\phi+\psi}}{[a(1-\Omega)]^{\psi}}\Gamma\left(1+\frac{\phi}{1-\Omega}\right)J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}\left(\frac{x}{a(1-\Omega)}\right)$$

$$\times\ {}_2\Psi_1\left[\begin{matrix}(\psi,1),(1,1)\\ (\psi+1+\frac{\phi}{1-\Omega},1)\end{matrix};\frac{-x}{a(1-\Omega)}\right]. \tag{18}$$

Proof. Using equation (5) & (12) in L.H.S, then we get

$$= x^{\phi}\int_0^{\frac{x}{a(1-\Omega)}} t^{\psi-1}\left(1-\frac{a(1-\Omega)t}{x}\right)^{\frac{\phi}{1-\Omega}}\sum_{m=0}^{\infty}\frac{(-t)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}dt,$$

$$= x^{\phi}\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\int_0^{\frac{x}{a(1-\Omega)}} t^{\psi+m-1}\left(1-\frac{a(1-\Omega)t}{x}\right)^{\frac{\phi}{1-\Omega}}dt.$$

If we now make the assumption that $\frac{a(1-\Omega)t}{x} = y$, then we are able to adjust the limit of integration to be as follows:

$$= x^{\phi}\left[\frac{x}{a(1-\Omega)}\right]^{\psi}\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\left[\frac{x}{a(1-\Omega)}\right]^m$$

$$\times\int_0^1 (1-y)^{\frac{\phi}{1-\Omega}}(y)^{\psi+m-1}dy,$$

$$= \frac{x^{\phi+\psi}}{[a(1-\Omega)]^{\psi}}\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\left[\frac{x}{a(1-\Omega)}\right]^m\frac{\Gamma(\psi+m)\Gamma(1+\frac{\phi}{1-\Omega})}{\Gamma(\psi+m+1+\frac{\phi}{1-\Omega})}.$$

By applying the Hadmard product (9) in above equation ,then we find the following result,

$$= \frac{x^{\phi+\psi}}{[a(1-\Omega)]^{\psi}}\Gamma\left(1+\frac{\phi}{1-\Omega}\right)J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}\left(\frac{x}{a(1-\Omega)}\right)$$

$$\times\ {}_2\Psi_1\left[\begin{matrix}(\psi,1),(1,1)\\ (\psi+1+\dfrac{\phi}{1-\Omega},1)\end{matrix};\frac{-x}{a(1-\Omega)}\right].$$

Corollary 2.2. *Let $a=1,\Omega=0,\phi=\phi-1$, then the theorem 2.1 hold the following result:*

$$I_{0+}^{\phi}[t^{\psi-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(t)](x)=x^{\phi+\psi-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\times\ {}_2\Psi_1\left[\begin{matrix}(\psi,1),(1,1)\\ (\psi+\phi,1)\end{matrix};-x\right]. \tag{19}$$

Theorem 2.3. *Let $\mu,\varpi,\lambda,\zeta,\tau,\sigma.\rho\in C$, be such that $\Re(\mu)\geq-1$, $\Re(\lambda)>0$, $\Re(\tau)>0$, $\Re(\varpi)>0$, $\Re(\zeta)>0,\Re(\xi)>0$, $\varrho,\sigma,q,\ \rho\geq0$ and $q<\Re(\lambda)+\varrho,\Omega>1$, $\Re\left(\frac{\phi}{\Omega-1}\right)>-1$. Then we have the following formula.*

$$P_{0+}^{\phi,\Omega}[t^{\psi-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(t)](x)=\frac{x^{\phi+\psi}}{[-a(\Omega-1)]^{\psi}}\Gamma\left(1-\frac{\phi}{\Omega-1}\right)J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}\left(\frac{-x}{a(\Omega-1)}\right)$$

$$\times\ {}_2\Psi_1\left[\begin{matrix}(\psi,1),(1,1)\\ (\psi+1-\dfrac{\phi}{\Omega-1},1)\end{matrix};\frac{x}{a(\Omega-1)}\right]. \tag{20}$$

Proof. Using equation (5) & (15) in L.H.S., then we have

$$=x^{\phi}\int_0^{\frac{-x}{a(\Omega-1)}}t^{\psi-1}\left(1+\frac{a(\Omega-1)t}{x}\right)^{\frac{-\phi}{\Omega-1}}\sum_{m=0}^{\infty}\frac{(-t)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}dt$$

$$=x^{\phi}\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\int_0^{\frac{-x}{a(\Omega-1)}}t^{\psi+m-1}\left(1+\frac{a(\Omega-1)t}{x}\right)^{\frac{-\phi}{\Omega-1}}dt$$

let us assume $\frac{-a(\Omega-1)t}{x}=y$, then we have

$$=x^{\phi}\left(\frac{-x}{a(\Omega-1)}\right)^{\psi}\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\left(\frac{-x}{a(\Omega-1)}\right)^{m}$$

$$\times\int_0^1(1-y)^{\frac{-\phi}{\Omega-1}}y^{\psi+m-1}dy$$

$$= \frac{x^{\phi+\psi}}{[-a(\Omega-1)]^{\psi}} \sum_{m=0}^{\infty} \frac{(-1)^m (\varpi)_{qm} (\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}} \left[\frac{x}{-a(\Omega-1)}\right]^m \frac{\Gamma(\psi+m)\Gamma(1-\frac{\phi}{\Omega-1})}{\Gamma(\psi+m+1-\frac{\phi}{\Omega-1})}.$$

Employing the Hadmard product (9) in the above equation, we arrive to the following conclusion:

$$= \frac{x^{\phi+\psi}}{[-a(\Omega-1)]^{\psi}} \Gamma\left(1-\frac{\phi}{\Omega-1}\right) J^{\lambda,\varpi,\zeta,\rho}_{\mu,q,\varrho,\sigma,\tau,\xi}\left(\frac{-x}{a(\Omega-1)}\right)$$

$$\times {}_2\Psi_1\left[\begin{matrix}(\psi,1),(1,1) \\ (\psi+1-\frac{\phi}{\Omega-1},1)\end{matrix}; \frac{x}{a(\Omega-1)}\right].$$

3. P-transform or pathway transform of $J^{\lambda,\varpi,\zeta,\rho}_{\mu,q,\varrho,\sigma,\tau,\xi}(x)$

In this section, we are going to figure out the pathway transform of the generalized Bessel function. The P-transform, also known as the pathway transform, is a generalization of the Krätzel transform as well as a number of other integral transforms that are already in use. In 2011, Kumar [13] introduced the pathway transform, which is obtained by applying the pathway model of Mathai [15] and further developed by Mathai and Haubold [17]. In addition, Mathai and Haubold [17] further developed the pathway model, which is represented as

$$(P^{\kappa,\delta,\phi}_{\chi} f)(x) = \int_0^{\infty} D^{\chi,\phi}_{\kappa,\delta}(xt) f(t) dt, \quad x>0, \tag{21}$$

where $D^{\chi,\phi}_{\kappa,\delta}(x)$ is given as

$$D^{\chi,\phi}_{\kappa,\delta}(x) = \int_0^{\infty} u^{\chi-1}(1+a(\phi-1)u^{\kappa})^{\frac{1}{-(\phi-1)}} e^{-xu^{-\delta}} du, \quad x>0. \tag{22}$$

If we utilize the kernel function given by (22), we may claim that (21)is a type 2 P-transform, where $\chi \in C, a>0, \delta>0$ and $\kappa \in \Re, \kappa \neq 0, \phi>1$.

$$D^{\chi,\phi}_{\kappa,\delta}(x) = \int_0^{\left[\frac{1}{a(1-\phi)}\right]^{\frac{1}{\kappa}}} u^{\chi-1}(1-a(1-\phi)u^{\kappa})^{\frac{1}{(1-\phi)}} e^{-xu^{-\delta}} du, \quad x>0. \tag{23}$$

In this instance, we may state that (22) is a type 1 P-transform because $\chi \in C, a>0, \delta>0$ and $\kappa>0, \phi<1$. Both versions of the P- transform are defined in the space $L_{v,r}$ of the measurable complex-valued lebesgue functions f for which

$$||f||_{v,r} = \left\{ \int_0^\infty |t^v f(t)|^r \frac{dt}{t} \right\}^{\frac{1}{r}} < \infty, \tag{24}$$

for $1 \leq r < \infty, v \in \Re$, when $a = 1, \delta = 1$ and $\phi \to 1$ than, we can observe that

$$lim_{\phi \to 1} D_{\kappa,1}^{\chi,\phi} = Z_\kappa^\chi(x), \tag{25}$$

where, as stated below, $Z_\kappa^\chi(x)$ is the kernal function of the Krätzel transform introduced by Krätzel [12].

$$K_\chi^\kappa f(x) = \int_0^\infty Z_\kappa^\chi(xt) f(t) dt, \quad x > 0, \tag{26}$$

and

$$Z_\kappa^\chi(x) = \int_0^\infty u^{\chi-1} e^{-u^\kappa - xu^{-1}} du. \tag{27}$$

The following lemma from [13] is used to generate the type-2 P-transform of the generalized Bessel function.

Lemma 3.1. *Let* $\chi, \eta \in C$, *such that* $a > 0, \delta > 0$ *and* $\kappa \in \Re, \kappa \neq 0, \phi > 1, \Re(\chi + \delta\eta) > 0$, $\Re\left(\frac{1}{\phi-1} - \frac{\chi+\delta\eta}{\kappa}\right) > 0$ *when* $\kappa > 0$ *and* $\Re(\chi + \delta\eta) < 0$, $\Re\left(\frac{1}{\phi-1} - \frac{\chi+\delta\eta}{\kappa}\right) > 0$ *when* $\kappa < 0$ *then type-2P transform of power function is given by:*

$$\left(P_\chi^{\kappa,\delta,\phi} x^{\eta-1}\right)(x) = \frac{\Gamma(\eta)\Gamma(\frac{\chi+\delta\eta}{\kappa})\Gamma\left(\frac{1}{\phi-1} - \frac{\chi+\delta\eta}{\kappa}\right)}{|\kappa|(x)^\eta [a(\phi-1)]^{\frac{\chi+\delta\eta}{\kappa}} \Gamma\frac{1}{(\phi-1)}}. \tag{28}$$

Theorem 3.2. *Let* $\chi, \eta \in C$, *such that* $a > 0, \delta > 0$, $\kappa \in \Re, \kappa \neq 0, \phi > 1, \Re(\chi + \delta(\eta+m)) > 0$, $\Re\left(\frac{1}{\phi-1} - \left(\frac{\chi+\delta(\eta+m)}{\kappa}\right)\right) > 0$ *when* $\kappa > 0$, *and* $\Re(\chi + \delta(\eta+m)) < 0$, $\Re\left(\frac{1}{\phi-1} - \left(\frac{\chi+\delta(\eta+m)}{\kappa}\right)\right) > 0$ *when* $\kappa < 0$, *then there holds the following formula:*

$$\left(P_\chi^{\kappa,\delta,\phi} x^{\eta-1} J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\right) = \frac{1}{|\kappa|(x)^\eta [a(\phi-1)]^{\frac{\chi+\delta\eta}{\kappa}} \Gamma\frac{1}{(\phi-1)}} J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}\left(\frac{1}{x[a(\phi-1)]^{\frac{\delta}{\kappa}}}\right)$$

$$\Gamma(\eta+m)\Gamma\left(\frac{1}{\phi-1}-\left(\frac{\chi+\delta(\eta+m)}{\kappa}\right)\right)\Gamma\left(\frac{\chi+\delta(\eta+m)}{\kappa}\right). \quad (29)$$

Proof. Using equation (5) & (21) in L.H.S,then we have

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\left[P_{\chi}^{\kappa,\delta,\phi}x^{m+\eta-1}\right](x)$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\int_0^{\infty}D_{\kappa,\delta}^{\chi,\phi}(xt)\quad t^{m+\eta-1}dt$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}$$

$$\int_0^{\infty}\int_0^{\infty}y^{\chi-1}(1+a(\phi-1)y^{\kappa})^{\frac{1}{-(\phi-1)}}e^{-xty^{-\delta}}t^{m+\eta-1}dy\quad dt.$$

Changing the order of integration

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}$$

$$\int_0^{\infty}y^{\chi-1}(1+a(\phi-1)y^{\kappa})^{\frac{1}{-(\phi-1)}}dy\int_0^{\infty}e^{-xty^{-\delta}}t^{m+\eta-1}\quad dt.$$

Now, let $xty^{-\delta}=z$ in the above equation, we have

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}$$

$$\int_0^{\infty}y^{\chi-1}(1+a(\phi-1)y^{\kappa})^{\frac{1}{-(\phi-1)}}dy\int_0^{\infty}e^{-z}\left(\frac{z}{xy^{-\delta}}\right)^{m+\eta-1}\frac{dz}{xy^{-\delta}}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\left(\frac{1}{x}\right)^{m+\eta}$$

$$\int_0^{\infty}y^{\chi+\delta(m+\eta)-1}(1+a(\phi-1)y^{\kappa})^{\frac{1}{-(\phi-1)}}dy\int_0^{\infty}e^{-z}z^{m+\eta-1}dz$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\frac{\Gamma(m+\eta)}{(x)^{m+\eta}}\int_0^{\infty}y^{\chi+\delta(m+\eta)-1}(1+a(\phi-1)y^{\kappa})^{\frac{1}{-(\phi-1)}}dy.$$

Now let $a(\phi-1)y^{\kappa}=n$ in the given equation then, we get

$$= \frac{1}{\kappa}\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\frac{\Gamma(m+\eta)}{(x)^{m+\eta}}$$

$$\frac{1}{[a(\phi-1)]^{\frac{\chi+\delta(m+\eta)}{\kappa}}}\int_0^{\infty}(n)^{\frac{\chi+\delta(m+\eta)}{\kappa}-1}(1+n)^{\frac{1}{-(\phi-1)}}dn$$

using the beta function, we have

$$= \frac{1}{\kappa}\sum_{m=0}^{\infty}\frac{(-1)^m(\varpi)_{qm}(\tau)_{\rho m}}{\Gamma(\lambda m+\mu+1)(\zeta)_{\varrho m}(\xi)_{\sigma m}}\frac{\Gamma(m+\eta)}{(x)^{m+\eta}}$$

$$\frac{1}{[a(\phi-1)]^{\frac{\chi+\delta(m+\eta)}{\kappa}}}\frac{\Gamma\left(\frac{\chi+\delta(m+\eta)}{\kappa}\right)\Gamma\left(\frac{1}{\phi-1}-\frac{\chi+\delta(m+\eta)}{\kappa}\right)}{\Gamma\left(\frac{1}{\phi-1}\right)},$$

we get the result given in equation (29).
In a similar vein, we are able to demonstrate the result for $\kappa<0$.

Corollary 3.3. *In accordance with the preexisting conditions of theorem 3.2, if we set $a=1$ and $\delta=1$, then the Krätzel transform of the generalized Bessel function is represented as follows:*

$$\lim_{\phi\to 1}\left(P_{\chi}^{\kappa,1,\phi}x^{\eta-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\right)(x)=\left(K_{\kappa}^{\chi}x^{\eta-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\right)(x)$$

$$=\frac{1}{\kappa}\frac{1}{(x)^{\eta}}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}\left(\frac{1}{x}\right)\Gamma(m+\eta)\Gamma\left(\frac{\chi+(m+\eta)}{\kappa}\right), \tag{30}$$

where $\Re(m+\eta)>0$ and $\Re\left(\chi+(m+\eta)\right)>0$ for $\kappa>0$.

We are now presenting the Mellin-Barnes integral representation of the generalized Bessel-Maitland function $J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(z)$

Theorem 3.4. *Let $\mu,\varpi,\lambda,\zeta,\tau,\rho,\xi\in C$, such that $\Re(\mu)>-1,\Re(\lambda)>0,\Re(\rho)>0,\Re(\zeta)>0,\Re(\tau)>0$, then $J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(z)$ can be expressed by the Mellin–Barnes integral type as follows:*

$$J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(z)=\frac{1}{2\pi i}\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}(-t)^{-s}ds, \tag{31}$$

where all poles are separated at $s=-n(n\in N_0=N\cup 0)$ to the left and at $s=n+1$ to the right, and the contour of integration L starts at $c-i\infty$ and ends at $c+i\infty$ for any $c>0$.

Proof. Equation (31)'s integrand contains poles at the points $s=0,-1,-2,...$ The poles are all placed to the left of the straight line contour $c-i\infty$ to $c+i\infty$ for any $c>0$. Any infinite semi-circle can therefore encompass all of these poles, and the residue theorem can be used to determine as

$$=\frac{1}{2\pi i}\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}(-t)^{-s}ds$$

$$=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\sum_{n=0}^{\infty}Res_{s=-n}\Big[\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}(-t)^{-s}\Big]$$

$$=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\sum_{n=0}^{\infty}\lim_{s\to\ -n}\Big[(s+n)\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}(-t)^{-s}\Big]$$

$$=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\sum_{n=0}^{\infty}\lim_{s\to\ -n}\Big[\frac{\Gamma(s+n+1)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)(-t)^{-s}}{(s+n-1)(s+n-2).....s\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}\Big]$$

$$=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\sum_{n=0}^{\infty}\Big[\frac{\Gamma(\varpi+qn)\Gamma(\tau+\rho n)}{\Gamma(\mu+1+\lambda n)\Gamma(\zeta+\varrho n)\Gamma(\xi+\sigma n)}(-t)^{n}\Big]=J^{\lambda,\varpi,\zeta,\rho}_{\mu,q,\varrho,\sigma,\tau,\xi}(z)$$

Theorem 3.5. *Let $\chi,\eta\in C$, such that $a>0,\delta>0$, $\kappa\in\Re,\kappa\neq 0,\phi>1,\Re\Big(\frac{\chi+\delta(\eta-s)}{\kappa}\Big)>0$, $\Re\Bigg(\frac{1}{\phi-1}-\Bigg(\frac{\chi+\delta(\eta-s)}{\kappa}\Bigg)\Bigg)>0$ when $\kappa>0$ and $\Re\Big(\frac{\chi+\delta(\eta-s)}{\kappa}\Big)<0$, $\Re\Bigg(\frac{1}{\phi-1}-\Bigg(\frac{\chi+\delta(\eta-s)}{\kappa}\Bigg)\Bigg)<0$, when $\kappa<0$ then, type-2 P-transform of generalized Bessel function holds the following formula:*

$$\text{for } \kappa>0\ \Bigg(P^{\kappa,\delta,\phi}_{\chi}x^{\eta-1}J^{\lambda,\varpi,\zeta,\rho}_{\mu,q,\varrho,\sigma,\tau,\xi}(t)\Bigg)(t)=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\frac{1}{|\kappa|t^{\eta}}\frac{1}{[a(\phi-1)]^{\frac{\chi+\delta\eta}{\kappa}}}\frac{1}{\Gamma\Bigg(\frac{1}{\phi-1}\Bigg)}$$

$$H^{2,5}_{5,5}\left[\begin{matrix}(0,1),(1-\varpi,q),(1-\tau,\rho),(1-\eta,1),\Big(1-(\frac{\chi+\delta\eta}{\kappa}),\frac{\delta}{\kappa}\Big)\\(0,1),(-\mu,\lambda),(1-\zeta,\varrho),(1-\xi,\sigma),\Big(\frac{1}{\phi-1}-\frac{\chi+\delta\eta}{\kappa},\frac{\delta}{\kappa}\Big)\end{matrix};\frac{1}{t[a(\phi-1)]^{\frac{\delta}{\kappa}}}\right],\tag{32}$$

$$\text{if } \kappa<0 \ \left(P_{\chi}^{\kappa,\delta,\phi}x^{\eta-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(t)\right)(t)=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\frac{1}{|\kappa|t^{\eta}}\frac{1}{[a(\phi-1)]^{\frac{\chi+\delta\eta}{\kappa}}}\frac{1}{\Gamma\left(\frac{1}{\phi-1}\right)}$$

$$H_{5,5}^{2,5}\left[\begin{matrix}(0,1),(1-\varpi,q),(1-\tau,\rho),(1-\eta,1),\left(1-\frac{1}{\phi-1}-(\frac{\chi+\delta\eta}{\kappa}),\frac{-\delta}{\kappa}\right)\\(0,1),(-\mu,\lambda),(1-\zeta,\varrho),(1-\xi,\sigma),\left(-\frac{\chi+\delta\eta}{\kappa},\frac{-\delta}{\kappa}\right)\end{matrix};\frac{1}{t[a(\phi-1)]^{\frac{\delta}{\kappa}}}\right]. \tag{33}$$

Proof. Let $\kappa>0$, using equation (31) & (21) in L.H.S, then we find

$$=\int_0^{\infty}D_{\kappa,\delta}^{\chi,\phi}(xt)t^{\eta-1}\frac{1}{2\pi i}\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}(-t)^{-s}dsdt$$
$$=\frac{1}{2\pi i}\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}\int_0^{\infty}D_{\kappa,\delta}^{\chi,\phi}(xt)t^{\eta-1-s}(-1)^{-s}dtds$$
$$=\frac{1}{2\pi i}\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}P_{\chi}^{\kappa,\delta,\phi}t^{\eta-1-s}ds.$$

Using the equation (28),we find

$$=\frac{1}{2\pi i}\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}$$
$$\frac{\Gamma(\eta-s)\Gamma(\frac{\chi+\delta(\eta-s)}{\kappa})\Gamma\left(\frac{1}{\phi-1}-\frac{\chi+\delta(\eta-s)}{\kappa}\right)}{|\kappa|(t)^{\eta-s}[a(\phi-1)]^{\frac{\chi+\delta(\eta-s)}{\kappa}}\Gamma\frac{1}{(\phi-1)}}$$

$$=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\frac{1}{|\kappa|}\frac{1}{(t)^{\eta}}\frac{1}{[a(\phi-1)]^{\frac{\chi+\delta\eta}{\kappa}}}\frac{1}{\Gamma\left(\frac{1}{\phi-1}\right)}\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\varpi-qs)\Gamma(\tau-\rho s)}{\Gamma(\mu+1-\lambda s)\Gamma(\zeta-\varrho s)\Gamma(\xi-\sigma s)}$$

$$\left(\frac{1}{t[a(\phi-1)]^{\frac{\delta}{\kappa}}}\right)^{-s}\Gamma(\eta-s)\Gamma\left(\frac{\chi+\delta(\eta-s)}{\kappa}\right)\Gamma\left(\frac{1}{\phi-1}-\frac{\chi+\delta(\eta-s)}{\kappa}\right)$$

then,we get the result given in equation (32).
Similarly, we can prove the result (33) for $\kappa<0$.

Corollary 3.6. *If the conditions of theorem 3.5 are met for $a=1$ and $\delta=1$, then* *the Krätzel transform of the generalized Bessel function is given for* $\kappa > 0$*,*

$$\lim_{\phi\to 1}\left(P_{\chi}^{\kappa,1,\phi}x^{\eta-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\right)(x)=\left(K_{\kappa}^{\chi}x^{\eta-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\right)(x)$$

$$=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\frac{1}{|\kappa|}\frac{1}{(t)^{\eta}}H_{5,4}^{1,5}\left[\begin{matrix}(0,1),(1-\varpi,q),(1-\tau,\rho),(1-\eta,1),\left(1-(\frac{\chi+\eta}{\kappa}),\frac{1}{\kappa}\right)\\(0,1),(-\mu,\lambda),(1-\zeta,\varrho),(1-\xi,\sigma)\end{matrix};\frac{1}{t}\right],\tag{34}$$

and for $\kappa < 0$,

$$\lim_{\phi\to 1}\left(P_{\chi}^{\kappa,1,\phi}x^{\eta-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\right)(x)=\left(K_{\kappa}^{\chi}x^{\eta-1}J_{\mu,q,\varrho,\sigma,\tau,\xi}^{\lambda,\varpi,\zeta,\rho}(x)\right)(x)$$

$$=\frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\tau)\Gamma(\varpi)}\frac{1}{|\kappa|}\frac{1}{(t)^{\eta}}H_{4,5}^{2,5}\left[\begin{matrix}(0,1),(1-\varpi,q),(1-\tau,\rho),(1-\eta,1)\\(0,1),(-\mu,\lambda),(1-\zeta,\varrho),(1-\xi,\sigma),\left(-(\frac{\chi+\eta}{\kappa}),\frac{1}{-\kappa}\right)\end{matrix};\frac{1}{t}\right].\tag{35}$$

4. Conclusion

We end this inquiry by noting that the findings are of a general nature and can be applied to the theory of the pathway fractional integration operator to derive various types of special functions. Characterising the pathway fractional integral operator of the generalized Bessel function is the goal of this research. The generalized Bessel function has certain defined specific cases, and further pathway transformations have been made.

Acknowledgement

The authors are obliged to the anonymous reviewers for the constructive criticism provided on the manuscript's structure.

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