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COMPUTATION OF b-CHROMATIC TOPOLOGICAL INDICES OF SOME GRAPHS AND ITS DERIVED GRAPHS

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Abstract: The two fastest-growing subfields of graph theory are graph coloring and topological indices. Graph coloring is assigning the colors/values to the edges/vertices or both. A proper coloring of the graph G is assigning colors/values to the vertices/edges or both so that no two adjacent vertices/edges share the same color/value. Recently, studies involving Chromatic Topological indices that dealt with different graph coloring were studied. In such studies, the vertex degrees get replaced with the colors, and the computation is carried out based on the topological index of our choice. We focus on b-Chromatic Zagreb indices and b-Chromatic irregularity indices in this work. This paper discusses the b-Chromatic Zagreb indices and b-Chromatic irregularity indices of the gear graph, star graph, and its derived graphs such as the line and middle graph.

Keywords and Phrases: b-coloring, b-Chromatic Zagreb indices, b-Chromatic irregularity index, b-Chromatic total irregularity index.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C78, 05C92.

1. Introduction

Graph coloring is one of the areas of research which has umpteen applications. There are mainly two types of graph coloring: vertex and edge-based. We focus on vertex coloring [3], mainly b-coloring, in our work. A b-coloring [2] is a type of proper vertex coloring subject to an additional property that each color class should have at least one vertex with a neighbor in all the other color classes. The topological index is a parameter that plays a vital role in Chemical Graph Theory. A molecular structure can be visualized as a graph, with the atoms and bonds resembling its vertices and edges, respectively. The term topological index refers to the numerical value assigned to these structures. In mathematical chemistry, topological indices are essential, particularly in studying QSPR and QSAR [11]. Topological indices are categorized using various criteria, including their degrees, spectrum, distances, etc [1, 4, 6, 8, 10, 17]. Our study focuses on the degree-based topological index, mainly the Zagreb index [5]. Unlike earlier proposed chromatic topological indices that followed proper, injective and equitable colorings, the proposed chromatic topological indices in this study adopt b-coloring. For computational purpose, let $C = \{c_1, c_2, ..., c_l\}$ represents the set of colors used in b-coloring and $\eta_{t,s}$ indicate the total number of edges with end points having the color c_t and c_s. Here, $t < s, 1 \leq t, s \leq \chi_b(G)$. The cardinality of the specific color used is denoted by $\theta(c_i)$. For the definition of chromatic number we refer to [18]. We examine the concept of b-Chromatic Zagreb indices and b-Chromatic irregularity indices for gear graph, star graph and its line and middle graphs, inspired by varieties of works [9, 13, 14, 15] on various types of graph colorings and chromatic topological indices. For the informations related to line graph, middle graph, gear graph and star graph we refer to [12, 16, 19, 20]. For other technical terms which is not defined in this paper we refer to [7, 21]. We deal with simple, undirected, finite, and connected graphs throughout our work.

Definition 1.1. [15] *The first b-Chromatic Zagreb index of G, represented by* $M_1^{\varphi_b t}(G)$*, is provided by* $M_1^{\varphi_b t}(G) = \sum_{u \in V(G)} c(u)^2$ *where c follows b-coloring of graph.*

Definition 1.2. [15] *The second b-Chromatic Zagreb index of G, represented by* $M_2^{\varphi_b t}(G)$*, is provided by* $M_2^{\varphi_b t}(G) = \sum_{uv \in E(G)} c(u) \cdot c(v)$ *where c follows b-coloring of graph.*

Definition 1.3. [15] *The b-Chromatic irregularity index of G, represented by* $M_3^{\varphi_b t}(G)$*, is provided by* $M_3^{\varphi_b t}(G) = \sum_{uv \in E(G)} |c(u) - c(v)|$ *where c follows b-coloring of graph.*

Definition 1.4. [15] *The b-Chromatic total irregularity index of G, represented by* $M_4^{\varphi_b t}(G)$, *is provided by* $M_4^{\varphi_b t}(G) = \frac{1}{2} \sum_{uv \in V(G)} |c(u) - c(v)|$ *where c follows b-coloring of graph.*

The minimum and maximum b-Chromatic Zagreb indices, b-Chromatic irregularity indices are specified below in relation to the definitions stated above.

$M_i^{\varphi_b^-}(G) = \min\{M_i^{\varphi_b t}(G) : 1 \leq t \leq l!\}$, for $1 \leq i \leq 4$

$M_i^{\varphi_b^+}(G) = \max\{M_i^{\varphi_b t}(G) : 1 \leq t \leq l!\}$, for $1 \leq i \leq 4$

The equations below describe the working rule for the first, second b-Chromatic Zagreb indices, b-Chromatic irregularity index and b-Chromatic total irregularity index respectively.

i. $M_1^{\varphi_b t}(G) = \sum_{u \in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j) \cdot j^2.$

ii. $M_2^{\varphi_b t}(G) = \sum_{uv \in E(G)} c(u) \cdot c(v) = \sum_{1 \leq t,s \leq \chi_b(G)}^{t<s} ts\eta_{ts}.$

iii. $M_3^{\varphi_b t}(G) = \sum_{uv \in E(G)} |c(u) - c(v)| = \sum_{1 \leq t,s \leq \chi_b(G)}^{t<s} \eta_{ts}|t - s|.$

iv. $M_4^{\varphi_b t}(G) = \frac{1}{2} \sum_{uv \in V(G)} |c(u) - c(v)| = \frac{1}{2} \sum_{t,s \in C}^{t<s} \theta(c_t) \cdot \theta(c_s)|t - s|.$

2. Main Results

Theorem 2.1. *For the gear graph* $G_n, n \geq 4$ *we have,*

$$i) \quad M_1^{\varphi_b^-}(G_n) = 5n + 29$$

$$ii) \quad M_2^{\varphi_b^-}(G_n) = 8n + 22$$

$$iii) \quad M_3^{\varphi_b^-}(G_n) = 5n - 1$$

$$iv) \quad M_4^{\varphi_b^-}(G_n) = \frac{n^2 + 9n - 8}{2}$$

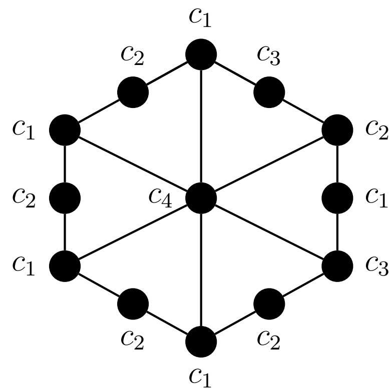

Figure 1: The b-coloring of G_6

Proof. We use 4 colors say c_1, c_2, c_3 and c_4 to color the vertices of the gear graph. The first color say c_1 and the second color say c_2 appears $n-1$ times, the third color say c_3 appears 2 times and the fourth color say c_4 appears once.

i) In order to calculate $M_1^{\varphi_b^-}$ of G_n, we first color the vertices as mentioned above and then we have $\theta(c_1) = n-1$, $\theta(c_2) = n-1$, $\theta(c_3) = 2$ and $\theta(c_4) = 1$. Thus, the associated first b-chromatic Zagreb index is provided by,

$$M_1^{\varphi_b^-}(G_n) = \sum_{u \in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j) \cdot j^2 = 5n + 29$$

ii) In order to calculate $M_2^{\varphi_b^-}$ of G_n, we first color the vertices as mentioned above and then we have $\eta_{12} = 2n-4$, $\eta_{13} = 2$, $\eta_{14} = n-2$, $\eta_{23} = 2$, $\eta_{24} = 1$ and $\eta_{34} = 1$. Thus, the associated second b-chromatic Zagreb index is provided by,

$$M_2^{\varphi_b^-}(G_n) = \sum_{uv \in E(G)} c(u) \cdot c(v) = \sum_{1 \le t,s \le \chi_b(G_n)}^{t<s} ts\eta_{ts} = 8n + 22$$

iii) In order to calculate $M_3^{\varphi_b^-}$ of G_n, we first color the vertices as mentioned above and then we have $\eta_{12} + \eta_{23} + \eta_{34} = 2n-1$ edges which contributes to 1 based on the color distance, $\eta_{13} + \eta_{24} = 3$ edges contributes to 2 based on the color distance $\eta_{14} = n-2$ edges contributes to 3 based on the color distance. Thus, the associated b-chromatic irregularity index is provided by,

$$M_3^{\varphi_b^-}(G_n) = \sum_{uv \in E(G)} |c(u) - c(v)| = \sum_{1 \le t,s \le \chi_b(G_n)}^{t<s} \eta_{ts}|t-s| = 5n - 1$$

iv) In order to calculate $M_4^{\varphi_b^-}$ of G_n, we first color the vertices as mentioned above. Then, we have to take into consideration the vertex pairs as well as all the color combinations which contributes non zero distances. The combinations $\{1,2\},\{2,3\},\{3,4\}$ contributes to the color distance 1, the combination $\{1,3\},\{2,4\}$ contributes to the color distance 2 and the combination $\{1,4\}$ contributes to the color distance 3. Also, we have $\theta(c_1) = n-1$, $\theta(c_2) = n-1$, $\theta(c_3) = 2$ and $\theta(c_4) = 1$. Thus, the associated b-chromatic total irregularity index is provided by,

$$M_4^{\varphi_b^-}(G_n) = \frac{1}{2}\sum_{uv\in V(G)} |c(u)-c(v)| = \frac{1}{2}\sum_{t,s\in C}^{t<s} \theta(c_t)\cdot\theta(c_s)|t-s| = \frac{n^2+9n-8}{2}$$

Theorem 2.2. *For the gear graph $G_n, n \geq 4$ we have,*

$$\begin{aligned}
&i) && M_1^{\varphi_b^+}(G_n) = 25n-16\\
&ii) && M_2^{\varphi_b^+}(G_n) = 28n-23\\
&iii) && M_3^{\varphi_b^+}(G_n) = 5n-1\\
&iv) && M_4^{\varphi_b^+}(G_n) = \frac{n^2+9n-8}{2}
\end{aligned}$$

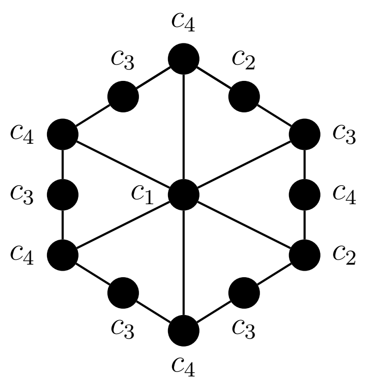

Figure 2: The b-coloring of G_6

Proof. We use 4 colors say c_1, c_2, c_3 and c_4 to color the vertices of the gear graph. The first color say c_1 appears once and the second color say c_2 appears 2 times. The third color say c_3 and the fourth color say c_4 appears $n-1$ times.

i) In order to calculate $M_1^{\varphi_b^+}$ of G_n, we first color the vertices as mentioned above and then we have $\theta(c_1) = 1$, $\theta(c_2) = 2$, $\theta(c_3) = n-1$ and $\theta(c_4) = n-1$. Thus, the associated first b-chromatic Zagreb index is provided by,

$$M_1^{\varphi_b^+}(G_n) = \sum_{u\in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j)\cdot j^2 = 25n - 16$$

ii) In order to calculate $M_2^{\varphi_b^+}$ of G_n, we first color the vertices as mentioned above and then we have $\eta_{12} = 1$, $\eta_{13} = 1$, $\eta_{14} = n-2$, $\eta_{23} = 2$, $\eta_{24} = 2$ and $\eta_{34} = 2n-4$. Thus, the associated second b-chromatic Zagreb index is provided by,

$$M_2^{\varphi_b^+}(G_n) = \sum_{uv\in E(G)} c(u)\cdot c(v) = \sum_{1\le t,s\le \chi_b(G_n)}^{t<s} ts\eta_{ts} = 28n - 23$$

iii) In order to calculate $M_3^{\varphi_b^+}$ of G_n, we first color the vertices as mentioned above and then we have $\eta_{12} + \eta_{23} + \eta_{34} = 2n-1$ edges which contributes to 1 based on the color distance, $\eta_{13} + \eta_{24} = 3$ edges contributes to 2 based on the color distance $\eta_{14} = n-2$ edges contributes to 3 based on the color distance. Thus, the associated b-chromatic irregularity index is provided by,

$$M_3^{\varphi_b^+}(G_n) = \sum_{uv\in E(G)} |c(u) - c(v)| = \sum_{1\le t,s\le \chi_b(G_n)}^{t<s} \eta_{ts}|t-s| = 5n - 1$$

iv) In order to calculate $M_4^{\varphi_b^+}$ of G_n, we first color the vertices as mentioned above. Then, we have to take into consideration the vertex pairs as well as all the color combinations which contributes non zero distances. The combinations $\{1,2\},\{2,3\},\{3,4\}$ contributes to the color distance 1, the combination $\{1,3\},\{2,4\}$ contributes to the color distance 2 and the combination $\{1,4\}$ contributes to the color distance 3. Also, we have $\theta(c_1) = 1$, $\theta(c_2) = 2$, $\theta(c_3) = n-1$ and $\theta(c_4) = n-1$. Thus, the associated b-chromatic total irregularity index is provided by,

$$M_4^{\varphi_b^+}(G_n) = \frac{1}{2}\sum_{uv\in V(G)} |c(u) - c(v)| = \frac{1}{2}\sum_{t,s\in C}^{t<s} \theta(c_t)\cdot\theta(c_s)|t-s| = \frac{n^2+9n-8}{2}$$

Theorem 2.3. *For the line graph of gear graph $L[G_n], n \geq 3$ we have,*

$$i) \qquad M_1^{\varphi_b^-}(L[G_n]) = \frac{2n^3 + 3n^2 + 31n + 78}{6}$$

$$ii) \qquad M_2^{\varphi_b^-}(L[G_n]) = \frac{3n^4 + 2n^3 + 33n^2 + 130n + 336}{24}$$

$$iii) \qquad M_3^{\varphi_b^-}(L[G_n]) = \frac{n^3 + 6n^2 - n + 42}{6}$$

$$iv) \qquad M_4^{\varphi_b^-}(L[G_n]) = \frac{7n^3 - 6n^2 + 17n + 30}{12}$$

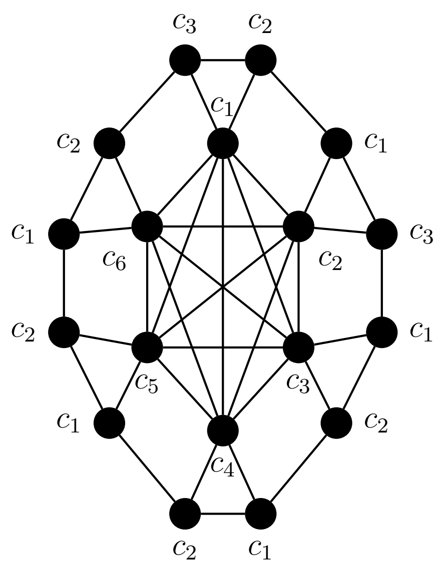

Figure 3: The b-coloring of $L[G_6]$

Proof. We use n colors say c_1, c_2, c_3,...,c_n to color the vertices of the line graph of gear graph. First, we color the inner complete graph with n colors. Then, we color the outer cycle with the colors c_1, c_2 and c_3 in such a way that the color c_1 and c_2 appears $n-1$ times and the color c_3 appears 2 times.

i) In order to calculate $M_1^{\varphi_b^-}$ of $L[G_n]$, we first color the vertices as mentioned above and then we have $\theta(c_1) = n$, $\theta(c_2) = n$, $\theta(c_3) = 3$, $\theta(c_4) = \theta(c_5) =, ..., \theta(c_n) = 1$. Thus, the associated first b-chromatic Zagreb index is provided by,

$$M_1^{\varphi_b^-}(L[G_n]) = \sum_{u \in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j) \cdot j^2 = \frac{2n^3+3n^2+31n+78}{6}$$

ii) In order to calculate $M_2^{\varphi_b^-}$ of $L[G_n]$, we first color the vertices as mentioned above and for $n \geq 3$, we have
$\eta_{12} = 2n - 1$, $\eta_{13} = 5$, $\eta_{14} = \eta_{15} =, ..., \eta_{1(n)} = 2$.
$\eta_{23} = 5$, $\eta_{24} = \eta_{25} =, ..., \eta_{2(n)} = 2$.
$\eta_{34} = \eta_{35} = \eta_{36}, ..., \eta_{3(n)} = 1$.
$,..., \eta_{(n-1)n} = 1$.
Thus, the associated second b-chromatic Zagreb index is provided by,

$$M_2^{\varphi_b^-}(L[G_n]) = \sum_{uv \in E(G)} c(u) \cdot c(v) = \sum_{1 \leq t,s \leq \chi_b(L[G_n])}^{t<s} ts\eta_{ts} = \frac{3n^4+2n^3+33n^2+130n+336}{24}$$

iii) In order to calculate $M_3^{\varphi_b^-}$ of $L[G_n]$, we first color the vertices as mentioned above and then we have $\eta_{12} + \eta_{23} + \eta_{34} + ... + \eta_{(n-1)n}$ edges which contributes to 1 based on the color distance and $\eta_{13} + \eta_{24} + \eta_{35} + ... + \eta_{(n-2)n}$ edges contributes to 2 based on the color distance and the procedure continues based on the selection of the graph. Thus, the associated b-chromatic irregularity index is provided by,

$$M_3^{\varphi_b^-}(L[G_n]) = \sum_{uv \in E(G)} |c(u) - c(v)| = \sum_{1 \leq t,s \leq \chi_b(L[G_n])}^{t<s} \eta_{ts}|t - s| = \frac{n^3+6n^2-n+42}{6}$$

iv) In order to calculate $M_4^{\varphi_b^-}$ of $L[G_n]$, we first color the vertices as mentioned above. Then, we have to take into consideration the vertex pairs as well as all the color combinations which contributes non zero distances. The combinations $\{1, 2\}, \{2, 3\}, \{3, 4\}, ..., \{n - 1, n\}$ contributes to the color distance 1 and the combination $\{1, 3\}, \{2, 4\}, \{3, 5\}, ..., \{n - 2, n\}$ contributes to the color distance 2 and same procedure is followed based on the selection of the graph. Also, we have $\theta(c_1) = \theta(c_2) = n$, $\theta(c_3) = 3, \theta(c_4) = \theta(c_5) =, ..., \theta(c_n) = 1$. Thus, the associated b-chromatic total irregularity index is provided by,

$$M_4^{\varphi_b^-}(L[G_n]) = \frac{1}{2} \sum_{uv \in V(G)} |c(u) - c(v)| = \frac{1}{2} \sum_{t,s \in C}^{t<s} \theta(c_t) \cdot \theta(c_s)|t - s| = \frac{7n^3-6n^2+17n+30}{12}$$

Theorem 2.4. *For the line graph of gear graph $L[G_n], n \geq 3$ we have,*

$$i) \qquad M_1^{\varphi_b^+}(L[G_n]) = \frac{14n^3 - 9n^2 - 29n + 42}{6}$$

$$ii) \qquad M_2^{\varphi_b^+}(L[G_n]) = \frac{3n^4 + 74n^3 - 39n^2 - 230n + 120}{24}$$

$$iii) \qquad M_3^{\varphi_b^+}(L[G_n]) = \frac{n^3+6n^2-n+42}{6}$$

$$iv) \qquad M_4^{\varphi_b^+}(L[G_n]) = \frac{7n^3-6n^2+17n+30}{12}$$

Proof. This proof is similar to the proof of Theorem 2.3.

Theorem 2.5. *For the middle graph of gear graph $M[G_n], n \geq 4$ we have,*

$$i) \qquad M_1^{\varphi_b^-}(M[G_n]) = \frac{2n^3+9n^2+103n+210}{6}$$

$$ii) \qquad M_2^{\varphi_b^-}(M[G_n]) = \frac{3n^4+14n^3+93n^2+610n+1200}{24}$$

$$iii) \qquad M_3^{\varphi_b^-}(M[G_n]) = \frac{n^3+12n^2+23n+144}{6}$$

$$iv) \qquad M_4^{\varphi_b^-}(M[G_n]) = \frac{13n^3+39n^2+92n+48}{12}$$

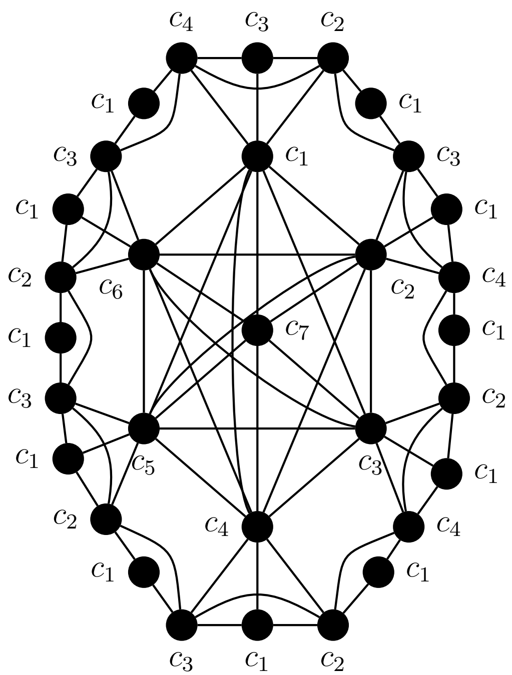

Figure 4: The b-coloring of $M[G_6]$

Proof. We use $n+1$ colors say c_1, c_2, c_3,...,c_{n+1} to color the vertices of the middle graph of gear graph. First, we color the inner cycle with n colors and its central vertex gets the color say c_{n+1}. Then, we color the outer cycle with the colors c_1, c_2, c_3 and c_4 in such a way that the color c_1 appears $2n-1$ times, c_2 and c_3 appears $n-1$ times and c_4 appears 3 times.

i)In order to calculate $M_1^{\varphi_b^-}$ of $M[G_n]$, we first color the vertices as mentioned above and then we have $\theta(c_1)=2n$, $\theta(c_2)=n$, $\theta(c_3)=n$, $\theta(c_4)=4$, $\theta(c_5)=\theta(c_6)=,...,\theta(c_{n+1})=1$.Thus, the associated first b-chromatic Zagreb index is provided by,

$$M_1^{\varphi_b^-}(M[G_n]) = \sum_{u\in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j)\cdot j^2 = \frac{2n^3+9n^2+103n+210}{6}$$

ii) In order to calculate $M_2^{\varphi_b^-}$ of $M[G_n]$, we first color the vertices as mentioned above and for $n=4$, we have $\eta_{12}=2n$, $\eta_{13}=2n-1$, $\eta_{14}=8$, $\eta_{15}=1$, $\eta_{23}=6$, $\eta_{24}=7$, $\eta_{25}=1$, $\eta_{34}=6$, $\eta_{35}=1$ and $\eta_{45}=1$.
$n=5$, we have $\eta_{12}=2n$, $\eta_{13}=2n-1$, $\eta_{14}=8$, $\eta_{15}=2$, $\eta_{16}=1$, $\eta_{23}=8$, $\eta_{24}=7$, $\eta_{25}=2$, $\eta_{26}=1$, $\eta_{34}=6$, $\eta_{35}=2$, $\eta_{36}=1$, $\eta_{45}=1$, $\eta_{46}=1$ and $\eta_{56}=1$.
and so on for $n=n$, we have
$\eta_{12}=2n$, $\eta_{13}=2n-1$, $\eta_{14}=8$, $\eta_{15}=\eta_{16}=,...,\eta_{1(n)}=2, \eta_{1(n+1)}=1$.
$\eta_{23}=2n-2$, $\eta_{24}=7$, $\eta_{25}=\eta_{26}=,...,\eta_{2(n)}=2, \eta_{2(n+1)}=1$.
$\eta_{34}=6$, $\eta_{35}=\eta_{36}=,...,\eta_{3(n)}=2, \eta_{3(n+1)}=1$.
$\eta_{45}=\eta_{46}=,...,\eta_{4(n+1)}=1$.
$,...,\ \eta_{n(n+1)}=1$.
Thus, the associated second b-chromatic Zagreb index is provided by,

$$M_2^{\varphi_b^-}(M[G_n]) = \sum_{uv\in E(G)} c(u)\cdot c(v) = \sum_{1\le t,s\le \chi_b(M[G_n])}^{t<s} ts\eta_{ts} = \frac{3n^4+14n^3+93n^2+610n+1200}{24}$$

iii) In order to calculate $M_3^{\varphi_b^-}$ of $M[G_n]$, we first color the vertices as mentioned above and then we have $\eta_{12}+\eta_{23}+\eta_{34}+...+\eta_{n(n+1)}$ edges which contributes to 1 based on the color distance and $\eta_{13}+\eta_{24}+\eta_{35}+...+\eta_{n-1(n+1)}$ edges contributes to 2 based on the color distance and the procedure continues based on the selection of the graph. Thus, the associated b-chromatic irregularity index is provided by,

$$M_3^{\varphi_b^-}(M[G_n]) = \sum_{uv\in E(G)} |c(u)-c(v)| = \sum_{1\le t,s\le \chi_b(M[G_n])}^{t<s} \eta_{ts}|t-s| = \frac{n^3+12n^2+23n+144}{6}$$

iv) In order to calculate $M_4^{\varphi_b^-}$ of $M[G_n]$, we first color the vertices as mentioned above. Then, we have to take into consideration the vertex pairs as well as all the color combinations which contributes non zero distances. The combinations

$\{1,2\},\{2,3\},\{3,4\},...,\{n,n+1\}$ contributes to the color distance 1 and the combination $\{1,3\},\{2,4\},\{3,5\},...,\{n-1,n+1\}$ contributes to the color distance 2 and same procedure is followed based on the selection of the graph. Also, we have $\theta(c_1) = 2n$, $\theta(c_2) = n$, $\theta(c_3) = n$, $\theta(c_4) = 4$, $\theta(c_5) = \theta(c_6) =, ..., \theta(c_{n+1}) = 1$. Thus, the associated b-chromatic total irregularity index is provided by,

$$M_4^{\varphi_b^-}(M[G_n]) = \frac{1}{2}\sum_{uv\in V(G)}|c(u)-c(v)| = \frac{1}{2}\sum_{t,s\in C}^{t<s}\theta(c_t)\cdot\theta(c_s)|t-s| = \frac{13n^3+39n^2+92n+48}{12}$$

Theorem 2.6. *For the middle graph of gear graph $M[G_n], n \geq 4$ we have,*

$$i)\qquad M_1^{\varphi_b^+}(M[G_n]) = \frac{26n^3+21n^2-41n+66}{6}$$

$$ii)\qquad M_2^{\varphi_b^+}(M[G_n]) = \frac{27n^4-358n^3+4785n^2-17750n+22992}{24}$$

$$iii)\qquad M_3^{\varphi_b^+}(M[G_n]) = \frac{n^3+12n^2+23n+144}{6}$$

$$iv)\qquad M_4^{\varphi_b^+}(M[G_n]) = \frac{13n^3+39n^2+92n+48}{12}$$

Proof. This proof is similar to the proof of Theorem 2.5.

Theorem 2.7. *For the star graph $K_{1,n}, n \geq 2$ we have,*

$$i)\qquad M_1^{\varphi_b^-}(K_{1,n}) = n+4$$

$$ii)\qquad M_2^{\varphi_b^-}(K_{1,n}) = 2n$$

$$iii)\qquad M_3^{\varphi_b^-}(K_{1,n}) = n$$

$$iv)\qquad M_4^{\varphi_b^-}(K_{1,n}) = \frac{n}{2}$$

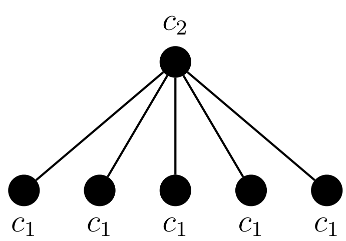

Figure 5: The b-coloring of $K_{1,5}$

Proof. We use 2 colors say c_1 and c_2 to color the vertices of the star graph. The first color say c_1 appears n times. The second color say c_2 appears 1 time.

i) In order to calculate $M_1^{\varphi_b^-}$ of $K_{1,n}$, we first color the vertices as mentioned above and then we have $\theta(c_1) = n$ and $\theta(c_2) = 1$. Thus, the associated first b-chromatic Zagreb index is provided by,

$$M_1^{\varphi_b^-}(K_{1,n}) = \sum_{u \in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j) \cdot j^2 = n + 4$$

ii) In order to calculate $M_2^{\varphi_b^-}$ of $K_{1,n}$, we first color the vertices as mentioned above and then we have $\eta_{12} = n$. Thus, the associated second b-chromatic Zagreb index is provided by,

$$M_2^{\varphi_b^-}(K_{1,n}) = \sum_{uv \in E(G)} c(u) \cdot c(v) = \sum_{1 \le t, s \le \chi_b(K_{1,n})}^{t<s} ts\eta_{ts} = 2n$$

iii) In order to calculate $M_3^{\varphi_b^-}$ of $K_{1,n}$, we first color the vertices as mentioned above and then we have $\eta_{12} = n$ edges which contributes to 1 based on the color distance. Thus, the associated b-chromatic irregularity index is provided by,

$$M_3^{\varphi_b^-}(K_{1,n}) = \sum_{uv \in E(G)} |c(u) - c(v)| = \sum_{1 \le t, s \le \chi_b(K_{1,n})}^{t<s} \eta_{ts}|t - s| = n$$

iv) In order to calculate $M_4^{\varphi_b^-}$ of $K_{1,n}$, we first color the vertices as mentioned above. Then, we have to take into consideration the vertex pair as well as all the color combination which contributes non zero distances. The combinations $\{1, 2\}$ contributes to the color distance 1. Also, we have $\theta(c_1) = n$ and $\theta(c_2) = 1$. Thus, the associated b-chromatic total irregularity index is provided by,

$$M_4^{\varphi_b^-}(K_{1,n}) = \frac{1}{2} \sum_{uv \in V(G)} |c(u) - c(v)| = \frac{1}{2} \sum_{t, s \in C}^{t<s} \theta(c_t) \cdot \theta(c_s)|t - s| = \frac{n}{2}$$

Theorem 2.8. *For the star graph $K_{1,n}, n \ge 2$ we have,*

$$\begin{aligned} i) \quad & M_1^{\varphi_b^+}(K_{1,n}) = 4n + 1 \\ ii) \quad & M_2^{\varphi_b^+}(K_{1,n}) = 2n \\ iii) \quad & M_3^{\varphi_b^+}(K_{1,n}) = n \\ iv) \quad & M_4^{\varphi_b^+}(K_{1,n}) = \frac{n}{2} \end{aligned}$$

Proof. This proof is similar to the proof of Theorem 2.7.

Theorem 2.9. *For the line graph of star graph $L[K_{1,n}], n \geq 2$ we have,*

$$i) \quad M_1^{\varphi_b^-}(L[K_{1,n}]) = \frac{2n^3+3n^2+n}{6}$$

$$ii) \quad M_2^{\varphi_b^-}(L[K_{1,n}]) = \frac{3n^4+2n^3-3n^2-2n}{24}$$

$$iii) \quad M_3^{\varphi_b^-}(L[K_{1,n}]) = \frac{n^3-n}{6}$$

$$iv) \quad M_4^{\varphi_b^-}(L[K_{1,n}]) = \frac{n^3-n}{12}$$

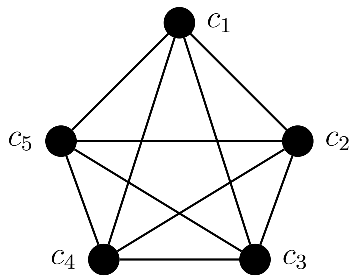

Figure 6: The b-coloring of $L[K_{1,5}]$

Proof. We use n colors say c_1, c_2, c_3,...,c_n to color the vertices of the line graph of star graph. The line graph of star graph gives rise to complete graph. Thus, the colors c_1, c_2, c_3,...,c_n appears 1 time.

i) In order to calculate $M_1^{\varphi_b^-}$ of $L[K_{1,n}]$, we first color the vertices as mentioned above and then we have $\theta(c_1) = \theta(c_2) = \theta(c_3) =, ..., \theta(c_n) = 1$. Thus, the associated first b-chromatic Zagreb index is provided by,

$$M_1^{\varphi_b^-}(L[K_{1,n}]) = \sum_{u \in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j) \cdot j^2 = \frac{2n^3+3n^2+n}{6}$$

ii) In order to calculate $M_2^{\varphi_b^-}$ of $L[K_{1,n}]$, we first color the vertices as mentioned above and we have $\eta_{12} = \eta_{13} = \eta_{14} =, ..., \eta_{1(n)} = 1$.

$\eta_{23} = \eta_{24} = \eta_{25} =, ..., \eta_{2(n)} = 1$.

$\eta_{34} = \eta_{35} = \eta_{36} =, ..., \eta_{3(n)} = 1$.

,..., $\eta_{(n-1)n} = 1$.

Thus, the associated second b-chromatic Zagreb index is provided by,

$$M_2^{\varphi_b^-}(L[K_{1,n}]) = \sum_{uv\in E(G)} c(u)\cdot c(v) = \sum_{1\le t,s\le \chi_b(L[K_{1,n}])}^{t<s} ts\eta_{ts} = \frac{3n^4+2n^3-3n^2-2n}{24}$$

iii) In order to calculate $M_3^{\varphi_b^-}$ of $L[K_{1,n}]$, we first color the vertices as mentioned above and then we have $\eta_{12}+\eta_{23}+\eta_{34}+...+\eta_{(n-1)n}$ edges which contributes to 1 based on the color distance and $\eta_{13}+\eta_{24}+\eta_{35}+...+\eta_{(n-2)n}$ edges contributes to 2 based on the color distance and the procedure continues based on the selection of the graph. Thus, the associated b-chromatic irregularity index is provided by,

$$M_3^{\varphi_b^-}(L[K_{1,n}]) = \sum_{uv\in E(G)} |c(u)-c(v)| = \sum_{1\le t,s\le \chi_b(L[K_{1,n}])}^{t<s} \eta_{ts}|t-s| = \frac{n^3-n}{6}$$

iv) In order to calculate $M_4^{\varphi_b^-}$ of $L[K_{1,n}]$, we first color the vertices as mentioned above. Then, we have to take into consideration the vertex pairs as well as all the color combinations which contributes non zero distances. The combinations $\{1,2\},\{2,3\},\{3,4\},...,\{n-1,n\}$ contributes to the color distance 1 and the combination $\{1,3\},\{2,4\},\{3,5\},...,\{n-2,n\}$ contributes to the color distance 2 and same procedure is followed based on the selection of the graph. Also, we have $\theta(c_1)=\theta(c_2)=\theta(c_3)=,...,\theta(c_n)=1$. Thus, the associated b-chromatic total irregularity index is provided by,

$$M_4^{\varphi_b^-}(L[K_{1,n}]) = \frac{1}{2}\sum_{uv\in V(G)} |c(u)-c(v)| = \frac{1}{2}\sum_{t,s\in C}^{t<s} \theta(c_t)\cdot\theta(c_s)|t-s| = \frac{n^3-n}{12}$$

Theorem 2.10. *For the line graph of star graph $L[K_{1,n}], n\ge 2$ we have,*

$$i)\quad M_1^{\varphi_b^+}(L[K_{1,n}]) = \frac{2n^3+3n^2+n}{6}$$

$$ii)\quad M_2^{\varphi_b^+}(L[K_{1,n}]) = \frac{3n^4+2n^3-3n^2-2n}{24}$$

$$iii)\quad M_3^{\varphi_b^+}(L[K_{1,n}]) = \frac{n^3-n}{6}$$

$$iv)\quad M_4^{\varphi_b^+}(L[K_{1,n}]) = \frac{n^3-n}{12}$$

Proof. This proof is similar to the proof of Theorem 2.9.

Theorem 2.11. *For the middle graph of star graph $M[K_{1,n}], n \geq 2$ we have,*

$$i) \quad M_1^{\varphi_b^-}(M[K_{1,n}]) = \frac{2n^3 + 9n^2 + 19n + 24}{6}$$

$$ii) \quad M_2^{\varphi_b^-}(M[K_{1,n}]) = \frac{3n^4 + 14n^3 + 33n^2 + 22n + 24}{24}$$

$$iii) \quad M_3^{\varphi_b^-}(M[K_{1,n}]) = \frac{n^3 + 6n^2 - n + 6}{6}$$

$$iv) \quad M_4^{\varphi_b^-}(M[K_{1,n}]) = \frac{2n^3 + 3n^2 + n}{6}$$

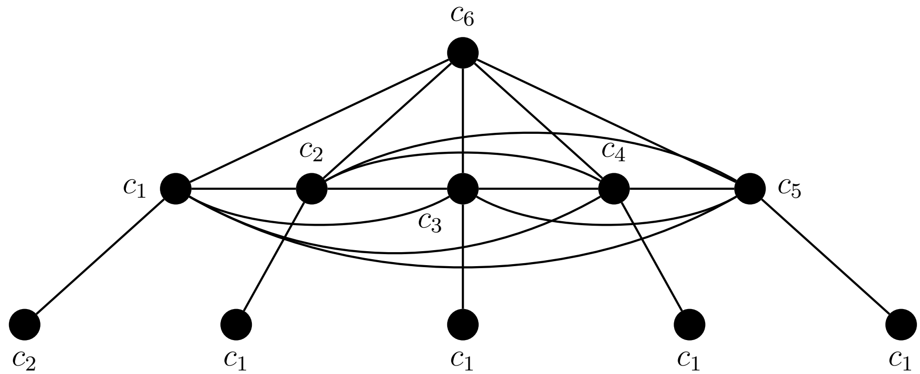

Figure 7: The b-coloring of $M[K_{1,5}]$

Proof. We use $n+1$ colors say c_1, c_2, c_3,..., c_n, c_{n+1} to color the vertices of the middle graph of star graph. The first color say c_1 appears n times, the second color say c_2 appears 2 times and the color c_3, c_4, c_5, ... , c_{n+1} appears 1 time.

i) In order to calculate $M_1^{\varphi_b^-}$ of $M[K_{1,n}]$, we first color the vertices as mentioned above and then we have $\theta(c_1) = n$, $\theta(c_2) = 2$, $\theta(c_3) = \theta(c_4) =, ..., \theta(c_{n+1}) = 1$. Thus, the associated first b-chromatic Zagreb index is provided by,

$$M_1^{\varphi_b^-}(M[K_{1,n}]) = \sum_{u \in V(G)} c(u)^2 = \sum_{j=1}^{l} \theta(c_j) \cdot j^2 = \frac{2n^3+9n^2+19n+24}{6}$$

ii) In order to calculate $M_2^{\varphi_b^-}$ of $M[K_{1,n}]$, we first color the vertices as mentioned above and for $n = 2$, we have $\eta_{12} = 3$, $\eta_{13} = 1$ and $\eta_{23} = 1$.

$n = 3$, we have $\eta_{12} = 3$, $\eta_{13} = 2$, $\eta_{14} = 1$, $\eta_{23} = 1$, $\eta_{24} = 1$ and $\eta_{34} = 1$.

$n = 4$, we have $\eta_{12} = 3$, $\eta_{13} = 2$, $\eta_{14} = 2$, $\eta_{15} = 1$, $\eta_{23} = 1$, $\eta_{24} = 1$, $\eta_{25} = 1$, $\eta_{34} = 1$,

$\eta_{35} = 1$ and $\eta_{45} = 1$.
and so on for $n = n$, we have
$\eta_{12} = 3$, $\eta_{13} = \eta_{14} = \eta_{15} =, ..., \eta_{1(n)} = 2, \eta_{1(n+1)} = 1$.
$\eta_{23} = \eta_{24} =, ..., \eta_{2(n)} = \eta_{2(n+1)} = 1$.
$\eta_{34} = \eta_{35} =, ..., \eta_{3(n)} = \eta_{3(n+1)} = 1$.
$,..., \eta_{n(n+1)} = 1$.
Thus, the associated second b-chromatic Zagreb index is provided by,

$$M_2^{\varphi_b^-}(M[K_{1,n}]) = \sum_{uv\in E(G)} c(u)\cdot c(v) = \sum_{1\le t,s\le \chi_b(M[K_{1,n}])}^{t<s} ts\eta_{ts} = \frac{3n^4+14n^3+33n^2+22n+24}{24}$$

iii) In order to calculate $M_3^{\varphi_b^-}$ of $M[K_{1,n}]$, we first color the vertices as mentioned above and then we have $\eta_{12} + \eta_{23} + \eta_{34} + ... + \eta_{n(n+1)}$ edges which contributes to 1 based on the color distance and $\eta_{13} + \eta_{24} + \eta_{35} + ... + \eta_{n-1(n+1)}$ edges contributes to 2 based on the color distance and the procedure continues based on the selection of the graph. Thus, the associated b-chromatic irregularity index is provided by,

$$M_3^{\varphi_b^-}(M[K_{1,n}]) = \sum_{uv\in E(G)} |c(u)-c(v)| = \sum_{1\le t,s\le \chi_b(M[K_{1,n}])}^{t<s} \eta_{ts}|t-s| = \frac{n^3+6n^2-n+6}{6}$$

iv) In order to calculate $M_4^{\varphi_b^-}$ of $M[K_{1,n}]$, we first color the vertices as mentioned above. Then, we have to take into consideration the vertex pairs as well as all the color combinations which contributes non zero distances. The combinations $\{1, 2\}, \{2, 3\}, \{3, 4\}, ..., \{n, n+1\}$ contributes to the color distance 1 and the combination $\{1, 3\}, \{2, 4\}, \{3, 5\}, ..., \{n-1, n+1\}$ contributes to the color distance 2 and same procedure is followed based on the selection of the graph. Also, we have $\theta(c_1) = n$, $\theta(c_2) = 2$, $\theta(c_3) = \theta(c_4) =, ..., \theta(c_{n+1}) = 1$. Thus, the associated b-chromatic total irregularity index is provided by,

$$M_4^{\varphi_b^-}(M[K_{1,n}]) = \frac{1}{2}\sum_{uv\in V(G)} |c(u)-c(v)| = \frac{1}{2}\sum_{t,s\in C}^{t<s} \theta(c_t)\cdot\theta(c_s)|t-s| = \frac{2n^3+3n^2+n}{6}$$

Theorem 2.12. *For the middle graph of star graph $M[K_{1,n}], n \ge 2$ we have,*

$$i) \quad M_1^{\varphi_b^+}(M[K_{1,n}]) = \frac{8n^3+21n^2+7n}{6}$$

$$ii) \quad M_2^{\varphi_b^+}(M[K_{1,n}]) = \frac{3n^4+26n^3+69n^2+22n-24}{24}$$

$$iii) \quad M_3^{\varphi_b^+}(M[K_{1,n}]) = \frac{n^3+6n^2-n+6}{6}$$

$$iv) \quad M_4^{\varphi_b^+}(M[K_{1,n}]) = \frac{2n^3+3n^2+n}{6}$$

Proof. This proof is similar to the proof of Theorem 2.11.

3. Conclusion

The study comprises the computation of the b-Chromatic Zagreb indices and b-Chromatic irregularity indices of the gear graph and star graph, as well as their line and middle graph. For several additional graph classes, we can make use of this idea. We can also explore this concept for several graph operations, including cartesian products, strong products, lexicographic products, etc. We can also investigate this concept for different graph colorings. Due to its applicability in various fields, including mathematical chemistry, distribution theory, optimization techniques, etc., studying b-chromatic topological indices is essential. In numerous other domains, similar investigations are feasible. These facts demonstrate the vast potential for additional study in this field.

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