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q-ANALOGUE OF HILFER-KATUGAMPOLA FRACTIONAL DERIVATIVES AND APPLICATIONS

Ishfaq Ahmad Mallah, Lata Chanchlani* and Subhash Alha

Department of Mathematics,
Maulana Azad National Urdu University,
Gachibowli, Hyderabad - 500032, Telangana, INDIA

E-mail : subhashalha@manuu.edu.in, ishfaqmalla@gmail.com

*Department of Mathematics,
University of Rajasthan, Jaipur - 302004, Rajasthan, INDIA

E-mail : chanchlani281106@gmail.com

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Abstract: A novel q^p-variant of the $q-$Mittag-Leffler function and a quantum analogue ${}^p\mathcal{D}^{\alpha,\beta}_{a\pm,q}$ of the Hilfer-Katugampola fractional derivative are defined. Then, generalizations of the $q-$Taylor's formula and the $q-$differential transform and its inverse are obtained using the operator ${}^p\mathcal{D}^{\alpha,\beta}_{a\pm,q}$. Additionally, a few properties of the newly defined q-differential transform are established. Finally, three proposed fractional q-difference equations are solved to show the effectiveness of the transform.

Keywords and Phrases: Hilfer-Katugampola fractional q-derivatives, q^p-Mittag-Leffler function, Generalized q-Taylor's formula, Generalized q-differential transform method.

2020 Mathematics Subject Classification: 26A33, 39A13.

1. Introduction

The theory of fractional q-difference calculus, which generalizes the concept of q-derivatives and q-integrals up to non-integer orders, emerged from the work of Al-Salam [3], Agarwal [2], Rajkovic *et al.* [26]. They presented a number of q-variants of the Riemann-Liouville fractional integral and derivative, as well as several well-known properties. Garg *et al.* [8] defined the generalized composite fractional q-derivative and obtained some of its key findings. The subject is explored and provided a number of intriguing findings in [4] and its sources.

In 2014, Katugampola [16] developed new fractional integral and derivative operators by using $t^p f(t)$ in the integrals concerned, which generalize and unify well-known Riemann-Liouville and Hadamard fractional integral and derivative operators. A novel class of fractional q-integral and q-derivative operators with the parameter p was also introduced by Momenzadeh *et al.* ([19], [20]) in the q-calculus. They described how their new classes of operators generalize all formerly known operators and can include Riemann-Liouville and Hadamard fractional q-integral and q-derivative operators. Chanchlani *et al.* then enhanced the work in [6] by using the findings from [19] as a basis.

The Hilfer-Katugampola (HK) fractional derivative, which resembles both the Hilfer and the Hilfer-Hadamard fractional derivatives, was introduced in 2017 by D. S. Oliveira and E. C. Oliveira [22] based on the generalized fractional integral and derivative operators defined by Katugmapola [16]. Jing and Fan [12] first proposed the idea of the q-differential transform method to address q-difference equations. Later Garg *et al.* [7] extended the method to solve q-difference equations with Caputo fractional q-derivative.

In this study, we define a q-analogue of the HK fractional derivative as a consequence of the research indicated above. This will unify all previously specified, well-known fractional q-derivatives. In addition, we develop a novel generalization of q-differential transform and provide a generalized HK fractional q-Taylor's formula.

2. Preliminaries

Definition 2.1. *For $\alpha > 0$, $0 < |q| < 1$, $p > 0$ and $f : [a, b] \to \mathbb{C}$, the Katugampola fractional q-integral is defined as* [6]*:*

$$\begin{aligned}\left({}^{p}\mathcal{J}^{\alpha}_{a,q}\phi\right)(x) &= \frac{(1-q)^{\alpha-1}}{(1-q^p)^{(\alpha-1)}_{q^p}} \int_a^x t^{p-1}\left(x^p - (tq)^p\right)^{(\alpha-1)}_{q^p}\phi(t)d_qt.\\ &= \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)} \int_a^x t^{p-1}\left(x^p - (tq)^p\right)^{(\alpha-1)}_{q^p}\phi(t)d_qt.\end{aligned} \tag{2.1}$$

Definition 2.2. *If $n-1 < \alpha \le n$, $0 < |q| < 1$ and $p > 0$, then the corresponding*

Katugampola fractional q-derivative is defined as [6]*:*

$$\begin{aligned}\left({}^{p}\mathcal{D}^{\alpha}_{a,q}\phi\right)(x) &= \left(x^{1-p}\mathcal{D}_q\right)^{n}\left({}^{p}\mathcal{J}^{n-\alpha}_{a,q}\right)\phi(x)\\ &= \frac{([p]_q)^{1-n+\alpha}}{\Gamma_{q^p}(n-\alpha)}\left(x^{1-p}\mathcal{D}_q\right)^{n}\int_a^x t^{p-1}\left(x^p-(tq)^p\right)_{q^p}^{(n-\alpha-1)}\phi(t)d_qt. \qquad (2.2)\\ \left({}^{p}\mathcal{D}^{0}_{a,q}\phi\right)(x) &= \phi(x).\end{aligned}$$

provided that $\phi \in L^1_{q,p}[a,b]$ *and* ${}^{p}\mathcal{J}^{n-\alpha}_{a,q}\phi \in AC^n_{p,q}[a,b]$*.*

Lemma 2.1. *For* $\alpha > 0$*,* $0 < |q| < 1$*,* $p > 0$ *and* $\lambda > -1$*, the following Jackson integral holds true* [20]*:*

$$\int_a^x t^{p-1}\left(x^p-(qt)^p\right)_{q^p}^{(\alpha-1)}\left(t^p-a^p\right)_{q^p}^{(\lambda)}d_qt = \frac{1}{[p]_q}\left(\frac{\Gamma_{q^p}(\alpha)\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}(\alpha+\lambda+1)}\right)\left[\left(x^p-a^p\right)_{q^p}^{(\alpha+\lambda)}\right]. \qquad (2.3)$$

Theorem 2.1. *If* $\alpha \in \mathbb{R}^+$*,* $0 < |q| < 1$*,* $p > 0$ *and* $\lambda \in (-1,\infty)$*, then the Images of power function* $(x^p-a^p)_{q^p}^{(\lambda)}$ *under* ${}^{p}\mathcal{J}^{\alpha}_{a,q}$ *is given by* [6]*:*

$${}^{p}\mathcal{J}^{\alpha}_{a,q}\left(x^p-a^p\right)_{q^p}^{(\lambda)} = \frac{1}{([p]_q)^{\alpha}}\left(\frac{\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}(\alpha+\lambda+1)}\right)\left(x^p-a^p\right)_{q^p}^{(\alpha+\lambda)}. \qquad (2.4)$$

Theorem 2.2. *For* $\alpha,\ \beta \in \mathbb{R}^+$*,* $0 < |q| < 1$ *and* $p > 0$*, if* $\phi \in L^1_{q,p}[a,b]$*, then the semi-group property for Katugampola fractional q-integral* ${}^{p}\mathcal{J}^{\alpha}_{a,q}$ *is given by* [6]*:*

$$\left({}^{p}\mathcal{J}^{\beta}_{a,q}\,{}^{p}\mathcal{J}^{\alpha}_{a,q}\phi\right)(x) = \left({}^{p}\mathcal{J}^{\alpha+\beta}_{a,q}\phi\right)(x), \quad (0 < a < x < b). \qquad (2.5)$$

Theorem 2.3. *For* $0 < |q| < 1$*,* $p > 0$ *and* $n-1 < \beta \le n$*,* $n \in \mathbb{N}$*, if* $f \in L^1_{q,p}[a,b]$ *and* ${}^{p}\mathcal{J}^{n-\beta}_{a,q}f \in AC^n_{p,q}[a,b]$*, then for any* $\alpha \ge 0$ [6]*:*

$$\left({}^{p}\mathcal{J}^{\alpha}_{a,q}\,{}^{p}\mathcal{D}^{\beta}_{a,q}\phi\right)(x) = {}^{p}\mathcal{D}^{-\alpha+\beta}_{a,q}\phi(x) - \sum_{k=1}^{n}\frac{([p]_q)^{k-\alpha}\left({}^{p}\mathcal{D}^{(\beta-k)}_{a,q}\phi\right)(a)}{\Gamma_{q^p}(\alpha-k+1)}\left(x^p-a^p\right)_{q^p}^{(\alpha-k)}, \qquad (2.6)$$

for $x \in (a,b]$*.*

Theorem 2.4. *Assume that* $\phi(x)$ *and* $\varphi(x)$ *be continuous on* $[a,b]$*. Then,* $\forall\ q \in (\hat{q},1)$ *where* $\hat{q} \in (0,1)$*,* $\exists\ \mu \in (a,b)$*, such that* [25]

$$\int_a^b (\phi\varphi)d_qt = \phi(\mu)\int_a^b \varphi(x)d_qt. \qquad (2.7)$$

3. Generalized q^p-Mittag Leffler Function

Here, we define generalized q^p-Mittag Leffler functions by adopting the concept from the formulations of q-Mittag Leffler functions provided in [7], as

Definition 3.1. *For $\alpha,\ \beta,\ \gamma \in \mathbb{C}$ with $Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\gamma) > 0$ and $|q| < 1,\ p > 0$, we define some $q-$analogues of Mittag-Leffler functions as*

$$ {}_{q^p}E_{\alpha}\left(\lambda, x-a\right) = \sum_{k=0}^{\infty} \lambda^k \frac{(x-a)_{q^p}^{(\alpha k)}}{\Gamma_{q^p}(\alpha k+1)}. \tag{3.1}$$

$$ {}_{q^p}E_{\alpha,\beta}\left(\lambda, x-a\right) = \sum_{k=0}^{\infty} \lambda^k \frac{(x-a)_{q^p}^{(\alpha k)}}{\Gamma_{q^p}(\alpha k+\beta)}. \tag{3.2}$$

and

$$ {}_{q^p}E_{\alpha,\beta}^{\gamma}\left(\lambda, x-a\right) = \sum_{k=0}^{\infty} \lambda^k \frac{(q^{p\gamma};q)_k\,(x-a)_{q^p}^{(\alpha k)}}{\Gamma_{q^p}(\alpha k+\beta)[k]_{q^p}!}. \tag{3.3}$$

In particular, for $\gamma = 2$, it gives

$$ {}_{q^p}E_{\alpha,\beta}^{2}\left(\lambda, x-a\right) = \sum_{k=0}^{\infty} \left(\lambda\left(1-q^p\right)\right)^k \frac{[k+1]_{q^p}\,(x-a)_{q^p}^{(\alpha k)}}{\Gamma_{q^p}(\alpha k+\beta)[k]_{q^p}!}. \tag{3.4}$$

Remark.
1. For $q \to 1^-$ and $a = 0$, we have the Mittag-Leffler functions introduced by G. M. Leffler [17], *Wiman* [28] *and Prabhakar* [23] *respectively.*
2. For $p = 1$, we have the q-variants of Mittag-Leffler functions given in ([4, 13, 27]).

4. Hilfer-Katugampola Fractional q-Derivative

Definition 4.1. *The Hilfer-Katugampola (HK) fractional q-derivative (left-sided / right-sided) of α order and β type, where $n-1 < \alpha \le n$ and $0 \le \beta \le 1$ with $p > 0$, $n \in \mathbb{N}$ and $0 < |q| < 1$, of a function ϕ is defined as follows*

$$\begin{aligned} \left({}^p\mathcal{D}_{a\pm,q}^{\alpha,\beta}\phi\right)(x) &= \left(\pm\ {}^p\mathcal{J}_{a\pm,q}^{\beta(n-\alpha)}\left(x^{1-p}\frac{d_q}{d_q x}\right)^n\ {}^p\mathcal{J}_{a\pm,q}^{(1-\beta)(n-\alpha)}\phi\right)(x) \\ &= \left(\pm\ {}^p\mathcal{J}_{a\pm,q}^{\beta(n-\alpha)}\ {}^p\delta_q^n\ {}^p\mathcal{J}_{a\pm,q}^{(1-\beta)(n-\alpha)}\phi\right)(x) \end{aligned} \tag{4.1}$$

where ${}^p\mathcal{J}_{a,q}^{\alpha}$ is the katugampola fractional q-integral defined by (2.1).
Only left-sided HK fractional q-derivatives are used to establish the results in this study. It is possible to establish similar results for the right-sided HK fractional q-derivative.

The derivative ${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}$ can be expressed in terms of the Katugampola fractional q-integral ${}^{p}\mathcal{J}^{\alpha}_{a,q}$ and Katugampola fractional q-derivative ${}^{p}\mathcal{D}^{\alpha}_{a,q}$ as

$$\left({}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi\right)(x) = \left({}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}\ {}^{p}\delta^{n}_{q}\ {}^{p}\mathcal{J}^{n-\gamma}_{a^+,q}\phi\right)(x) = \left({}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}\ {}^{p}\mathcal{D}^{\gamma}_{a^+,q}\phi\right)(x), \tag{4.2}$$

where $\gamma = \alpha + \beta(n-\alpha)$.
The operator ${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}$ in view of the (2.6) can be expressed as:

$$\begin{aligned}\left({}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi\right)(x) &= \left({}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}\ {}^{p}\mathcal{D}^{\gamma}_{a^+,q}\phi\right)(x)\\ &= {}^{p}\mathcal{D}^{\alpha}_{a^+,q}\phi(x) - \sum_{k=1}^{n}\frac{([p]_q)^{k-\beta(n-\alpha)}\left({}^{p}\mathcal{D}^{(\gamma-k)}_{a^+,q}\phi\right)(a)}{\Gamma_{q^p}(\beta(n-\alpha)-k+1)}\left(x^p-a^p\right)^{(\beta(n-\alpha)-k)}_{q^p}\end{aligned}$$

where $\phi \in L^{1}_{q,p}[a,b]$ and ${}^{p}\mathcal{J}^{n-\beta}_{a,q}\phi \in AC^{n}_{p,q}[a,b]$, $L^{1}_{q,p}\,[a,b]$ is the Banach space of all the functions defined on $[a,b]$, satisfying [4]

$$\|f\| = \int_a^b t^{p-1}|f(t)|d_qt < \infty.$$

$AC^{n}_{p,q}[a,b]$ is the space of all the functions f for which, f, ${}^{p}\delta_q(f),\ldots,({}^{p}\delta_q)^{n-1}(f)$ are q-regular at a and $({}^{p}\delta_q)^{n-1}(f) \in AC_{p,q}[a,b]$ [18].
For $p \to 1$, ${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}$ gives the q-extension of Hilfer (also called composite) fractional derivative $\left(\mathcal{D}^{\alpha,\beta}_{a^+}\phi\right)(x)$ defined by Hilfer [11] and for $p \to 0$, ${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}$ gives the q-extension of Hilfer-Hadamard fractional derivative [24].
In particular, for $\beta = 0$, we have Katugampola fractional q-derivative (2.2), and if we further let $p \to 1$, then from (4.1), we get Riemann-Liouville type fractional q-derivative of order α [4] and for $p \to 0^+$ and $q \to 1^-$, we obtain the Hadamard's fractional derivative [15].
Also, for $\beta = 1$ in (4.1), we obtain Caputo Katugampola fractional q-derivative [20]. In which, further on taking limit as $p \to 1$, we get Caputo fractional q-derivative of order α [4] and for $p \to 0$ and limit as $q \to 1^-$, we reach at Caputo-Hadamard's fractional derivative [14].

Theorem 4.1. *For $\lambda \in (-1,\infty)$, $n-1 < \alpha \le n, 0 \le \beta \le 1$ and $0 < |q| < 1$, $p > 0$, the image of power function $\left(x^p-a^p\right)^{(\lambda)}_{q^p}$ under ${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}$ is:*

$${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\left(x^p-a^p\right)^{(\lambda)}_{q^p} = ([p]_q)^{\alpha}\frac{\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}(\lambda-\alpha+1)}\left(x^p-a^p\right)^{(\lambda-\alpha)}_{q^p}.$$

Proof. On using (4.1) and (2.4) , we have

$$
\begin{aligned}
&{}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\Big(x^p-a^p\Big)^{(\lambda)}_{q^p}\\
&=([p]_q)^{\gamma}\frac{\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}(\lambda-\gamma+1)}\left\{([p]_q)^{-\beta(n-\alpha)}\frac{\Gamma_{q^p}(\lambda-\gamma+1)}{\Gamma_{q^p}\Big(\beta(n-\alpha)+\lambda-\gamma+1\Big)}\Big(x^p-a^p\Big)^{(\beta(n-\alpha)+\lambda-\gamma)}_{q^p}\right\}\\
&=\Big([p]_q\Big)^{\alpha}\frac{\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}(\lambda-\alpha+1)}\Big(x^p-a^p\Big)^{(\lambda-\alpha)}_{q^p}
\end{aligned}
$$

Particularly, for $\lambda=0$, we have $\Big({}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}1\Big)(x)=\frac{\Big([p]_q\Big)^{\alpha}}{\Gamma_{q^p}(1-\alpha)}\Big(x^p-a^p\Big)^{(-\alpha)}_{q^p}$, and for $\lambda=\alpha$, we have ${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\Big(x^p-a^p\Big)^{(\alpha)}_{q^p}=\Big([p]_q\Big)^{\alpha}\,\Gamma_{q^p}(\alpha+1)$, which is constant.

Theorem 4.2. *For $n-1<\alpha\le n, 0\le\beta\le 1,\ 0<|q|<1,\ p>0\ \gamma=\alpha+\beta(n-\alpha)$ and $0<a<b<\infty$. If $\phi\in L^1_{q,p}[a,b]$ and $\Big({}^{p}\mathcal{J}^{n-\gamma}_{a^+,q}\phi\Big)(x)\in AC^n_{p,q}[a,b]$, then for $x\in(a,b]$*

$$
\begin{aligned}
{}^{p}\mathcal{J}^{\alpha}_{a^+,q}\Big({}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi\Big)(x)&={}^{p}\mathcal{J}^{\gamma}_{a^+,q}\Big({}^{p}\mathcal{D}^{\gamma}_{a^+,q}\phi\Big)(x)\\
&=\phi(x)-\sum_{k=1}^{n}\frac{([p]_q)^{k-\gamma}\Big({}^{p}\mathcal{D}^{(\gamma-k)}_{a^+,q}\phi\Big)(a)}{\Gamma_{q^p}(\gamma-k+1)}\Big(x^p-a^p\Big)^{(\gamma-k)}_{q^p}
\end{aligned}
\tag{4.3}
$$

Proof. The proof of this theorem follows with the help of (2.5) and (4.2).

Theorem 4.3. *For $n-1<\alpha\le n, 0\le\beta\le 1,\ 0<|q|<1,\ p>0$ and $\gamma=\alpha+\beta(n-\alpha)$. Let $\phi\in L^1_{q,p}[a,b]$ and if ${}^{p}\mathcal{D}^{\beta(n-\alpha)}_{a^+,q}\phi\in L^1_{q,p}[a,b]$ exists, then*

$$
{}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\,{}^{p}\mathcal{J}^{\alpha}_{a^+,q}\phi={}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}{}^{p}\mathcal{D}^{\beta(n-\alpha)}_{a^+,q}\phi \tag{4.4}
$$

Proof. Using the (2.2), (2.5) and (4.1), we obtain

$$
\begin{aligned}
{}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\,{}^{p}\mathcal{J}^{\alpha}_{a^+,q}\phi&={}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}\mathcal{D}^{\gamma}_{a^+,q}\,{}^{p}\mathcal{J}^{\alpha}_{a^+,q}\phi={}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}\,{}^{p}\delta^{n}_{q}\,{}^{p}\mathcal{J}^{n-\gamma}_{a^+,q}{}^{p}\mathcal{J}^{\alpha}_{a^+,q}\phi\\
&={}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}\,{}^{p}\delta^{n}_{q}\,{}^{p}\mathcal{J}^{n-\beta(n-\alpha)}_{a^+,q}\phi={}^{p}\mathcal{J}^{\beta(n-\alpha)}_{a^+,q}{}^{p}\mathcal{D}^{\beta(n-\alpha)}_{a^+,q}\phi.
\end{aligned}
$$

5. Law of Exponent for Hilfer-Katugampola Fractional q-Derivative

Here, we present the law of exponents for HK fractional q-derivative, which holds under particular conditions. Various fractional derivatives follow similar laws of exponents [9, 10, 21].

Theorem 5.1. *For* $\phi(x) = \frac{\left(x^p - a^p\right)_{q^p}^{(\lambda)}}{\left(\left[p\right]_q\right)^{\lambda}} \psi(x)$ *with* $a,\ \lambda > 0$ *and* $0 < |q| < 1,\ p > 0,$ $\psi(x)$ *having the generalized series expansion* $\psi(x) = \sum\limits_{k=0}^{\infty} a_k \frac{\left(x^p - a^p\right)_{q^p}^{(k\alpha)}}{\left(\left[p\right]_q\right)^{k\alpha}}$ *with a radius of convergence* $R > 0,\ 0 < \alpha \leq 1,$ *we have*

$$ {}^{p}\mathcal{D}_{a^+,q}^{\eta,\beta}\, {}^{p}\mathcal{D}_{a^+,q}^{\delta,\beta} \phi(x) = {}^{p}\mathcal{D}_{a^+,q}^{\eta+\delta,\beta} \phi(x), \quad for\ all\ \ \frac{\left(x^p - a^p\right)_{q^p}^{(\lambda)}}{\left(\left[p\right]_q\right)^{\lambda}} \in (0, R) \tag{5.1}$$

$\mu = max\left(\delta + \eta + \beta\left(\lfloor \eta \rfloor + 1 - \eta\right) - 1, \delta + \eta + \beta\left(\lfloor \delta + \eta \rfloor + 1 - \eta - \delta\right) - 1\right)$ *and either*

(a) $\lambda > \mu$, *or*

(b) $\lambda = \mu,\ a_0 = 0$, *or*

(c) $\lambda < \mu,\ a_k = 0$, *for* $k = 0, 1, 2, \cdots, -\lfloor -\frac{\mu-\lambda}{\alpha} \rfloor - 1.$

Here $\lfloor \alpha \rfloor$ *denotes the greatest integer less than or equal to* α.

Proof. For part (a), by the definition of HK fractional q-derivative (4.1), we have

$$\left({}^{p}\mathcal{D}_{a^+,q}^{\delta,\beta} \phi\right)(x) = \left({}^{p}\mathcal{J}_{a^+,q}^{\beta(\lfloor \delta \rfloor + 1 - \delta)}\, {}^{p}\mathcal{D}_{a^+,q}^{\delta + \beta(\lfloor \delta \rfloor + 1 - \delta)}\right) \sum_{k=0}^{\infty} a_k \frac{\left(x^p - a^p\right)_{q^p}^{(k\alpha+\lambda)}}{\left(\left[p\right]_q\right)^{k\alpha+\lambda}}, \tag{5.2}$$

Differentiating terms one from another is permitted under the conditions $\lambda > \mu \geq -1$, the derivatives of order $\delta + \beta\left(\lfloor \delta \rfloor + 1 - \delta\right)$ of the series involved $\frac{\left(x^p - a^p\right)_{q^p}^{(k\alpha+\lambda)}}{\left(\left[p\right]_q\right)^{k\alpha+\lambda}}$ are uniformly convergent for $\frac{\left(x^p - a^p\right)_{q^p}^{(\lambda)}}{\left(\left[p\right]_q\right)^{\lambda}} \in (0, R)$, thus on using ${}^{p}\mathcal{D}_{a^+,q}^{\alpha}\left(x^p - a^p\right)_{q^p}^{(\lambda)} =$

$\frac{\left(\left[p\right]_q\right)^{\alpha}\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}\left(\lambda-\alpha+1\right)}\left(x^p-a^p\right)_{q^p}^{(\lambda-\alpha)}$, we obtain

$$\left({}^p\mathcal{D}_{a^+,q}^{\delta,\beta}\phi\right)(x) = {}^p\mathcal{J}_{a^+,q}^{\beta(\lfloor\delta\rfloor+1-\delta)}$$
$$\sum_{k=0}^{\infty} a_k \frac{\Gamma_{q^p}(k\alpha+\lambda+1)}{\Gamma_{q^p}(k\alpha+\lambda-\delta-\beta(\lfloor\delta\rfloor+1-\delta)+1)} \frac{\left(x^p-a^p\right)_{q^p}^{(k\alpha+\lambda-\delta-\beta(\lfloor\delta\rfloor+1-\delta))}}{\left(\left[p\right]_q\right)^{k\alpha+\lambda-\delta-\beta(\lfloor\delta\rfloor+1-\delta)}},$$

Additionally, we have $\lambda > \mu \geq \delta + \beta(\lfloor\delta\rfloor + 1 - \delta) - 1$ and uniformly convergence of series for $\frac{\left(x^p-a^p\right)_{q^p}^{(\lambda)}}{\left(\left[p\right]_q\right)^{\lambda}} \in (0, R)$, and by reversing the integration and summation orders and using ${}^p\mathcal{J}_{a^+,q}^{\alpha}\left(x^p-a^p\right)_{q^p}^{(\lambda)} = \frac{1}{\left(\left[p\right]_q\right)^{\alpha}}\left(\frac{\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}(\alpha+\lambda+1)}\right)\left(x^p-a^p\right)_{q^p}^{(\alpha+\lambda)}$, we are able to obtain

$$\left({}^p\mathcal{D}_{a^+,q}^{\delta,\beta}\phi\right)(x) = \sum_{k=0}^{\infty} a_k \frac{\Gamma_{q^p}(k\alpha+\lambda+1)}{\Gamma_{q^p}(k\alpha+\lambda-\delta+1)} \frac{\left(x^p-a^p\right)_{q^p}^{(k\alpha+\lambda-\delta)}}{\left(\left[p\right]_q\right)^{(k\alpha+\lambda-\delta)}} \tag{5.3}$$

Using the same premise as before with $\lambda > \mu \geq \delta - 1$, $\lambda > \mu \geq \delta + \eta + \beta\left(\lfloor\eta\rfloor + 1 - \eta\right) - 1$, we now have

$$\begin{aligned} {}^p\mathcal{D}_{a^+,q}^{\eta,\beta}\,{}^p\mathcal{D}_{a^+,q}^{\delta,\beta}\phi(x) &= {}^p\mathcal{D}_{a^+,q}^{\eta,\beta}\sum_{k=0}^{\infty} a_k \frac{\Gamma_{q^p}(k\alpha+\lambda+1)}{\Gamma_{q^p}(k\alpha+\lambda-\delta+1)} \frac{\left(x^p-a^p\right)_{q^p}^{(k\alpha+\lambda-\delta)}}{\left(\left[p\right]_q\right)^{(k\alpha+\lambda-\delta)}} \\ &= \sum_{k=0}^{\infty} a_k \frac{\Gamma_{q^p}(k\alpha+\lambda+1)}{\Gamma_{q^p}(k\alpha+\lambda-\eta-\delta+1)} \frac{\left(x^p-a^p\right)_{q^p}^{(k\alpha+\lambda-\eta-\delta)}}{\left(\left[p\right]_q\right)^{(k\alpha+\lambda-\eta-\delta)}} \end{aligned} \tag{5.4}$$

Then, for $\lambda > \mu \geq -1$ and $\lambda > \mu \geq \delta + \eta + \beta\left(\lfloor\delta+\eta\rfloor + 1 - \eta - \delta\right) - 1$

$$\left({}^p\mathcal{D}_{a^+,q}^{\eta+\delta,\beta}\phi\right)(x) = \sum_{k=0}^{\infty} a_k \frac{\Gamma_{q^p}(k\alpha+\lambda+1)}{\Gamma_{q^p}(k\alpha+\lambda-\eta-\delta+1)} \frac{\left(x^p-a^p\right)_{q^p}^{(k\alpha+\lambda-\eta-\delta)}}{\left(\left[p\right]_q\right)^{(k\alpha+\lambda-\eta-\delta)}}, \tag{5.5}$$

which is exactly ${}^p\mathcal{D}^{\eta,\beta}_{a^+,q}\,{}^p\mathcal{D}^{\delta,\beta}_{a^+,q}\phi(x)$, according to (5.4).
For parts (b) and (c) i.e. $\lambda \leq \mu$, we begin with $a_k = 0$, for $k = 0, 1, \cdots, l-1$, where $l = -\lfloor -\frac{\mu-\lambda}{\alpha}\rfloor$, we take into account the uniform convergence of derived series up to order $\lfloor \delta \rfloor + 1$

$$\begin{aligned}\left({}^p\mathcal{D}^{\delta,\beta}_{a^+,q}\phi\right)(x) &= \sum_{k=l}^{\infty} a_k \frac{\Gamma_{q^p}(k\alpha+\lambda+1)}{\Gamma_{q^p}(k\alpha+\lambda-\delta+1)} \frac{\left(x^p - a^p\right)_{q^p}^{(k\alpha+\lambda-\delta)}}{\left(\left[p\right]_q\right)^{(k\alpha+\lambda-\delta)}} \\ &= \sum_{i=0}^{\infty} a_{i+j} \frac{\Gamma_{q^p}((i+j)\alpha+\lambda+1)}{\Gamma_{q^p}((i+j)\alpha+\lambda-\delta+1)} \frac{\left(x^p - a^p\right)_{q^p}^{((i+j)\alpha+\lambda-\delta)}}{\left(\left[p\right]_q\right)^{((i+j)\alpha+\lambda-\delta)}}.\end{aligned} \tag{5.6}$$

If we assume $\lambda' = j\alpha + \lambda$, then (5.6) becomes identical to (5.3) (with λ substituted by λ'), and the proof continues as in part (a).

6. Hilfer-Katugampola Fractional q-Taylor's Formula

Lemma 6.1. *For $n-1 < \alpha \leq n, 0 \leq \beta \leq 1$, $0 < |q| < 1$, $p > 0$, $\gamma = \alpha + \beta(n-\alpha)$ and $0 < a < b < \infty$. If $\phi \in L^1_{q,p}[a,b]$ and $\left({}^p\mathcal{J}^{n-\gamma}_{a^+,q}\phi\right)(x) \in AC^n_{p,q}[a,b]$, ${}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi \in C[a,b]$, $x \in (a,b]$, then there exists $c \in (a,x)$, such that*

$$\phi(x) = \sum_{k=1}^{n} \frac{([p]_q)^{k-\gamma}\left({}^p\mathcal{D}^{(\gamma-k)}_{a^+,q}\phi\right)(a)}{\Gamma_{q^p}(\gamma-k+1)}\left(x^p - a^p\right)_{q^p}^{(\gamma-k)} + \frac{\left([p]_q\right)^{1-2\alpha}}{\Gamma_{q^p}(\alpha+1)}\,{}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi(c)\left(x^p - a^p\right)_{q^p}^{(\alpha)}$$

Proof. On using (2.1) and (2.7), we have

$${}^p\mathcal{J}^{\alpha}_{a^+,q}\,{}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi(x) = \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\,{}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi(c)\int_a^x t^{p-1}\left(x^p - (tq)^p\right)_{q^p}^{(\alpha-1)} d_q t. \tag{6.1}$$

By making use of (2.3) (with $\lambda = 0$), (6.1) gives

$${}^p\mathcal{J}^{\alpha}_{a^+,q}\,{}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi(x) = \frac{\left([p]_q\right)^{1-2\alpha}}{\Gamma_{q^p}(\alpha+1)}\,{}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi(c)\left(x^p - a^p\right)_{q^p}^{(\alpha)}$$

On taking Theorem 4.2 in account, we get

$$\phi(x) = \sum_{k=1}^{n} \frac{([p]_q)^{k-\gamma}\left({}^p\mathcal{D}^{(\gamma-k)}_{a^+,q}\phi\right)(a)}{\Gamma_{q^p}(\gamma-k+1)}\left(x^p - a^p\right)_{q^p}^{(\gamma-k)} + \frac{\left([p]_q\right)^{1-2\alpha}}{\Gamma_{q^p}(\alpha+1)}\,{}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}\phi(c)\left(x^p - a^p\right)_{q^p}^{(\alpha)}$$

Lemma 6.2. *For $n-1<\alpha\le n, 0\le\beta\le 1$, $0<|q|<1$, $p>0$, $\gamma=\alpha+\beta(n-\alpha)$ and $0<a<b<\infty$. If ${}^p\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi\in L^1_{q,p}[a,b]$ and ${}^p\mathcal{J}^{n-\gamma}_{a^+,q}{}^p\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi\in AC^n_{p,q}[a,b]$, $k=0,1,2,\ldots,m+1$, $m\in\mathbb{N}$, then we have*

$$\begin{aligned}{}^p\mathcal{J}^{m\alpha}_{a^+,q}{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(x)-{}^p\mathcal{J}^{(m+1)\alpha}_{a^+,q}{}^p\mathcal{D}^{(m+1)\alpha,\beta}_{a^+,q}\phi(x)&\\ =\sum_{j=1}^{n}\frac{\left([p]_q\right)^{j-m\alpha-\gamma}\left(x^p-a^p\right)^{(m\alpha+\gamma-j)}_{q^p}}{\Gamma_{q^p}(m\alpha+\gamma-j+1)}{}^p\mathcal{D}^{\gamma-j}_{a^+,q}{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(a)&\end{aligned}\tag{6.2}$$

Proof. By using (2.5), we can write the left hand side of (6.2) as

$$\begin{aligned}&{}^p\mathcal{J}^{m\alpha}_{a^+,q}\left[{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(x)-{}^p\mathcal{J}^{\alpha}_{a^+,q}{}^p\mathcal{D}^{(m+1)\alpha,\beta}_{a^+,q}\phi(x)\right]\\ &\qquad={}^p\mathcal{J}^{m\alpha}_{a^+,q}\left[{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(x)-{}^p\mathcal{J}^{\alpha}_{a^+,q}{}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(x)\right]\end{aligned}\tag{6.3}$$

Using Theorem 4.2, (6.3) becomes

$$={}^p\mathcal{J}^{m\alpha}_{a^+,q}\left[{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(x)-{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(x)+\sum_{j=1}^{n}\frac{\left([p]_q\right)^{j-\gamma}\left(x^p-a^p\right)^{(\gamma-j)}_{q^p}}{\Gamma_{q^p}(\gamma-j+1)}{}^p\mathcal{D}^{\gamma-j}_{a^+,q}{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(a)\right]$$

$$={}^p\mathcal{J}^{m\alpha}_{a^+,q}\left[\sum_{j=1}^{n}\frac{\left([p]_q\right)^{j-\gamma}\left(x^p-a^p\right)^{(\gamma-j)}_{q^p}}{\Gamma_{q^p}(\gamma-j+1)}{}^p\mathcal{D}^{\gamma-j}_{a^+,q}{}^p\mathcal{D}^{m\alpha,\beta}_{a^+,q}\phi(a)\right]$$

Finally, we obtain the right hand side of (6.2) by using (2.4).

Theorem 6.1. *For $n-1<\alpha\le n, 0\le\beta\le 1$, $0<|q|<1$, $p>0$, $\gamma=\alpha+\beta(n-\alpha)$ and $0<a<b<\infty$. If ${}^p\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi\in L^1_{q,p}[a,b], {}^p\mathcal{J}^{n-\gamma}_{a^+,q}{}^p\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi\in AC^n_{p,q}[a,b]$, $k=0,1,2,\ldots,m+1$, ${}^p\mathcal{D}^{(n+1)\alpha,\beta}_{a^+,q}\phi\in C[a,b]$, then there exists $c\in(a,x)$ such that*

$$\begin{aligned}\phi(x)=\sum_{k=0}^{m}\sum_{j=1}^{n}\frac{\left([p]_q\right)^{k\alpha-\gamma+j}\left(x^p-a^p\right)^{(k\alpha+\gamma-j)}_{q^p}}{\Gamma_{q^p}(k\alpha+\gamma-j+1)}{}^p\mathcal{D}^{\gamma-j}_{a^+,q}{}^p\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi(a)&\\ +\frac{{}^p\mathcal{D}^{(m+1)\alpha,\beta}_{a^+,q}\phi(c)}{\Gamma_{q^p}\big((m+1)\alpha+1\big)}\left(x^p-a^p\right)^{((m+1)\alpha)}_{q^p}&\end{aligned}$$

Proof. From (6.2), we have

$$\begin{aligned}&\sum_{k=0}^{m}\left\{\left({}^p\mathcal{J}^{k\alpha}_{a^+,q}{}^p\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi\right)(x)-\left({}^p\mathcal{J}^{(k+1)\alpha}_{a^+,q}{}^p\mathcal{D}^{(k+1)\alpha,\beta}_{a^+,q}\phi\right)(x)\right\}\\ &\qquad=\sum_{k=0}^{m}\sum_{j=1}^{n}\frac{\left([p]_q\right)^{j-k\alpha-\gamma}\left(x^p-a^p\right)^{(k\alpha+\gamma-j)}_{q^p}}{\Gamma_{q^p}(k\alpha+\gamma-j+1)}{}^p\mathcal{D}^{\gamma-j}_{a^+,q}{}^p\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi(a)\end{aligned}$$

After simplifying, we obtain

$$\phi(x)=\sum_{k=0}^{m}\sum_{j=1}^{n}\frac{\big([p]_q\big)^{j-k\alpha-\gamma}\big(x^p-a^p\big)_{q^p}^{(k\alpha+\gamma-j)}}{\Gamma_{q^p}(k\alpha+\gamma-j+1)}{}^{p}\mathcal{D}_{a^+,q}^{\gamma-j}\,{}^{p}\mathcal{D}_{a^+,q}^{k\alpha,\beta}\phi(a) + \big({}^{p}\mathcal{J}_{a^+,q}^{(m+1)\alpha}\,{}^{p}\mathcal{D}_{a^+,q}^{(m+1)\alpha,\beta}\phi\big)(x)$$

On using (2.1), we get

$$\phi(x)=\sum_{k=0}^{m}\sum_{j=1}^{n}\frac{\big([p]_q\big)^{j-k\alpha-\gamma}\big(x^p-a^p\big)_{q^p}^{(k\alpha+\gamma-j)}}{\Gamma_{q^p}(k\alpha+\gamma-j+1)}{}^{p}\mathcal{D}_{a^+,q}^{\gamma-j}\,{}^{p}\mathcal{D}_{a^+,q}^{k\alpha,\beta}\phi(a) + \frac{([p]_q)^{1-(m+1)\alpha}}{\Gamma_{q^p}\big((m+1)\alpha\big)}\int_a^x t^{p-1}\big(x^p-(tq)^p\big)_{q^p}^{((m+1)\alpha-1)}\,{}^{p}\mathcal{D}_{a^+,q}^{(m+1)\alpha,\beta}\phi(t)d_qt$$

Using (2.3) (with $\lambda=0$) and (2.7), we have

$$\phi(x)=\sum_{k=0}^{m}\sum_{j=1}^{n}\frac{\big([p]_q\big)^{j-k\alpha-\gamma}\big(x^p-a^p\big)_{q^p}^{(k\alpha+\gamma-j)}}{\Gamma_{q^p}(k\alpha+\gamma-j+1)}{}^{p}\mathcal{D}_{a^+,q}^{\gamma-j}\,{}^{p}\mathcal{D}_{a^+,q}^{k\alpha,\beta}\phi(a) + \frac{{}^{p}\mathcal{D}_{a^+,q}^{(m+1)\alpha,\beta}\phi(c)}{\Gamma_{q^p}\big((m+1)\alpha+1\big)}\left[\frac{\big(x^p-a^p\big)_{q^p}^{((m+1)\alpha)}}{\big([p]_q\big)^{((m+1)\alpha)}}\right];\ c\in(a,x)$$

Now, if the remainder term $\frac{{}^{p}\mathcal{D}_{a^+,q}^{(m+1)\alpha,\beta}\phi(c)}{\Gamma_{q^p}\big((m+1)\alpha+1\big)}\left[\frac{\big(x^p-a^p\big)_{q^p}^{((m+1)\alpha)}}{\big([p]_q\big)^{((m+1)\alpha)}}\right]\to 0$ as $m\to\infty$, we have the generalized q-Taylor's formula involving HK fractional q-derivative as

$$\phi(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{n}\frac{\big([p]_q\big)^{j-k\alpha-\gamma}\big(x^p-a^p\big)_{q^p}^{(k\alpha+\gamma-j)}}{\Gamma_{q^p}(k\alpha+\gamma-j+1)}{}^{p}\mathcal{D}_{a^+,q}^{\gamma-j}\,{}^{p}\mathcal{D}_{a^+,q}^{k\alpha,\beta}\phi(a)$$

In particular, for $0<\alpha\le 1$, the HK fractional q-Taylor's formula is given by

$$\phi(x)=\sum_{k=0}^{\infty}\frac{\big([p]_q\big)^{1-k\alpha-\gamma}\big(x^p-a^p\big)_{q^p}^{(k\alpha+\gamma-1)}}{\Gamma_{q^p}(k\alpha+\gamma)}{}^{p}\mathcal{D}_{a^+,q}^{\gamma-1}\,{}^{p}\mathcal{D}_{a^+,q}^{k\alpha,\beta}\phi(a) \tag{6.4}$$

Remark. *(i) For $p\to 1$, we have the generalized Taylor's formula for involving composite fractional q-derivative* $\mathcal{D}_{a^+,q}^{\alpha,\beta}$ [5].

(ii) For $q \to 1^-$, we get the results for HK fractional derivative ${}^{p}\mathcal{D}^{\alpha,\beta}_{a^+}$ [22].

7. Generalized HK Fractional q-Differential Transform

Here, in this section, we make use of generalized q-taylor's formula obtained in previous section, for ${}^{p}\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi \in L^{1}_{q,p}[a,b], {}^{p}\mathcal{D}^{\gamma}_{a^+,q}\,{}^{p}\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi \in AC^{n}_{p,q}[a,b]$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < |q| < 1$, $p > 0$, we define the HK generalized q-differential transform ${}^{p}_{q}\Phi_{\alpha,\beta}(k)$ of function $\phi(x)$ at point $x = a$ as follows

$$ {}^{p}_{q}\Phi_{\alpha,\beta}(k) = \frac{1}{\Gamma_{q^p}(k\alpha+\gamma)}\left[{}^{p}\mathcal{D}^{\gamma-1}_{a^+,q}\,{}^{p}\mathcal{D}^{k\alpha,\beta}_{a^+,q}\phi(x)\right]_{x=a} \tag{7.1} $$

where ${}^{p}\mathcal{D}^{k\alpha,\beta}_{a^+,q} = {}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\,{}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}\cdots{}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}$ $(k-times)$, and the inverse HK generalized q-differential transform of ${}^{p}_{q}\Phi_{\alpha,\beta}(k)$ in view of (6.4) is given as follows

$$ \phi(x) = \sum_{k=0}^{\infty} {}^{p}_{q}\Phi_{\alpha,\beta}(k)\left[\frac{\left(x^p - a^p\right)^{(k\alpha+\gamma-1)}_{q^p}}{\left([p]_q\right)^{k\alpha+\gamma-1}}\right] \tag{7.2} $$

Remark. *1. If we let $p \to 1$ in (7.1) and (7.2), we have generalized q-differential transform and its inverse for composite fractional q-derivative which are same as obtained in* [5].
2. If $q \to 1^-$, $p \to 1$, we get the results for composite fractional derivative obtained in [9].
The HK generalized q-differential transform's fundamental characteristics are described here.

Theorem 7.1. *The following results hold true if ${}^{p}_{q}\Phi_{\alpha,\beta}(k)$, ${}^{p}_{q}U_{\alpha,\beta}(k)$ and ${}^{p}_{q}V_{\alpha,\beta}(k)$ are generalized q-differential transforms of functions $\phi(x)$, $u(x)$ and $v(x)$, respectively, at point $x = a$*

1. *In the case if $\phi(x) = u(x) \pm v(x)$, then ${}^{p}_{q}\Phi_{\alpha,\beta}(k) = {}^{p}_{q}U_{\alpha,\beta}(k) \pm {}^{p}_{q}V_{\alpha,\beta}(k)$ will follow.*

2. *${}^{p}_{q}\Phi_{\alpha,\beta}(k) = c\,{}^{p}_{q}U_{\alpha,\beta}(k)$ follows if $\phi(x) = cu(x)$, where c is a constant.*

3. *For $\phi(x) = \frac{\left(x^p-a^p\right)^{(n\alpha+\gamma-1)}_{q^p}}{\left([p]_q\right)^{n\alpha+\gamma-1}}$, with $n \in \mathbb{N}$, then ${}^{p}_{q}U_{\alpha,\beta}(k) = \delta(k-n)$, where*

$$ \delta(k) = \begin{cases} 1, & when\ k = 0 \\ 0, & otherwise \end{cases} $$

4. *For* $\phi(x) = {}^{p}\mathcal{D}^{\alpha.\beta}_{a^+,q}u(x)$, *with* $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, *the following equation hold true*

$$ {}^{p}_{q}U_{\alpha,\beta}(k) = \frac{\Gamma_{q^p}(k\alpha + \alpha + \gamma)}{\Gamma_{q^p}(k\alpha + \gamma)} {}^{p}_{q}U_{\alpha,\beta}(k+1) $$

Proof. The generalized q-differential transform's linearity property enables it simple to obtain the findings 1. and 2.

3. With the help of (7.2), $\phi(x) = \frac{\left(x^p - a^p\right)^{(n\alpha+\gamma-1)}_{q^p}}{\left([p]_q\right)^{n\alpha+\gamma-1}}$, we can write

$\phi(x) = \sum\limits_{k=0}^{\infty} \delta(k) \left[\frac{\left(x^p - a^p\right)^{(k\alpha+\gamma-1)}_{q^p}}{\left([p]_q\right)^{k\alpha+\gamma-1}} \right].$

On using inverse HK generalized q-differential transform (7.2), we have ${}^{p}_{q}\Phi_{\alpha,\beta}(k) = \delta(k-n)$.

4. By taking $\phi(x) = {}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}u(x)$ in (7.1), we have

$$ \begin{aligned} {}^{p}_{q}\Phi_{\alpha,\beta}(k) =& \frac{1}{\Gamma_{q^p}(k\alpha + \gamma)} \left[{}^{p}\mathcal{D}^{\gamma-1}_{a^+,q} {}^{p}\mathcal{D}^{k\alpha,\beta}_{a^+,q} {}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q} u(x) \right]_{x=a} \\ =& \frac{1}{\Gamma_{q^p}(k\alpha + \gamma)} \left[{}^{p}\mathcal{D}^{\gamma-1}_{a^+,q} {}^{p}\mathcal{D}^{(k+1)\alpha,\beta}_{a^+,q} u(x) \right]_{x=a} = \frac{\Gamma_{q^p}(k\alpha + \alpha + \gamma)}{\Gamma_{q^p}(k\alpha + \gamma)} {}^{p}_{q}U_{\alpha,\beta}(k+1) \end{aligned} $$

The next finding is highly helpful in solving differential equations including fractional derivatives of order δ.

Theorem 7.2. *If* $u(x)$ *satisfies the conditions stated in law of exponents [Theorem 5.1] and* $\phi(x) = {}^{p}\mathcal{D}^{\delta,\beta}_{a^+,q}u(x)$, *then*

$$ {}^{p}_{q}\Phi_{\alpha,\beta}(k) = \frac{\Gamma_{q^p}(k\alpha + \delta + \gamma)}{\Gamma_{q^p}(k\alpha + \gamma)} {}^{p}_{q}U_{\alpha,\beta}(k + \delta/\alpha) \tag{7.3} $$

8. Applications

In this part, we will use the HK generalized q-differential transform to solve certain fractional q-difference equations incorporating HK fractional q-derivative for ${}^{p}\mathcal{D}^{k\alpha,\beta}_{a^+,q}y \in L^{1}_{q,p}[a,b]$, ${}^{p}\mathcal{D}^{\gamma}_{a^+,q} {}^{p}\mathcal{D}^{k\alpha,\beta}_{a^+,q}y \in AC^{n}_{p,q}[a,b]$, $\gamma = \alpha + \beta(1-\alpha)$, and $0 < |q| < 1$, $p > 0$.

Problem 1. For $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, we consider the following fractional q-initial value problem

$$ {}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q}y(x) - \lambda y(x) = 0, \lambda \in R, \tag{8.1} $$

with initial condition

$$ {}^{p}\mathcal{D}^{\gamma-1}_{a^+,q} y(a) = y_0. \tag{8.2}$$

Solution to the problem is provided by

$$ y(x) = y_0 \frac{\left(x^p - a^p\right)^{(\gamma-1)}_{q^p}}{\left([p]_q\right)^{\gamma-1}} {}_{q^p}E_{\alpha,\gamma}\left[\frac{\lambda}{\left([p]_q\right)^{\alpha}}, \left(x^p - a^p q^{p(\gamma-1)}\right)\right]. \tag{8.3}$$

Solution. By applying the generalized $q-$differential transform (7.1) to both sides of (8.1) and (8.2), and then utilizing the results outlined in the Theorem 7.1, we are able to derive

$$ \frac{\Gamma_{q^p}(k\alpha+\alpha+\gamma)}{\Gamma_{q^p}(k\alpha+\gamma)} {}^{p}_{q}Y_{\alpha,\beta}(k+1) - \lambda {}^{p}_{q}Y_{\alpha,\beta}(k) = 0 \tag{8.4}$$

and

$$ {}^{p}_{q}Y_{\alpha,\beta}(0) = \frac{1}{\Gamma_{q^p}(\gamma)} y_0 \tag{8.5}$$

We have obtained the following values of ${}^{p}_{q}Y_{\alpha,\beta}(k)$ by the application of recurrence relation (8.4) and transformed initial condition (8.5)

$$ {}^{p}_{q}Y_{\alpha,\beta}(1) = \lambda\frac{1}{\Gamma_{q^p}(\alpha+\gamma)} y_0, {}^{p}_{q}Y_{\alpha,\beta}(2) = \lambda^2\frac{1}{\Gamma_{q^p}(2\alpha+\gamma)} y_0, {}^{p}_{q}Y_{\alpha,\beta}(3) = \lambda^3\frac{1}{\Gamma_{q^p}(3\alpha+\gamma)} y_0$$

and so on.

In view of inverse HK generalized q-differential transform defined by (7.2) and using the values of ${}^{p}_{q}Y_{\alpha,\beta}(k)$, we get

$$ y(x) = y_0 \frac{\left(x^p - a^p\right)^{(\gamma-1)}_{q^p}}{\left([p]_q\right)^{\gamma-1}} \left\{ \frac{1}{\Gamma_{q^p}(\gamma)} + \frac{\lambda}{\Gamma_{q^p}(\alpha+\gamma)}\left[\frac{\left(x^p - a^p q^{p(\gamma-1)}\right)^{(\alpha)}_{q^p}}{\left([p]_q\right)^{\alpha}}\right] \right.$$
$$ \left. + \frac{\lambda^2}{\Gamma_{q^p}(2\alpha+\gamma)}\left[\frac{\left(x^p - a^p q^{p(\gamma-1)}\right)^{(2\alpha)}_{q^p}}{\left([p]_q\right)^{2\alpha}}\right] + \frac{\lambda^3}{\Gamma_{q^p}(3\alpha+\gamma)}\left[\frac{\left(x^p - a^p q^{p(\gamma-1)}\right)^{(3\alpha)}_{q^p}}{\left([p]_q\right)^{3\alpha}}\right] + \cdots \right\}$$

Which in view of the definition (3.2) of q^p-Mittag-Leffler function gives (8.3) as the solution.

Problem 2. Next, for $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, we consider the following fractional q-initial value problem

$$ {}^{p}\mathcal{D}^{\alpha,\beta}_{a^+,q} y(x) - q^p y(x) = \frac{\left(x^p - a^p\right)^{(\gamma-1)}_{q^p}}{\left([p]_q\right)^{\gamma-1}} {}_{q^p}E_{\alpha}\left[\frac{1}{\left([p]_q\right)^{\alpha}}, \left(x^p - a^p q^{p(\gamma-1)}\right)\right]. \tag{8.6}$$

with initial condition

$$ {}^{p}\mathcal{D}^{\gamma-1}_{a^+,q}y(a) = y_0. \tag{8.7}$$

Solution to the problem is provided by

$$\begin{aligned} y(x) =& y_0\frac{\left(x^p-a^p\right)^{(\gamma-1)}_{q^p}}{\left([p]_q\right)^{\gamma-1}}{}_{q^p}E_{\alpha,\gamma}\left[\frac{q^p}{\left([p]_q\right)^{\alpha}},\left(x^p-a^pq^{p(\gamma-1)}\right)\right]\\ &+\frac{\left(x^p-a^p\right)^{(\alpha+\gamma-1)}_{q^p}}{\left([p]_q\right)^{\alpha+\gamma-1}}{}_{q^p}E^2_{\alpha,\alpha+\gamma}\left[\frac{1}{(1-q)\left([p]_q\right)^{\alpha+1}},\left(x^p-a^pq^{p(\alpha+\gamma-1)}\right)\right]. \end{aligned}\tag{8.8}$$

Solution. With the help of the results listed in the Theorem 7.1 and generalized q-differential transform (7.1) applied to both sides of (8.6), (8.7), we get

$$\frac{\Gamma_{q^p}(k\alpha+\alpha+\gamma)}{\Gamma_{q^p}(k\alpha+\gamma)}{}^{p}_{q}Y_{\alpha,\beta}(k+1)-q^{p}{}^{p}_{q}Y_{\alpha,\beta}(k)=\frac{1}{\Gamma_{q^p}(\alpha k+\gamma)}. \tag{8.9}$$

and

$${}^{p}_{q}Y_{\alpha,\beta}(0)=\frac{1}{\Gamma_{q^p}(\gamma)}y_0 \tag{8.10}$$

By making use of recurrence relation (8.9) and transformed initial condition (8.10), we have

$$\begin{aligned} {}^{p}_{q}Y_{\alpha,\beta}(1) &= \frac{1}{\Gamma_{q^p}(\alpha+\gamma)}+\frac{q^p}{\Gamma_{q^p}(\alpha+\gamma)}y_0,\\ {}^{p}_{q}Y_{\alpha,\beta}(2) &= \frac{[2]_{q^p}}{\Gamma_{q^p}(\alpha+(\alpha+\gamma))}+\frac{q^{2p}}{\Gamma_{q^p}(2\alpha+\gamma)}y_0,\\ {}^{p}_{q}Y_{\alpha,\beta}(3) &= \frac{[3]_{q^p}}{\Gamma_{q^p}(2\alpha+(\alpha+\gamma))}+\frac{q^{3p}}{\Gamma_{q^p}(3\alpha+\gamma)}y_0 \end{aligned}$$

and so on. Therefore

$${}^{p}_{q}Y_{\alpha,\beta}(k+1)=\frac{[k+1]_{q^p}}{\Gamma_{q^p}(k\alpha+(\alpha+\gamma))}+\frac{q^{k+1}}{\Gamma_{q^p}((k+1)\alpha+\gamma)}y_0,\quad k=0,1,2,3,\cdots.$$

In view of inverse HK generalized q-differential transform defined by (7.2) and using the values of ${}^{p}_{q}Y_{\alpha,\beta}(k)$, we get

$$\begin{aligned} y(x) =& \frac{\left(x^p-a^p\right)^{(\gamma-1)}_{q^p}}{\left([p]_q\right)^{\gamma-1}}y_0\left\{\frac{1}{\Gamma_{q^p}(\gamma)}+\frac{q^p}{\Gamma_{q^p}(\alpha+\gamma)}\frac{\left(x^p-a^pq^{p(\gamma-1)}\right)^{(\alpha)}_{q^p}}{\left([p]_q\right)^{\alpha}}+\cdots\right\}\\ &+\frac{\left(x^p-a^p\right)^{(\gamma-1)}_{q^p}}{\left([p]_q\right)^{\gamma-1}}\left\{\frac{[1]_{q^p}}{\Gamma_{q^p}(\alpha+\gamma)}\frac{\left(x^p-a^pq^{p(\gamma-1)}\right)^{(\alpha)}_{q^p}}{\left([p]_q\right)^{\alpha}}\right. \end{aligned}$$

$$+\frac{[2]_{q^p}}{\Gamma_{q^p}(\alpha+\alpha+\gamma)}\frac{\left(x^p-a^pq^{p(\gamma-1)}\right)_{q^p}^{(2\alpha)}}{\left([p]_q\right)^{2\alpha}}+\cdots.\Bigg\}$$

$$y(x)=y_0\frac{\left(x^p-a^p\right)_{q^p}^{(\gamma-1)}}{\left([p]_q\right)^{\gamma-1}}\sum_{k=0}^{\infty}\frac{q^{kp}}{\Gamma_{q^p}(k\alpha+\alpha+\gamma)}\frac{\left(x^p-a^pq^{p(\gamma-1)}\right)_{q^p}^{(k\alpha)}}{\left([p]_q\right)^{k\alpha}}$$
$$+\frac{\left(x^p-a^p\right)_{q^p}^{(\alpha+\gamma-1)}}{\left([p]_q\right)^{\alpha+\gamma-1}}\sum_{k=0}^{\infty}\frac{[k+1]_{q^p}}{\Gamma_{q^p}(k\alpha+\alpha+\gamma)}\frac{\left(x^p-a^pq^{p(\alpha+\gamma-1)}\right)_{q^p}^{(k\alpha)}}{\left([p]_q\right)^{k\alpha}}.$$

Which in view of the definitions (3.2) and (3.4) of q^p-Mittag-Leffler functions gives (8.8) as the solution.

Problem 3. For $1<2\alpha\leq 2$, $0\leq\beta\leq 1$, we take the following q-initial value problem

$$ {}^p\mathcal{D}_{a^+,q}^{2\alpha,\beta}y(x)-(1+q^p)\,{}^p\mathcal{D}_{a^+,q}^{\alpha,\beta}y(x)+q^py(x)=\frac{\left(x^p-a^p\right)_{q^p}^{(n\alpha+\gamma-1)}}{\left(\left[p\right]_q\right)^{n\alpha+\gamma-1}}. \tag{8.11}$$

with initial conditions

$$ {}^p\mathcal{D}_{a^+,q}^{\gamma-1}y(a)=y_0,\quad {}^p\mathcal{D}_{a^+,q}^{\alpha+\gamma-1}y(a)=y_1 \tag{8.12}$$

Solution to the problem is provided by

$$\begin{aligned}y(x)&=y_0\frac{\left(x^p-a^p\right)_{q^p}^{(\gamma-1)}}{\left([p]_q\right)^{\gamma-1}}\frac{1}{\Gamma_{q^p}(\gamma)}\\
&-y_0\frac{q^p\left(x^p-a^p\right)_{q^p}^{(2\alpha+\gamma-1)}}{\left([p]_q\right)^{2\alpha+\gamma-1}}\frac{1}{\Gamma_{q^p}(\gamma)}\,{}_{q^p}E_{\alpha,2\alpha+\gamma}^2\left[\frac{1}{(1-q)(\left[p\right]_q)^{\alpha+1}},\left(x^p-a^pq^{p(2\alpha+\gamma-1)}\right)\right]\\
&+y_1\frac{\left(x^p-a^p\right)_{q^p}^{(\alpha+\gamma-1)}}{\left([p]_q\right)^{\alpha+\gamma-1}}\,{}_{q^p}E_{\alpha,\alpha+\alpha+\gamma}^2\left[\frac{1}{(1-q)(\left[p\right]_q)^{\alpha+1}},\left(x^p-a^pq^{p(\alpha+\gamma-1)}\right)\right]\\
&+\frac{\Gamma_{q^p}\left(n\alpha+\gamma\right)}{\Gamma_{q^p}\left(n\alpha+\alpha+\gamma\right)}\left[\frac{\left(x^p-a^p\right)_{q^p}^{((n+1)\alpha+\gamma-1)}}{\left(\left[p\right]_q\right)^{(n+1)\alpha+\gamma-1}}\right]\end{aligned} \tag{8.13}$$

Solution. With the help of Theorem 7.2 and generalized q-differential transform (7.1) applied to both sides of (8.11), (8.12) we get

$$\frac{\Gamma_{q^p}(k\alpha+2\alpha+\gamma)}{\Gamma_{q^p}(k\alpha+\gamma)} {}_q^pY_{\alpha,\beta}(k+2)$$
$$= \left(1+q^p\right)\frac{\Gamma_{q^p}(k\alpha+\alpha+\gamma)}{\Gamma_{q^p}(k\alpha+\gamma)} {}_q^pY_{\alpha,\beta}(k+1) - q^p {}_q^pY_{\alpha,\beta}(k) + \delta(k-n) \quad (8.14)$$

and

$${}_q^pY_{\alpha,\beta}(0) = \frac{1}{\Gamma_{q^p}(\gamma)} y_0, \quad {}_q^pY_{\alpha,\beta}(1) = \frac{1}{\Gamma_{q^p}(\alpha+\gamma)} y_1 \quad (8.15)$$

By making use of recurrence relation (8.14) and transformed initial conditions (8.15), we have

$${}_q^pY_{\alpha,\beta}(2) = \frac{[2]_{q^p}}{\Gamma_{q^p}(2\alpha+\gamma)} y_1 - \frac{q^p[1]_{q^p}}{\Gamma_{q^p}(2\alpha+\gamma)} y_0,$$
$${}_q^pY_{\alpha,\beta}(3) = \frac{[3]_{q^p}}{\Gamma_{q^p}(3\alpha+\gamma)} y_1 - \frac{q^p[2]_{q^p}}{\Gamma_{q^p}(3\alpha+\gamma)} y_0$$
$${}_q^pY_{\alpha,\beta}(4) = \frac{[4]_{q^p}}{\Gamma_{q^p}(4\alpha+\gamma)} y_1 - \frac{q^p[3]_{q^p}}{\Gamma_{q^p}(4\alpha+\gamma)} y_0$$

and so on. Which gives

$${}_q^pY_{\alpha,\beta}(k+2) = \frac{[k+2]_{q^p}}{\Gamma_{q^p}((k+2)\alpha+\gamma)} y_1 - \frac{q^p[k+1]_{q^p}}{\Gamma_{q^p}((k+2)\alpha+\gamma)} y_0, \ \ k = 0,1,2,\cdots, k \neq n.$$
$${}_q^pY_{\alpha,\beta}(n+2) = \frac{[n+2]_{q^p}}{\Gamma_{q^p}((n+2)\alpha+\gamma)} y_1 - \frac{q^p[n+1]_{q^p}}{\Gamma_{q^p}((n+2)\alpha+\gamma)} y_0 + \frac{\Gamma_{q^p}(n\alpha+\gamma)}{\Gamma_{q^p}(n\alpha+\alpha+\gamma)}.$$

Putting these values of ${}_q^pY_{\alpha,\beta}(k)$ in (7.2), we have

$$y(x) = y_0 \frac{\left(x^p-a^p\right)_{q^p}^{(\gamma-1)}}{\left([p]_q\right)^{\gamma-1}} \left\{ \frac{1}{\Gamma_{q^p}(\gamma)} - \frac{q^p[1]_{q^p}}{\Gamma_{q^p}(2\alpha+\gamma)} \left[\frac{\left(x^p - a^p q^{p(\gamma-1)}\right)_{q^p}^{(2\alpha)}}{\left([p]_q\right)^{2\alpha}} \right] \right.$$
$$+ y_1 \frac{\left(x^p-a^p\right)_{q^p}^{(\gamma-1)}}{\left([p]_q\right)^{\gamma-1}} \left\{ \frac{1}{\Gamma_{q^p}(\alpha+\gamma)} \left[\frac{\left(x^p - a^p q^{p(\gamma-1)}\right)_{q^p}^{(\alpha)}}{\left([p]_q\right)^{\alpha}} \right] + \frac{[2]_{q^p}}{\Gamma_{q^p}(2\alpha+\gamma)} \left[\frac{\left(x^p - a^p q^{p(\gamma-1)}\right)_{q^p}^{(2\alpha)}}{\left([p]_q\right)^{2\alpha}} \right] \right.$$
$$\left. + \frac{[3]_{q^p}}{\Gamma_{q^p}(3\alpha+\gamma)} \left[\frac{\left(x^p - a^p q^{p(\gamma-1)}\right)_{q^p}^{(3\alpha)}}{\left([p]_q\right)^{3\alpha}} \right] + \cdots \right\} + \frac{\Gamma_{q^p}\left(n\alpha+\gamma\right)}{\Gamma_{q^p}\left(n\alpha+\alpha+\gamma\right)} \left[\frac{\left(x^p-a^p\right)_{q^p}^{((n+1)\alpha+\gamma-1)}}{\left(\left[p\right]_q\right)^{(n+1)\alpha+\gamma-1}} \right]$$

In view of the q^p-Mittag-Leffler function, we arrive at (8.13).

Remark. *By allowing $p \to 1$ in Sections 4 and 6, we arrive to the identical results for the generalized composite fractional q-derivative $\mathcal{D}^{\alpha,\beta}_{a^+,q}$ as carried out in* [5].

9. Conclusion

In this study, we have introduced the Hilfer-Katugampola Fractional q-derivative ${}^p\mathcal{D}^{\alpha,\beta}_{a^\pm,q}$ of order α and type β in the function space $L^1_{q,p}[a,b]$. The operator ${}^p\mathcal{D}^{\alpha,\beta}_{a^\pm,q}$ serves as a q-extension the Hilfer-Katugampola fractional derivative initially defined in [22]. Then, we have given the Hilfer-Katugampola fractional q-Taylor's formula involving the operator ${}^p\mathcal{D}^{\alpha,\beta}_{a^+,q}$. Also, generalized Hilfer-Katugampola fractional q-differential transform method has been developed and applied to solve three fractional q-difference equations.

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