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AN INVESTIGATION OF $\mathfrak{F}$-CLOSURE OF INTUITIONISTIC FUZZY SUBMODULES OF A MODULE

P. K. Sharma

Post-Graduate Department of Mathematics,
D. A. V. College, Jalandhar - 144008, Punjab, INDIA

E-mail : pksharma@davjalandhar.com

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Abstract: In this paper, we introduce the notion of $\mathfrak{F}$-closure of intuitionistic fuzzy submodules of a module M. Our attempt is to investigate various characteristics of such an $\mathfrak{F}$-closure. If $\mathfrak{F}$ is a non-empty set of intuitionistic fuzzy ideals of a commutative ring R and A is an intuitionistic fuzzy submodule of M, then the $\mathfrak{F}$-closure of A is denoted by $Cl_{\mathfrak{F}}^{M}(A)$. If $\mathfrak{F}$ is weak closed under intersection, then (1) $\mathfrak{F}$-closure of A exhibits the submodule character, and (2) the intersection of $\mathfrak{F}$-closure of two intuitionistic fuzzy submodules equals the $\mathfrak{F}$-closure of intersection of the intuitionistic fuzzy submodules. If $\mathfrak{F}$ is weak closed under intersection, then the submodule property of $\mathfrak{F}$-closure implies that $\mathfrak{F}$ is closed. Moreover, if $\mathfrak{F}$ is inductive, then $\mathfrak{F}$ is a topological filter if and only if $Cl_{\mathfrak{F}}^{M}(A)$ is an intuitionistic fuzzy submodule for any intuitionistic fuzzy submodule A of M.

Keywords and Phrases: Intuitionistic fuzzy ideals(submodules), $\mathfrak{F}$-closure, $\mathfrak{F}$-torsion, $\mathfrak{F}$-closed, topological filter.

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1. Introduction

In several branches of mathematics, closure operators have been extremely important. The closure operators T-closed and T-honest, which have been researched by Fay and Joubert [7], are two examples of the various closure operators that can be used for categories of modules. When studying different facets of rings and modules, these operators are helpful. Abian and Rinehart [1], created the theory of honest subgroups. Researchers Fay, Joubert, and Jara [7], [9] have examined the ideas of isolated submodules and honest submodules respectively. Honest submodules and isolated submodules both have the same meaning for skew fields. A novel characterization of the ore domain results from the honest submodules. Additionally, according to the hypothesis put forth by Fay and Joubert, isolated subgroups are helpful in the investigation of torsion-free groups in terms of the category of groups. Joubert and Schoeman [10] developed the idea of super-honest submodules. Cheng [6] has researched the super-honest submodules of quasi-injective modules. Kashu in [11] has given a complete characterizations of closure operators in modules.

In this paper, an attempt has been made to extend the notion of honest and superhonest submodules to intuitionistic fuzzy submodules. The only a few of the terms like intuitionistic fuzzy honest submodules, intuitionistic fuzzy closure, intuitionistic fuzzy torsion, and intuitionistic fuzzy superhonest submodules are defined. In this study, various properties of honest and superhonest submodules are intuitionistically fuzzified.

2. Preliminaries

Throughout this paper R is a non commutative ring with unity and M is a R-module. The zero elements of R and M are 0 and θ, respectively.

Definition 2.1. ([2, 3, 4]) *An intuitionistic fuzzy set (IFS) A in X can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.*

Remark 2.2. ([3, 4])

(i) When $\mu_A(x) + \nu_A(x) = 1, \forall x \in X$. Then A is called a fuzzy set.

(ii) An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is briefly written as $A(x) = (\mu_A(x), \nu_A(x))$, $\forall x \in X$. We denote by $IFS(X)$ the set of all IFSs of X.

(iii) If $p, q \in [0, 1]$ such that $p + q \leq 1$. Then $A \in IFS(X)$ defined by $\mu_A(x) = p$ and $\nu_A(x) = q$, for all $x \in X$, is called a constant intuitionistic fuzzy set of X. Any IFS of X defined other than this is referred to as a non-constant intuitionistic fuzzy set.

If $A, B \in IFS(X)$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$. For any subset Y of X, the intuitionistic fuzzy characteristic function χ_Y is an intuitionistic fuzzy set of X, defined as $\chi_Y(x) = (1, 0), \forall x \in Y$ and $\chi_Y(x) = (0, 1), \forall x \in X \backslash Y$. Let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then the crisp set $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$ is called the (α, β)-level cut subset of A. Moreover, the set $\{x \in X : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}$ is called the support of the IFS A and is denoted by A^*. Also the IFS $x_{(\alpha,\beta)}$ of X defined as $x_{(\alpha,\beta)}(y) = (\alpha, \beta)$, if $y = x$, otherwise $(0, 1)$ is called the intuitionistic fuzzy point (IFP) in X with support x. By $x_{(\alpha,\beta)} \in A$ we mean $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$.

Definition 2.3. ([5, 15]) *Let $A \in IFS(R)$. Then A is called an intuitionistic fuzzy ideal (IFI) of ring R, if for all $x, y \in R$, the followings are satisfied*

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$; *(ii)* $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$;

(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$; *(iv)*$\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Note that $\mu_A(0_R) \geq \mu_A(x) \geq \mu_A(1_R), \mu_A(0_R) \leq \mu_A(x) \leq \nu_A(1_R), \forall x \in R$. The set of all intuitionistic fuzzy ideals of R is denoted by $IFI(R)$.

Definition 2.4. ([5, 8, 13, 14, 15]) *Let $A \in IFS(M)$. Then A is called an intuitionistic fuzzy submodule ($IFSM$) of M if for all $x, y \in M, r \in R$, the followings are satisfied*

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ *(ii)* $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$,

(iii) $\mu_A(xr) \geq \mu_A(x)$ *(iv)* $\nu_A(xr) \leq \nu_A(x)$,

(v) $\mu_A(\theta) = 1$ *(vi)* $\nu_A(\theta) = 0$.

Clearly, $\chi_{\{\theta\}}$, χ_M are IFSMs of M and these are called trivial IFSMs of M. Any IFSM of M other than these is called non-trivial proper IFSM of M. Let $IFSM(M)$ denote the set of all intuitionistic fuzzy R-submodules of M and $IFI(R)$ denote the set of all intuitionistic fuzzy ideals of R. We note that when $R = M$, then $A \in IFSM(M)$ if and only if $\mu_A(\theta) = 1, \nu_A(\theta) = 0$ and $A \in IFI(R)$.

Definition 2.5. ([15, 16]) *Let $A \in IFSM(M)$ and $C \in IFI(R)$. Then the intuitionistic fuzzy product AC of A and C is defined as: For all $x \in M$*

$$(\mu_{AC}(x), \nu_{AC}(x)) = \begin{cases} (Sup_{x=mr}(\mu_A(m) \wedge \mu_C(r)), Inf_{x=mr}(\nu_A(m) \vee \nu_C(r)), & \text{if } x = mr \\ (0, 1), & \text{otherwise} \end{cases}$$

Remark 2.6. ([15, 17]) *Let M be an R-module. Then for any $x_{(p,q)} \in IFP(M), y_{(t,s)} \in IFP(R)$, we have $x_{(p,q)}y_{(t,s)} = (xy)_{(p \wedge t, q \vee s)}$.*

Theorem 2.7. ([5, 15]) *Let $A \in IFS(R)$. Then A is an intuitionistic fuzzy ideal* *if and only if* $A_{(\alpha,\beta)}$ *is an ideal of* R*, for all* $\alpha \leq \mu_A(0), \beta \geq \nu_A(0)$ *with* $\alpha + \beta \leq 1$.

Definition 2.8. ([12, 17]) *An non-zero IFI* A *of a ring* R *is called an intuitionistic fuzzy essential ideal of* R*, denoted by* $A \subseteq_e R$*, if for every nonzero IFI* B *of* R*,* $A \cap B \neq \chi_0$*, i.e., there exist* $0 \neq x \in R$ *such that* $x_{(s,t)} \in A \cap B$*, for some* $s, t \in (0, 1]$ *such that* $s + t \leq 1$.

Definition 2.9. ([12, 17]) *Let* A, B *be two non-zero IFIs of a ring* R *such that* $A \subseteq B$*. Then* A *is called an intuitionistic fuzzy essential ideal in* B*, denoted by* $A \subseteq_e B$*, if for every non-zero IFI* C *of* R *satisfying* $C \subseteq B$*,* $A \cap C \neq \chi_0$*, i.e., there exist* $0 \neq x \in R$ *such that* $x_{(s,t)} \in A \cap C$*, for some* $s, t \in (0, 1]$ *such that* $s + t \leq 1$.

Lemma 2.10. ([12, 17]) *An IFI* A *of a ring* R *is an intuitionistic fuzzy essential ideal of* R*, if and only if* A *is an intuitionistic fuzzy essential ideal of* χ_R.

Theorem 2.11. ([12, 17]) *Let* A, B *and* C *be non-zero IFIs of ring* R *such that* $A \subseteq B \subseteq C$*. Then* $A \subseteq_e C$ *if and only if* $A \subseteq_e B \subseteq_e C$.

Definition 2.12. ([15]) *For* $A, B \in IFS(M)$ *and* $C \in IFS(R)$*, define the residual quotient* $(A : B)$ *and* $(A : C)$ *as follows:* $(A : B) = \bigcup\{D : D \in IFS(R)$ *such that* $D \cdot B \subseteq A\}$ *and* $(A : C) = \bigcup\{E : E \in IFS(M)$ *such that* $C \cdot E \subseteq A\}$*. Clearly,* $(A : B) \in IFS(R)$ *and* $(A : C) \in IFS(M)$*. Further, when* $A, B \in IFSM(M)$ *and* $C \in IFI(R)$*, then* $(A : B) \in IFI(R)$ *and* $(A : C) \in IFSM(M)$.

Definition 2.13. ([15]) *For* $A, B \in IFS(M)$ *and* $C \in IFS(R)$*. Then we have*

(i) $(A : B) \cdot B \subseteq A$;

(ii) $C \cdot (A : C) \subseteq A$;

(iii) $C \cdot B \subseteq A \Leftrightarrow C \subseteq (A : B) \Leftrightarrow B \subseteq (A : C)$.

Definition 2.14. ([15]) *Suppose* A *and* B *be two IFSMs of an R-module* M*. Then* $(A : B) = \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in (0, 1]$ *with* $\alpha + \beta \leq 1, r_{(\alpha,\beta)}B \subseteq A\}$ *is an IFI in* R.

Definition 2.15. ([15]) *Let* M *be a R-module and* $A \in IFS(M)$*, then the annihilator of* A *is denoted by* $Ann(A)$ *and is defined as:*

$$Ann(A) = \bigcup\{B : B \in IFS(R) \text{ such that } BA \subseteq \chi_\theta\}$$

Note that $(\chi_\theta : A) = Ann(A)$.

Lemma 2.16. ([15]) *Let* M *be a R-module, then* $Ann(\chi_\theta) = \chi_R$.

3. $\mathfrak{F}$-closure of Intuitionistic Fuzzy Submodules

In this section let $\mathfrak{F}$ be a non empty set of intuitionistic fuzzy ideals of R.

Definition 3.1. *Let M be an R-module and A be an IFSM of M. Then we define an intuitionistic fuzzy torsion of A as follows:*

$$T(A) = \bigcup\{B : B \in IFS(M), B \subseteq A, BC \subseteq \chi_\theta, \text{ for some } C \in IFI(R)\}.$$

Also we define the $\mathfrak{F}$-torsion of A as follows:

$$T_{\mathfrak{F}}^{M}(A) = \bigcup\{B : B \in IFS(M), B \subseteq A, \text{ there is } C \in \mathfrak{F} \text{ such that } BC \subseteq \chi_\theta\}.$$

χ_M *is $\mathfrak{F}$-torsion if* $T_{\mathfrak{F}}^{M}(\chi_M) = \chi_M$ *and $\mathfrak{F}$-torsion free if* $T_{\mathfrak{F}}^{M}(\chi_M) = \chi_\theta$

Definition 3.2. *Let M be an R-module and A be an IFSM of M. Then we define the intuitionistic fuzzy closure of A as follows:*

$$Cl(A) = \bigcup\{B : B \in IFS(M), B \subseteq A, BC \subseteq A, \text{ for some } C \in IFI(R)\}.$$

Also we define the $\mathfrak{F}$-closure of A as follows:

$$Cl_{\mathfrak{F}}^{M}(A) = \bigcup\{B : B \in IFS(M), B \subseteq A, \text{ there is } C \in \mathfrak{F} \text{ such that } BC \subseteq A\}.$$

Lemma 3.3. *Let M be an R-module and A be an IFSM of M. Then*

$$T(A) = \bigcup\{m_{(\alpha,\beta)} : m_{(\alpha,\beta)} \in A, m_{(\alpha,\beta)}C \subseteq \chi_\theta, \text{ for some } C \in IFI(R)\}.$$

Proof. Clearly, $\{m_{(\alpha,\beta)} : m_{(\alpha,\beta)} \in A, m_{(\alpha,\beta)}C \subseteq \chi_\theta, \text{ for some } C \in IFI(R)\}$
$\subseteq \{B : B \in IFS(M), B \subseteq A, \text{ there is } C \in \mathfrak{F} \text{ such that } BC \subseteq \chi_\theta\}$.
Therefore, $\bigcup\{m_{(\alpha,\beta)} : m_{(\alpha,\beta)} \in A, m_{(\alpha,\beta)}C \subseteq \chi_\theta, \text{ for some } C \in IFI(R)\}$
$\subseteq \bigcup\{B : B \in IFS(M), B \subseteq A, \text{ there is } C \in \mathfrak{F} \text{ such that } BC \subseteq \chi_\theta\} = T(A)$.
Let $B \in IFS(M)$ such that $BC \subseteq \chi_\theta$ for some $C \in IFI(R)$. Let $m \in M$ such that $\mu_B(m) = \alpha, \nu_B(m) = \beta$.

$$\begin{aligned}
\mu_{m_{(\alpha,\beta)}C}(x) &= \vee\{m_{(\alpha,\beta)}(s) \wedge \mu_C(y) | x = sy; s \in M, y \in R\} \\
&= \vee\{\mu_B(m) \wedge \mu_C(y) | x = my; m \in M, y \in R\} \\
&\leq \vee\{\mu_B(s) \wedge \mu_C(y) | x = sy; s \in M, y \in R\} \\
&= \mu_{BC}(x) \\
&\leq \mu_{\chi_\theta}(x).
\end{aligned}$$

i.e., $\mu_{m_{(\alpha,\beta)}C}(x) \leq \mu_{\chi_\theta}(x)$. Similarly, we can show that $\nu_{m_{(\alpha,\beta)}C}(x) \geq \nu_{\chi_\theta}(x)$.
Thus $m_{(\alpha,\beta)}C \subseteq \chi_\theta$. Therefore $T(A) \subseteq \bigcup\{m_{(\alpha,\beta)} : m_{(\alpha,\beta)} \in A, m_{(\alpha,\beta)}C \subseteq \chi_\theta,$ for some $C \in IFI(R)\}$. Hence the proof of the lemma completed.

The proofs of the following Lemmas are similar

Lemma 3.4. *Let M be an R-module and A be an IFSM of M. Then*

$$T^M_{\mathfrak{F}}(A) = \bigcup\{m_{(\alpha,\beta)} : m_{(\alpha,\beta)} \in A,\ \text{there is } C \in \mathfrak{F} \text{ such that } m_{(\alpha,\beta)}C \subseteq \chi_\theta\}.$$

Lemma 3.5. *Let M be an R-module and A be an IFSM of M. Then*

$$Cl(A) = \bigcup\{m_{(\alpha,\beta)} : m_{(\alpha,\beta)} \in A, m_{(\alpha,\beta)}C \subseteq A,\ \text{for some } C \in IFI(R)\}.$$

Lemma 3.6. *Let M be an R-module and A be an IFSM of M. Then*

$$Cl^M_{\mathfrak{F}}(A) = \bigcup\{m_{(\alpha,\beta)} : m_{(\alpha,\beta)} \in A,\ \text{there is } C \in \mathfrak{F} \text{ such that } m_{(\alpha,\beta)}C \subseteq A\}.$$

Definition 3.7. *Let M be an R-module and $A \in IFSM(M)$. We call A is $\mathfrak{F}$-closed if $Cl^M_{\mathfrak{F}}(A) = A$.*

Definition 3.8. *$\mathfrak{F}$ is called weak closed under intersection if for any $C_1, C_2 \in \mathfrak{F}$ there exists $C \in \mathfrak{F}$ such that $C \subseteq C_1 \cap C_2$.*

Definition 3.9. *$\mathfrak{F}$ is called inductive if for any $C_1 \in \mathfrak{F}$ and any ideal $C_2 \supseteq C_1$, we have $C_2 \in \mathfrak{F}$.*

Definition 3.10. *$\mathfrak{F}$ is called closed if for any $r_{(s,t)} \in IFP(R)$ and any $C_1 \in \mathfrak{F}$, there is $C_2 \in \mathfrak{F}$ such that $C_2 r_{(s,t)} \subseteq C_1$, i.e., $C_2 \subseteq (C_1 : r_{(s,t)})$.*

Definition 3.11. *$\mathfrak{F}$ is called a topological filter if it is closed under intersection, inductive and closed.*

Theorem 3.12. *If $\mathfrak{F}$ is the set of all intuitionistic fuzzy essential ideals of R, then $\mathfrak{F}$ is inductive.*

Proof. Let $C_1 \in \mathfrak{F}$. If C_2 is any intuitionistic fuzzy ideal of R such that $C_2 \supseteq C_1$. Then $C_1 \subseteq_e R$ and so $C_1 \subseteq_e \chi_R$. Thus from $C_1 \subseteq C_2 \subseteq \chi_R$, it follows that $C_2 \subseteq_e R$ and hence $C_2 \in \mathfrak{F}$.

Theorem 3.13. *Let M be an R-module.*

(a) If $\mathfrak{F}$ is weak closed under intersection, then for any $A \in IFSM(M)$ we have that $Cl^M_{\mathfrak{F}}(A)$ is an intuitionistic fuzzy submodule of M.

(b) If $\mathfrak{F}$ is weak closed under intersection if and only if $Cl^M_{\mathfrak{F}}(A_1) \cap Cl^M_{\mathfrak{F}}(A_2) = Cl^M_{\mathfrak{F}}(A_1 \cap A_2)$ for any two $A_1, A_2 \in IFSM(M)$.

(c) If $\mathfrak{F}$ is weak closed under intersection, then $\mathfrak{F}$ is closed if and only if $Cl^M_{\mathfrak{F}}(B)$ is an intuitionistic fuzzy submodule of M for any $B \in IFSM(M)$.

Proof. For (a) Let $m_1, m_2 \in M$. Now, $\mu_{Cl^M_{\mathfrak{F}}(A)}(m_1) \wedge \mu_{Cl^M_{\mathfrak{F}}(A)}(m_2)$
$= (\vee\{\mu_{B_1}(m_1) : B_1 \in IFS(M), B_1 \subseteq A,\ \text{there is } C_1 \in \mathfrak{F} \text{ such that } B_1C_1 \subseteq A\})$
$\wedge(\vee\{\mu_{B_2}(m_2) : B_1 \in IFS(M), B_2 \subseteq A,\ \text{there is } C_2 \in \mathfrak{F} \text{ such that } B_2C_2 \subseteq A\})$

$= \vee(\{\mu_{B_1}(m_1) \wedge \mu_{B_2}(m_2) : B_i \in IFS(M), B_i \subseteq A,$ there is $C_i \in \mathfrak{F}$ such that $B_iC_i \subseteq A, i = 1, 2\})$
$\leq \{\mu_{(B_1+B_2)}(m_1) \wedge \mu_{(B_1+B_2)}(m_2) : B_i \in IFS(M), B_i \subseteq A,$ there is $C_i \in \mathfrak{F}$ such that $B_iC_i \subseteq A, i = 1, 2\}$
$\leq \{\mu_{(B_1+B_2)}(m_1 - m_2) : B_i \in IFS(M), B_i \subseteq A,$ there is $C_i \in \mathfrak{F}$ such that $B_iC_i \subseteq A, i = 1, 2\}$.

Since $\mathfrak{F}$ is weak closed, it follows that $C_1, C_2 \in \mathfrak{F}$ implies there is $C \in \mathfrak{F}$ such that $C \subseteq C_1 \cap C_2$. Therefore we have,

$$\begin{aligned}(B_1 + B_2)C &\subseteq (B_1 + B_2)(C_1 \cap C_2)\\ &\subseteq B_1(C_1 \cap C_2) + B_2(C_1 \cap C_2)\\ &\subseteq B_1C_1 + B_2C_2\\ &\subseteq A + A = A.\end{aligned}$$

Thus, $\mu_{Cl^M_{\mathfrak{F}}(A)}(m_1) \wedge \mu_{Cl^M_{\mathfrak{F}}(A)}(m_2) \leq \{\mu_{(B_1+B_2)}(m_1 - m_2) : (B_1 + B_2)C \subseteq A$ for some $C \in \mathfrak{F}\} = \mu_{Cl^M_{\mathfrak{F}}(A)}(m_1 - m_2)$.

Therefore, $\mu_{Cl^M_{\mathfrak{F}}(A)}(m_1) \wedge \mu_{Cl^M_{\mathfrak{F}}(A)}(m_2) \leq \mu_{Cl^M_{\mathfrak{F}}(A)}(m_1 - m_2)$. Similarly, we can show that $\nu_{Cl^M_{\mathfrak{F}}(A)}(m_1) \vee \nu_{Cl^M_{\mathfrak{F}}(A)}(m_2) \geq \nu_{Cl^M_{\mathfrak{F}}(A)}(m_1 - m_2)$. Also,
$\mu_{Cl^M_{\mathfrak{F}}(A)}(rm) = \vee\{\mu_B(rm) : B \in IFS(M), B \subseteq A,$ there is $C \in \mathfrak{F}$ s.t. $BC \subseteq A\}$
$\geq \vee\{\mu_B(m) : B \in IFS(M), B \subseteq A,$ there is $C \in \mathfrak{F}$ s.t. $BC \subseteq A\} = \mu_{Cl^M_{\mathfrak{F}}(A)}(m)$.
Similarly, we can show that $\nu_{Cl^M_{\mathfrak{F}}(A)}(rm) \leq \nu_{Cl^M_{\mathfrak{F}}(A)}(m)$.
Hence $Cl^M_{\mathfrak{F}}(A) \in IFSM(M)$.

For (b)($\Rightarrow$) For any $m \in M$, $\mu_{Cl^M_{\mathfrak{F}}(A_1 \cap A_2)}(m)$
$= \vee\{\mu_B(m) : B \in IFS(M), B \subseteq A_1 \cap A_2,$ there is $C \in \mathfrak{F}$ s.t. $BC \subseteq A_1 \cap A_2\}$
$= \vee\{\mu_B(m) \wedge \mu_B(m) : B \in IFS(M), B \subseteq A_1 \cap A_2,$ there is $C \in \mathfrak{F}$ s.t. $BC \subseteq A_1, BC \subseteq A_2\}$
$= (\vee\{\mu_B(m) : B \in IFS(M), B \subseteq A_1,$ there is $C \in \mathfrak{F}$ s.t. $BC \subseteq A_1\}) \wedge (\vee\{\mu_B(m) : B \in IFS(M), B \subseteq A_2,$ there is $C \in \mathfrak{F}$ s.t. $BC \subseteq A_2\})$.
$\leq \mu_{Cl^M_{\mathfrak{F}}(A_1)}(m) \wedge \mu_{Cl^M_{\mathfrak{F}}(A_2)}(m)$. Thus, $\mu_{Cl^M_{\mathfrak{F}}(A_1 \cap A_2)}(m) \leq \mu_{Cl^M_{\mathfrak{F}}(A_1)}(m) \wedge \mu_{Cl^M_{\mathfrak{F}}(A_2)}(m)$.
Similarly, we can show
$\nu_{Cl^M_{\mathfrak{F}}(A_1 \cap A_2)}(m) \geq \nu_{Cl^M_{\mathfrak{F}}(A_1)}(m) \vee \nu_{Cl^M_{\mathfrak{F}}(A_2)}(m)$. Thus we have
$Cl^M_{\mathfrak{F}}(A_1 \cap A_2) \subseteq Cl^M_{\mathfrak{F}}(A_1) \cap Cl^M_{\mathfrak{F}}(A_2)$. Also,

$\mu_{Cl^M_{\mathfrak{F}}(A_1) \cap Cl^M_{\mathfrak{F}}(A_2)}(m) = \mu_{Cl^M_{\mathfrak{F}}(A_1)}(m) \wedge \mu_{Cl^M_{\mathfrak{F}}(A_2)}(m)$
$= (\vee\{\mu_{B_1}(m) : B_1 \in IFS(M), B_1 \subseteq A_1,$ there is $C_1 \in \mathfrak{F}$ s.t. $B_1C_1 \subseteq A_1\})$
$\wedge(\vee\{\mu_{B_2}(m) : B_2 \in IFS(M), B_2 \subseteq A_2,$ there is $C \in \mathfrak{F}$ s.t. $B_2C_2 \subseteq A_2\})$
$= \vee\{\mu_{B_1}(m) \wedge \mu_{B_2}(m) : B_i \in IFS(M), B_i \subseteq A_i,$ there is $C_i \in \mathfrak{F}$ s.t. $B_iC_i \subseteq A_i, i = 1, 2\}$

$= \vee\{\mu_{B_1\cap B_2}(m) : B_i \in IFS(M), B_i \subseteq A_i,$ there is $C_i \in \mathfrak{F}$ s.t. $B_1C_1 \cap B_2C_2 \subseteq A_1 \cap A_2, i = 1, 2\}$. Now we have

$$\begin{aligned}(B_1 \cap B_2)(C_1 \cap C_2) &\subseteq B_1(C_1 \cap C_2) \cap B_2(C_1 \cap C_2)\\ &\subseteq B_1C_1 \cap B_2C_2\\ &\subseteq A_1 \cap A_2.\end{aligned}$$

Since $\mathfrak{F}$ is weak closed, for $C_1, C_2 \in \mathfrak{F}$ there exists $C \in \mathfrak{F}$ such that $C \subseteq B_1 \cap B_2$. Therefore, $(B_1 \cap B_2)C \subseteq (B_1 \cap B_2)(C_1 \cap C_2) \subseteq A_1 \cap A_2$. Thus we have $\mu_{Cl^M_{\mathfrak{F}}(A_1)\cap Cl^M_{\mathfrak{F}}(A_2)}(m) \leq \vee\{\mu_B(m) : B \in IFS(M), B \subseteq A_1 \cap A_2,$ there is $C \in \mathfrak{F}$ s.t. $BC \subseteq A_1 \cap A_2, i = 1, 2\} = \mu_{Cl^M_{\mathfrak{F}}(A_1\cap A_2)}(m)$. Thus, we have $\mu_{Cl^M_{\mathfrak{F}}(A_1)\cap Cl^M_{\mathfrak{F}}(A_2)}(m) \leq \mu_{Cl^M_{\mathfrak{F}}(A_1\cap A_2)}(m)$. Similarly, we can show that $\nu_{Cl^M_{\mathfrak{F}}(A_1)\cap Cl^M_{\mathfrak{F}}(A_2)}(m) \geq \nu_{Cl^M_{\mathfrak{F}}(A_1\cap A_2)}(m)$, i.e., $Cl^M_{\mathfrak{F}}(A_1) \cap Cl^M_{\mathfrak{F}}(A_2) \subseteq Cl^M_{\mathfrak{F}}(A_1 \cap A_2)$. Hence the result follows.

For (b)($\Leftarrow$) Let $C_1, C_2 \in \mathfrak{F}$, we have

$$\begin{aligned}\mu_{C_i1_{(\alpha,\beta)}}(x) &= \vee\{\mu_{C_i}(y) \wedge 1_{(\alpha,\beta)}(z) : x = yz, y, z \in R\}\\ &\leq \mu_{C_i}(x) \wedge \alpha\\ &\leq \mu_{C_i}(x).\end{aligned}$$

Thus, $\mu_{C_i1_{(\alpha,\beta)}}(x) \leq \mu_{C_i}(x)$. Similarly, we can show $\nu_{C_i1_{(\alpha,\beta)}}(x) \geq \nu_{C_i}(x)$. So we have, $C_i1_{(\alpha,\beta)} \subseteq C_i, i = 1, 2$ and therefore, $1_{(\alpha,\beta)} \in Cl^R_{\mathfrak{F}}(A_1) \cap Cl^R_{\mathfrak{F}}(A_2) = Cl^R_{\mathfrak{F}}(A_1 \cap A_2)$, hence there exist $C \in \mathfrak{F}$ such that $C \subseteq A_1 \cap A_2$.

For (c)($\Rightarrow$) Follows from part (a).

For (c)($\Leftarrow$) Let $C \in \mathfrak{F}$ and $r_{(s,t)} \in IFP(R)$, then $Cl^R_{\mathfrak{F}}(C) = \chi_R$, hence $r_{(s,t)} \in Cl^R_{\mathfrak{F}}(C)$ and therefore there is $D \in \mathfrak{F}$ such that $Dr_{(s,t)} \subseteq C$.

Theorem 3.14. *Let $\mathfrak{F}$ be an inductive set of intuitionistic fuzzy ideals, then the following statements are equivalent:*

(a) $\mathfrak{F}$ is a topological filter.

(b) $Cl^M_{\mathfrak{F}}(A)$ is an intuitionistic fuzzy submodule for any $A \in IFSM(M)$

Proof. (a)$\Rightarrow$ (b) $\mathfrak{F}$ closed under intersection, so weak closed under intersection. It is given that $\mathfrak{F}$ is closed, hence by part (c) of the above theorem the result follows.

(b)$\Rightarrow$ (a) $\mathfrak{F}$ is weak closed under intersection and closed as $Cl^M_{\mathfrak{F}}(A)$ is an intuitionistic fuzzy submodule for any $A \in IFSM(M)$. Since $\mathfrak{F}$ is inductive, therefore it is closed under intersection. Hence $\mathfrak{F}$ is a topological filter.

Definition 3.15. *Let M be an R-module. Then an intuitionistic fuzzy submodule A of M is said to $\mathfrak{F}$-closed in χ_M if for any intuitionistic fuzzy ideal $C \in \mathfrak{F}$ and any $m_{(\alpha,\beta)} \in IFP(M)$, $\chi_\theta \neq m_{(\alpha,\beta)}C \subseteq A$ implies $m_{(\alpha,\beta)} \in A$, $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$.*

Theorem 3.16. *If $A \subseteq B \subseteq \chi_M$. If A is $\mathfrak{F}$-closed in B and B is $\mathfrak{F}$-closed in χ_M, then A is $\mathfrak{F}$-closed in χ_M.*

Proof. Let $C \in \mathfrak{F}$ and $m_{(\alpha,\beta)} \in IFP(M)$ such that $\chi_\theta \neq m_{(\alpha,\beta)}C \subseteq A$. Thus $m_{(\alpha,\beta)}C \subseteq A \subseteq B$. Since B is $\mathfrak{F}$-closed in χ_M, it follows that $m_{(\alpha,\beta)} \in B$. Now, $m_{(\alpha,\beta)} \in B$ and $m_{(\alpha,\beta)}C \subseteq A \subseteq A$. Since A is $\mathfrak{F}$-closed in B, so it gives $m_{(\alpha,\beta)} \in A$. Hence A is $\mathfrak{F}$-closed in χ_M.

Theorem 3.17. *Let $A \in IFSM(M)$. If A is $\mathfrak{F}$-closed in χ_M and inductive then, for any $m_{(\alpha,\beta)} \in Cl^M_{\mathfrak{F}}(A)\backslash A$, we have $(A : m_{(\alpha,\beta)}) = Ann(m_{(\alpha,\beta)})$.*

Proof. Let A is $\mathfrak{F}$-closed in χ_M and inductive. Clearly, $Ann(m_{(\alpha,\beta)}) \subseteq (A : m_{(\alpha,\beta)})$.

Let $m_{(\alpha,\beta)} \in Cl^M_{\mathfrak{F}}(A)\backslash A$. Then there exists $C \in \mathfrak{F}$ such that $m_{(\alpha,\beta)}C \subseteq \chi_\theta$ and this implies $C \in Ann(m_{(\alpha,\beta)})$. Since $\mathfrak{F}$ is inductive, so we have $Ann(m_{(\alpha,\beta)}) \in \mathfrak{F}$. Now $Ann(m_{(\alpha,\beta)}) \subseteq (A : m_{(\alpha,\beta)})$ implies $(A : m_{(\alpha,\beta)}) \in \mathfrak{F}$.

Next let $r_{(s,t)} \in (A : m_{(\alpha,\beta)})$, then $r_{(s,t)}m_{(\alpha,\beta)} \subseteq A$, $\therefore (A : m_{(\alpha,\beta)})m_{(\alpha,\beta)} \subseteq A$ and thus $(A : m_{(\alpha,\beta)})m_{(\alpha,\beta)} \subseteq \chi_\theta$. This implies $(A : m_{(\alpha,\beta)}) \subseteq Ann(m_{(\alpha,\beta)})$.
Hence $(A : m_{(\alpha,\beta)}) = Ann(m_{(\alpha,\beta)})$.

Theorem 3.18. *Let $A \in IFSM(M)$. For any $m_{(\alpha,\beta)} \in Cl^M_{\mathfrak{F}}(A)\backslash A$, we have $(A : m_{(\alpha,\beta)}) = Ann(m_{(\alpha,\beta)})$, then $m_{(\alpha,\beta)}\chi_R \cap A = \chi_\theta$.*

Proof. We have

$$\begin{aligned}
\mu_{m_{(\alpha,\beta)}\chi_R}(x) &= \vee\{\mu_{m_{(\alpha,\beta)}}(y) \wedge \mu_{\chi_R}(z), y \in M, z \in R, x = yz\} \\
&= \vee\{\mu_{m_{(\alpha,\beta)}}(y), y \in M, z \in R, x = yz\} \\
&= \begin{cases} 0, & \text{if } x \notin mR \\ \alpha, & \text{if } x \in mR. \end{cases} \\
&= \mu_{(mR)_{(\alpha,\beta)}}(x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{m_{(\alpha,\beta)}\chi_R}(x) &= \wedge\{\nu_{m_{(\alpha,\beta)}}(y) \vee \nu_{\chi_R}(z), y \in M, z \in R, x = yz\} \\
&= \wedge\{\nu_{m_{(\alpha,\beta)}}(y), y \in M, z \in R, x = yz\} \\
&= \begin{cases} 1, & \text{if } x \notin mR \\ \beta, & \text{if } x \in mR. \end{cases} \\
&= \nu_{(mR)_{(\alpha,\beta)}}(x).
\end{aligned}$$

Then $(mR)_{(\alpha,\beta)}$ is an intuitionistic fuzzy submodule of M. Let $C = (A : m_{(\alpha,\beta)})$ and $L = \{r \in R : mr \in A^*\}$. Let $x \in C^*$, then there exists $p, q \in [0,1]$ such that $p + q \leq 1$ and $\mu_A(mx) \geq \alpha \wedge p > 0, \nu_A(mx) \leq \beta \vee q < 1$. Consequently, $mx \in A^*$ and thus $x \in L$. Hence $A^* \subseteq L$.

Again let $x \in L$, then $x \in A^*$ and this imply that $\mu_A(mx) \geq 0, \nu_A(mx) \leq 1$. Let $\mu_A(mx) = \alpha, \nu_A(mx) = \beta$, then $\mu_A(mx) \geq \alpha \wedge p, \nu_A(mx) \leq \beta \vee q$, which gives $(mx)_{(\alpha\wedge p,\beta\vee q)} \in A$. Thus $x_{(\alpha\wedge p,\beta\vee q)} \in C$, i.e., $\mu_C(x) \geq \alpha > 0, \nu_C(x) \leq \beta < 1$ therefore, $x \in C^*$, thus $L \subseteq C^*$. Hence $L = C^*$. Thus $(A : m_{(\alpha,\beta)})^* = L = \{r \in R : mr \in A^*\} = (A^* : m_{(\alpha,\beta)})$. Also, $[Ann(m_{(\alpha,\beta)})]^* = Ann(m_{(\alpha,\beta)})$.

Now, $[(mR)_{(\alpha,\beta)} \cap A]^* = \{x|\mu_{(mR)_{(\alpha,\beta)}}(x) > 0, \nu_{(mR)_{(\alpha,\beta)}}(x) < 1\} = \{x \in mR \text{ and } x \in A^*\} = mR \cap A^*$. By hypothesis we have, $(A : m_{(\alpha,\beta)}) = Ann(m_{(\alpha,\beta)})$, this implies $(A^* : m) = [Ann(m_{(\alpha,\beta)})]^* = Ann(m)$ and therefore $mR \cap A^* = 0$ so $[(mR)_{(\alpha,\beta)} \cap A^*] = 0$. Hence $m_{(\alpha,\beta)}\chi_R \cap A = \chi_\theta$.

Theorem 3.19. *Let $A \in IFSM(M)$. If for any $m_{(\alpha,\beta)} \in Cl^M_{\mathfrak{F}}(A)\backslash A$, we have $m_{(\alpha,\beta)}\chi_R \cap A = \chi_\theta$, then A is $\mathfrak{F}$-closed.*
Proof. Let $C \in \mathfrak{F}, m_{(\alpha,\beta)} \in IFP(M)$ such that $\chi_\theta \neq m_{(\alpha,\beta)}C \subseteq A$. If $m_{(\alpha,\beta)} \notin A$ then we have, by hypothesis $m_{(\alpha,\beta)}\chi_R \cap A = \chi_\theta$. Now $m_{(\alpha,\beta)}C \subseteq A$ implies $m_{(\alpha,\beta)}C \cap A = m_{(\alpha,\beta)}C \subseteq m_{(\alpha,\beta)}\chi_R$. Therefore $m_{(\alpha,\beta)}C = m_{(\alpha,\beta)} \cap A \subseteq m_{(\alpha,\beta)}\chi_R \cap A = \chi_\theta$, a contradiction. Hence the result follows.

As the consequences of the Theorems (3.17), (3.18), (3.19) we obtain the following:

Theorem 3.20. *Let $A \in IFSM(M)$ and $\mathfrak{F}$ is inductive. Then the following statements are equivalent:*

(a) A is $\mathfrak{F}$-closed in χ_M.

(b) For any $m_{(\alpha,\beta)} \in Cl^M_{\mathfrak{F}}(A)\backslash A$, we have $(A : m_{(\alpha,\beta)}) = Ann(m_{(\alpha,\beta)})$.

(c) For any $m_{(\alpha,\beta)} \in Cl^M_{\mathfrak{F}}(A)\backslash A$, we have $\chi_R \cap m_{(\alpha,\beta)} = \chi_\theta$.

Theorem 3.21. *Let $A \in IFSM(M)$ be an $\mathfrak{F}$-closed, then $Cl^M_{\mathfrak{F}}(A) = A \cup T^M_{\mathfrak{F}}(\chi_M)$.*
Proof. Clearly, $A \cup T^M_{\mathfrak{F}}(\chi_M) \subseteq Cl^M_{\mathfrak{F}}(A)$. Now let $m_{(\alpha,\beta)} \in Cl^M_{\mathfrak{F}}(A)\backslash A$, there exists $C \in \mathfrak{F}$ such that $m_{(\alpha,\beta)}C = \chi_\theta$, thus $m_{(\alpha,\beta)} \in T^M_{\mathfrak{F}}(\chi_M)$. This complete the proof.

4. Conclusion

In this paper, we have introduced the notion of $\mathfrak{F}$-closure of intuitionistic fuzzy submodules of a module M. We investigated various characteristics of such an $\mathfrak{F}$-closure. It is proved that if $\mathfrak{F}$ is weak closed under intersection, then $\mathfrak{F}$-closure of A is an intuitionistic fuzzy submodule of M. Further, the intersection of $\mathfrak{F}$-closure of two intuitionistic fuzzy submodules equals the $\mathfrak{F}$-closure of intersection of the intuitionistic fuzzy submodules. Also, if $\mathfrak{F}$ is weak closed under intersection, then the submodule property of $\mathfrak{F}$-closure asserts that $\mathfrak{F}$ is closed. Furthermore, we have shown that, if $\mathfrak{F}$ is inductive, then $\mathfrak{F}$ is a topological filter if and only if $Cl^M_{\mathfrak{F}}(A)$ is an intuitionistic fuzzy submodule of any intuitionistic fuzzy submodule A of M.

References

[1] Abian, A., Reinhart, D., Honest subgroups of abelian groups, Rend Circ Math Palermo., 12 (1963), 353-356.

[2] Atanassov, K. T., Intuitionistic fuzzy sets, In: Sgurev v(ed) vii ITKR′s session, Central Science and Technology Library of the Bulgarian Academy of Sci, Sofia, 1983.

[3] Atanassov, K. T., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1) (1986), 87-96.

[4] Atanassov, K. T., Intuitionistic Fuzzy Sets Theory and Applications, Studies on Fuzziness and Soft Computing, 35 (1999), Physica-Verlag, Heidelberg.

[5] Basnet, D. K., Topics in intuitionistic fuzzy algebra, Lambert Academic Publishing, 2011.

[6] Cheng, M. H., Notes on superhonest submodules, Chinese J. Math., 14 (2) (1986), 109-119.

[7] Fay, T. H., Joubert, S. V., Isolated submodules and skew fields, Appl Categorial Struct., 8 (2000), 317-326.

[8] Isaac, P. and John, P. P., On intuitionistic fuzzy submodules of a modules, International Journal of Mathematical Sciences and Applications, 1 (3) (2011), 1447-1454.

[9] Jara, P., Honest submodules, Czochoslovak Math J., 57 (132) (2007), 225-241.

[10] Jouvert, S. V., Schoeman, M. J., Superhonesty for modules and abelian groups, J Math., 12 (2) (1984), 87-95.

[11] Kashu A. I., Closure operators in modules and their characterizations, Bul. Acad. Stiinte Repub. Moldova, Mat., 1 (92) (2020), 31-62.

[12] Medhi, U., Saikia, H. K., On T-fuzzy essential ideals of rings, International Journal of Pure and Applied Mathematics, 89 (3) (2013), 343-352.

[13] Rahman, S. and Saikia, H. K., Some Aspects of Atanassov's Intuitionistic Fuzzy Submodules, International Journal of Pure and Applied Mathematics, 77 (3) (2013), 369-383.

[14] Sharma, P. K., (α, β)-cut of intuitionistic fuzzy modules, International Journal of Mathematical Sciences and Applications, 1 (3) (2011), 1489-1492.

[15] Sharma, P. K. and Kaur, G., Residual quotient and annihilator of intuitionistic fuzzy sets of ring and module, International Journal of Computer Science and Information Technology, 9 (4) (2017), 1-15.

[16] Sharma, P. K., intuitionistic fuzzification of essential submodule with respect to an arbitrary submodule, Annals of Fuzzy Mathematics and Informatics, 20 (3) (2020), 223-234.

[17] Sharma, P. K., Kanchan and Kaur, G., Intuitionistic L-fuzzy essential and closed submodules, Notes on Intuitionistic Fuzzy Sets, Vol. 27 (4) (2021), 44-54.