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FIXED POINT APPROXIMATION USING UNIFIED ITERATION SCHEME FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN $CAT(0)$ SPACES

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Abstract: In the present paper, we focus on reexamination of unified three step iteration scheme in more general infinite-dimensional manifolds i.e. in geodesic $CAT(0)$ spaces for asymptotically non-expansive mappings. The findings hold true for both asymptotically non-expansive type mappings and asymptotically quasi nonexpansive mappings. Since, numerous iteration schemes have been introducing for so long and also claimed new and different from other which shows huge lacking of existing iteration based literature. It is to be noted that there are several iteration schemes which are claimed to be different and unique but is special case of some existing scheme. Our results improve the existing iteration scheme based literature.

Keywords and Phrases: $CAT(0)$ spaces, asymptotically nonexpansive mappings, fixed points, unified iteration, $\Delta-$convergence, strong convergence.

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1. Introduction and Preliminaries

A metric space $(\mathfrak{C}, \mathfrak{d})$, that is geodesically connected with the property that every geodesic triangle in $\mathfrak{C}$ is at least as thin as its comparison triangle in the

Euclidean plane is a $CAT(0)$ space. If a metric space $(\mathfrak{C}, \mathfrak{d})$ is locally a $CAT(0)$ space, it is said to have *curvature* ≤ 0. In this case $(\mathfrak{C}, \mathfrak{d})$ is said to be *non-positively curved*. Any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space [13]. Other examples are R-tress, pre-Hilbert spaces (see [10]) and many others. For more discussion on $CAT(0)$ spaces, one can consult [3].

Kirk [14] poineered fixed point theory in $CAT(0)$ spaces and established some theorems for nonexpansive mappings. In 2008, Kirk and Panyanak [15] employed the Lim [18] concept of $\Delta-$convergence to show $CAT(0)$ space results that involve weak convergence under some appropriate conditions $\Delta-$convergence theorems for the Picard, Mann and Ishikawa iterations were also established for nonexpansive mappings by Dhompongsa and Panyanak [7]. Recent research by Nanjaras and Panyanak [20] extend Suzuki's results on fixed point theorems and convergence theorem for asymptotically nonexpansive mappings in $CAT(0)$ spaces. In this paper, inspired by the above results, we prove strong convergence theorems of the Noor iterative schemes for asymptotically nonexpansive mappings in the $CAT(0)$ space setting. Our findings extend and improve some results revealed in [7, 20, 28] and many others.

The following definitions are required:

Definition 1.1. *Let* $\mathfrak{C}$ *be a* $CAT(0)$ *space and* $\mathfrak{C}_s$ *be its nonempty subset. A mapping* $\wp : \mathfrak{C}_s \to \mathfrak{C}_s$ *is said to be asymptotically nonexpansive if there exists a sequence* $\{\kappa_{\mathfrak{n}}\}$ *of positive numbers with* $\lim_{\mathfrak{n}\to\infty} \kappa_{\mathfrak{n}} = 1$ *such that*

$$\mathfrak{d}(\wp^{\mathfrak{n}}(\mathfrak{m}^*), \wp^{\mathfrak{n}}(\mathfrak{j})) \leq \kappa_{\mathfrak{n}} \mathfrak{d}(\mathfrak{m}, \mathfrak{j}),$$

for all $n \geq 1$ *and* $\mathfrak{m}, \mathfrak{j} \in \mathfrak{C}$.

A point $\mathfrak{m} \in \mathfrak{C}$ is said to a fixed point of $\wp$ if $\mathfrak{m} = \wp\mathfrak{m}$. We will use $\mathfrak{P}_{\wp}$ to represent the collection of fixed points of $\wp$. Kirk [16] provided the proof that fixed point exists for asymptotically nonexpansive mappings in $CAT(0)$ spaces as the following statement.

Theorem 1.2. [16] *Let* $\mathfrak{C}_s$ *be a nonempty bounded closed and convex subset of a complete* $CAT(0)$ *space* $\mathfrak{C}$ *and* $\wp : \mathfrak{C}_s \to \mathfrak{C}_s$ *be asymptotically nonexpansive. Then* $\wp$ *has a fixed point.*

We gather some fundamental concepts and necessary outcomes in order to make our presentation self contained. A geodesic path joining $\mathfrak{m} \in \mathfrak{C}$ and $\mathfrak{j} \in \mathfrak{C}$ in a metric space $(\mathfrak{C}, \mathfrak{d})$, is a map $\alpha : [0, r] \subset R$ to $\mathfrak{C}$ such that $\alpha(0) = \mathfrak{m}, \alpha(r) = \mathfrak{j}$ and for all $s,\ t \in [0,\ r]$, $\mathfrak{d}(\alpha(s), \alpha(t)) = |s - t|$. Particularly, the mapping α is an isometry and $\mathfrak{d}(\mathfrak{m},\ \mathfrak{j}) = r$. For more details about geodesic path, see [3, 14, 15]. The image of α

is called a geodesic segment. Any two arbitrary points in the space $(\mathfrak{C},\ \mathfrak{d})$ that are connected by a geodesic path is said to be geodesic space, and if there is exactly one geodesic joining $\mathfrak{m}$ and $\mathfrak{j}$ for each $\mathfrak{m}, \mathfrak{j} \in \mathfrak{C}$ then $\mathfrak{C}$ is said to be uniquely geodesic and this unique geodesic segment is denoted by $[\mathfrak{m}, \mathfrak{j}]$. Whenever such a segment exists uniquely. We represent the point $z \in [\mathfrak{m}, \mathfrak{j}]$ for any $\mathfrak{m}, \mathfrak{j} \in \mathfrak{C}$ by $z = (1-\beta)\mathfrak{m} \oplus \beta\mathfrak{j}$, where $0 \leq \beta \leq 1$ if $\mathfrak{d}(\mathfrak{m},\ z) = \beta\mathfrak{d}(\mathfrak{m},\ \mathfrak{j})$ and $\mathfrak{d}(z,\ \mathfrak{j}) = (1-\beta)\mathfrak{d}(\mathfrak{m},\ \mathfrak{j})$. A subset $\mathfrak{C}_1$ of $\mathfrak{C}$ is said to be convex if $\mathfrak{C}_1$ contains every geodesic segment connecting any two of its points.

A geodesic triangle $\Delta(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ in a geodesic space $(\mathfrak{C}, \mathfrak{d})$ consists of three points $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ in $\mathfrak{C}$ and a geodesic segment between each pair of vertices. A comparison triangle of a geodesic triangle $\Delta(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ in $(\mathfrak{C},\ \mathfrak{d})$ is a triangle $\overline{\Delta}(\ \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3) := \Delta(\overline{\mathfrak{m}}_1,\ \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_3)$ in the Euclidean space $\mathbb{R}^2$ where $\mathfrak{d}_{\mathbb{R}^2}(\overline{\mathfrak{m}}_{\mathfrak{i}},\ \overline{\mathfrak{m}}_j) = \mathfrak{d}(\mathfrak{m}_i, \mathfrak{m}_j)$ for each $i,\ j \in \{1, 2, 3\}$.

A geodesic space $\mathfrak{C}$ is a $CAT(0)$ space if for each geodesic triangle $\Delta(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ in $\mathfrak{C}$ and its comparison triangle $\overline{\Delta} : \Delta(\overline{\mathfrak{m}}_1,\ \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_3)$ in $\mathbb{R}^2$, the $CAT(0)$ inequality:

$$\mathfrak{d}(\mathfrak{m},\ \mathfrak{j}) \leq \mathfrak{d}_{\mathbb{R}^2}(\overline{\mathfrak{m}},\ \bar{\mathfrak{j}}).$$

is satisfied by all $\mathfrak{m}, \mathfrak{j} \in \Delta$ and comparison points $\overline{\mathfrak{m}}, \bar{\mathfrak{j}} \in \overline{\Delta}$. If $\mathfrak{m}, m_1$ and m_2 are points of $CAT(0)$ space and m_0 is the midpoint of the segment $[m_1,\ m_2]$, then the $CAT(0)$ inequality implies

$$\mathfrak{d}(\mathfrak{m}, m_0)^2 \ \leq \ \frac{1}{2}\mathfrak{d}(\mathfrak{m}, m_1)^2 + \frac{1}{2}\mathfrak{d}(\mathfrak{m}, m_2)^2 - \frac{1}{4}\mathfrak{d}(m_1, m_2)^2.$$

The aforementioned disparity, known as the $(\mathcal{CN})$ inequality, and was proposed by Bruhat and Tits [4].

Let $\{\mathfrak{m}_\mathfrak{n}\}$ be a bounded sequence in a closed convex subset $\mathfrak{C}_1$ of $CAT(0)$ space $\mathfrak{C}$. For $\mathfrak{m} \in \mathfrak{C}$, we set:

$$r(\mathfrak{m}, \{\mathfrak{m}_\mathfrak{n}\} = \limsup_{n\to\infty} \mathfrak{d}(\mathfrak{m}, \mathfrak{m}_\mathfrak{n}).$$

The asymptotic radius $r(\{\mathfrak{m}_\mathfrak{n}\})$ is given by

$$r(\{\mathfrak{m}_\mathfrak{n}\}) = \inf\{r(\mathfrak{m}, \mathfrak{m}_\mathfrak{n}) : \mathfrak{m}\ \in \mathfrak{C}\},$$

and the asymptotic center $A(\{\mathfrak{m}_\mathfrak{n}\})$ of $\{\mathfrak{m}_\mathfrak{n}\}$ is defined as:

$$A(\{\mathfrak{m}_\mathfrak{n}\}) = \{\mathfrak{m} \in \mathfrak{C} : r(\mathfrak{m}, \mathfrak{m}_\mathfrak{n}) = r(\{\mathfrak{m}_\mathfrak{n}\})\}.$$

If $\mathfrak{C}$ be a complete $CAT(0)$ space, then $A(\{\mathfrak{m}_\mathfrak{n}\})$ contains exactly one point (see Proposition 5 of [6]).

Definition 1.3. ([15], [18]) *A sequence* $\{\mathfrak{m}_{\mathfrak{n}}\}$ *in a* $CAT(0)$ *space* $\mathfrak{C}$ *is said to* $\Delta-$*converge to* $\mathfrak{m} \in \mathfrak{C}$ *if* $\mathfrak{m}$ *is the unique asymptotic center of* $\{\mathfrak{m}_{\mathfrak{n}_{\mathfrak{k}}}\}$ *for every subsequence* $\{\mathfrak{m}_{\mathfrak{n}_{\mathfrak{k}}}\}$ *of* $\{\mathfrak{m}_{\mathfrak{n}}\}$. *In this case we write* $\Delta - \lim_{\mathfrak{n}\to\infty} \mathfrak{m}_{\mathfrak{n}} = \mathfrak{m}$.

We now collect some relevant facts followed previous research results about $CAT(0)$ spaces which will be used in the proofs of our main results.

Lemma 1.4. [15] *In a complete* $CAT(0)$ *space, every bounded sequence admits a* Δ*-convergent subsequence.*

Lemma 1.5. [8] *Let* $\mathfrak{C}_s$ *be closed convex subset of a complete* $CAT(0)$ *space* $\mathfrak{C}$ *and if* $\{\mathfrak{m}_{\mathfrak{n}}\}$ *is a bounded sequence in* $\mathfrak{C}_s$, *then the asymptotic center of* $\{\mathfrak{m}_{\mathfrak{n}}\}$ *is in* $\mathfrak{C}_s$.

Lemma 1.6. [8] *Let* $\mathfrak{C}$ *be a complete* $CAT(0)$ *space and* $\mathfrak{C}_s$ *be its closed convex subset and let* $\wp : \mathfrak{C}_s \to \mathfrak{C}$ *be an asymptotically nonexpansive mapping. If* $\{\mathfrak{m}_{\mathfrak{n}}\}$ *is a bounded sequence in* $\mathfrak{C}_s$ *such that* $\lim\limits_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp\mathfrak{m}_{\mathfrak{n}}) = 0$ *and* $\Delta - \lim\limits_{\mathfrak{n}\to\infty} \mathfrak{m}_{\mathfrak{n}} = \mathfrak{m}$. *Then* $\mathfrak{m} = \wp\mathfrak{m}$.

Lemma 1.7. [7] *Let* $\mathfrak{C}$ *be a* $CAT(0)$ *space. For* $\mathfrak{m},\ \mathfrak{j} \in \mathfrak{C}$ *and* $\rho \in [0,1]$, *there exists a unique* $\mathfrak{l} \in [\mathfrak{m},\ \mathfrak{j}]$ *such that*

$$\mathfrak{d}(\mathfrak{m},\mathfrak{l}) = \rho\ \mathfrak{d}(\mathfrak{m},\mathfrak{j}) \quad \text{and} \quad \mathfrak{d}(\mathfrak{j},\mathfrak{l}) = (1-\rho)\mathfrak{d}(\mathfrak{m},\mathfrak{j}).$$

We use the notation $(1-\rho)\mathfrak{m} \oplus \rho\mathfrak{j}$ for the unique point $\mathfrak{l}$ of the above lemma.

Lemma 1.8. *For* $\mathfrak{m},\mathfrak{j},\mathfrak{l} \in \mathfrak{C}$ *and* $\rho \in [0,1]$ *we have*

$$\mathfrak{d}((1-\rho)\mathfrak{m} \oplus \rho\mathfrak{j}, \mathfrak{l}) \le (1-\rho)\mathfrak{d}(\mathfrak{m},\mathfrak{l}) + \rho\mathfrak{d}(\mathfrak{j},\mathfrak{l}).$$

Lemma 1.9. *For* $\mathfrak{m},\mathfrak{j},\mathfrak{l} \in \mathfrak{C}$ *and* $\rho \in [0,1]$ *we have*

$$\mathfrak{d}((1-\rho)\mathfrak{m} \oplus \rho\mathfrak{j}, \mathfrak{l})^2 \le (1-\rho)\mathfrak{d}^2(\mathfrak{m},\mathfrak{l}) + \rho\mathfrak{d}^2(\mathfrak{j},\mathfrak{l}) - \rho(1-\rho)\mathfrak{d}^2(\mathfrak{m},\mathfrak{j}).$$

Inspired and motivated by the results of existing three step iteration schemes, Nisha *et al*. [25] introduced a new iteration scheme namely, Standard three-step iteration scheme which is an unification of many existing iteration schemes and defined as follows :
for any $\mathfrak{m}_0 \in \mathfrak{C}$,

$$\begin{cases} \mathfrak{l}_{\mathfrak{n}} & = \mathfrak{a}^0_{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} + \mathfrak{a}^1_{\mathfrak{n}}\wp\mathfrak{m}_{\mathfrak{n}} + \mathfrak{a}^2_{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}} + \mathfrak{a}^3_{\mathfrak{n}}\wp\mathfrak{j}_{\mathfrak{n}}; \\ \mathfrak{j}_{\mathfrak{n}} & = \mathfrak{b}^0_{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} + \mathfrak{b}^1_{\mathfrak{n}}\wp\mathfrak{m}_{\mathfrak{n}} + \mathfrak{b}^2_{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}} + \mathfrak{b}^3_{\mathfrak{n}}\wp\mathfrak{l}_{\mathfrak{n}}; \\ \mathfrak{m}_{\mathfrak{n}+1} & = \mathfrak{c}^0_{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} + \mathfrak{c}^1_{\mathfrak{n}}\wp\mathfrak{m}_{\mathfrak{n}} + \mathfrak{c}^2_{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}} + \mathfrak{c}^3_{\mathfrak{n}}\wp\mathfrak{j}_{\mathfrak{n}} + \mathfrak{c}^4_{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}} + \mathfrak{c}^5_{\mathfrak{n}}\wp\mathfrak{l}_{\mathfrak{n}}, \end{cases} \tag{1.1}$$

where sequence $\{\mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}}\}$, $\{\mathfrak{b}^{\mathfrak{i}}_{\mathfrak{n}}\}$ for $i = 0,1,2,3$ and $\{\mathfrak{c}^{\mathfrak{i}}_{\mathfrak{n}}\}$ for $i = 0,1,2,3,4,5$ are sequences in [0,1] such that one of the following condition holds such that $\sum_{i=0}^{3} \mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}} \le$

$1, \sum_{i=0}^{3} \mathfrak{b}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1, \sum_{i=0}^{5} \mathfrak{c}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1$.
For $x_1 \in \mathfrak{C}$, the Standard three-step iteration scheme in the framework of $CAT(0)$ spaces is defined as

$$\begin{cases} \mathfrak{l}_{\mathfrak{n}} & = \mathfrak{a}_{\mathfrak{n}}^{0}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{a}_{\mathfrak{n}}^{1}\wp\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{a}_{\mathfrak{n}}^{2}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{a}_{\mathfrak{n}}^{3}\wp\mathfrak{j}_{\mathfrak{n}}; \\ \mathfrak{j}_{\mathfrak{n}} & = \mathfrak{b}_{\mathfrak{n}}^{0}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{b}_{\mathfrak{n}}^{1}\wp\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{l}_{\mathfrak{n}} \oplus \mathfrak{b}_{\mathfrak{n}}^{3}\wp\mathfrak{l}_{\mathfrak{n}}; \\ \mathfrak{m}_{\mathfrak{n}+1} & = \mathfrak{c}_{\mathfrak{n}}^{0}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{1}\wp\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{2}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{3}\wp\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{4}\mathfrak{l}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{5}\wp\mathfrak{l}_{\mathfrak{n}}, \end{cases} \tag{1.2}$$

where sequence $\{\mathfrak{a}_{\mathfrak{n}}^{\mathfrak{i}}\}$, $\{\mathfrak{b}_{\mathfrak{n}}^{\mathfrak{i}}\}$ for $i = 0, 1, 2, 3$ and $\{\mathfrak{c}_{\mathfrak{n}}^{\mathfrak{i}}\}$ for $i = 0, 1, 2, 3, 4, 5$ are sequences in [0,1] such that one of the following condition holds such that $\sum_{i=0}^{3} \mathfrak{a}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1, \sum_{i=0}^{3} \mathfrak{b}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1, \sum_{i=0}^{5} \mathfrak{c}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1$.

Now, we are in the position of iteration scheme analysis.

Remark. *For distinct values of $\mathfrak{a}_{\mathfrak{n}}^{\mathfrak{i}}$, $\mathfrak{b}_{\mathfrak{n}}^{\mathfrak{i}}$, $\mathfrak{c}_{\mathfrak{n}}^{\mathfrak{i}}$, for $i = 0, 1, 2$; $\mathfrak{c}_{\mathfrak{n}}^{4}$ and $\mathfrak{c}_{\mathfrak{n}}^{5}$ we have well-known distinct iteration schemes. On substituting*

($\mathcal{B}_1$) $\mathfrak{a}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{b}_{\mathfrak{n}}^{2} = \mathfrak{b}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{2} = \mathfrak{c}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{4} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$, $\mathfrak{a}_{\mathfrak{n}}^{0} = (1 - \mathfrak{a}_{\mathfrak{n}}^{1})$, $\mathfrak{b}_{\mathfrak{n}}^{0} = (1 - \mathfrak{a}_{\mathfrak{n}}^{3})$, $\mathfrak{b}_{\mathfrak{n}}^{0} = (1 - \mathfrak{b}_{\mathfrak{n}}^{3})$ *in the n_v iteration, we obtain the Noor iterative scheme [21].*

($\mathcal{B}_2$) $\mathfrak{a}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{b}_{\mathfrak{n}}^{2} = \mathfrak{b}_{\mathfrak{n}}^{0} = \mathfrak{b}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{2} = \mathfrak{b}_{\mathfrak{n}}^{3} = \mathfrak{c}_{\mathfrak{n}}^{4} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$, $\mathfrak{a}_{\mathfrak{n}}^{0} = (1 - \mathfrak{a}_{\mathfrak{n}}^{1})$, $\mathfrak{b}_{\mathfrak{n}}^{2} = (1 - \mathfrak{a}_{\mathfrak{n}}^{3})$ *and* $\mathfrak{c}_{\mathfrak{n}}^{2} = (1 - \mathfrak{b}_{\mathfrak{n}}^{3})$ *in the n_v iteration, we obtain the SP iterative scheme [22].*

($\mathcal{B}_3$) $\mathfrak{a}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{b}_{\mathfrak{n}}^{2} = \mathfrak{b}_{\mathfrak{n}}^{0} = \mathfrak{b}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{0} = \mathfrak{c}_{\mathfrak{n}}^{2} = \mathfrak{b}_{\mathfrak{n}}^{3} = \mathfrak{c}_{\mathfrak{n}}^{4} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$, $\mathfrak{a}_{\mathfrak{n}}^{2} = (1 - \mathfrak{a}_{\mathfrak{n}}^{1})$, $\mathfrak{b}_{\mathfrak{n}}^{1} = (1 - \mathfrak{a}_{\mathfrak{n}}^{3})$ *and* $\mathfrak{c}_{\mathfrak{n}}^{1} = 1$ *in the n_v iteration, we obtain the $Picard - S$ iterative scheme [11].*

($\mathcal{B}_4$) $\mathfrak{a}_{\mathfrak{n}}^{0} = \mathfrak{a}_{\mathfrak{n}}^{1} = \mathfrak{a}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{b}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{c}_{\mathfrak{n}}^{0} = \mathfrak{c}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{4} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$ *and* $\mathfrak{a}_{\mathfrak{n}}^{0} = (1 - \mathfrak{a}_{\mathfrak{n}}^{1})$, $\mathfrak{b}_{\mathfrak{n}}^{1} = (1 - \mathfrak{a}_{\mathfrak{n}}^{3})$ *and* $\mathfrak{c}_{\mathfrak{n}}^{2} = (1 - \mathfrak{b}_{\mathfrak{n}}^{3})$ *in the n_v iteration, we obtain the CR iterative scheme [5].*

($\mathcal{B}_5$) $\mathfrak{a}_{\mathfrak{n}}^{0} = \mathfrak{a}_{\mathfrak{n}}^{1} = \mathfrak{b}_{\mathfrak{n}}^{2} = \mathfrak{b}_{\mathfrak{n}}^{0} = \mathfrak{c}_{\mathfrak{n}}^{0} = \mathfrak{c}_{\mathfrak{n}}^{2} = \mathfrak{c}_{\mathfrak{n}}^{4} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$, $\mathfrak{b}_{\mathfrak{n}}^{0} = (1 - \mathfrak{b}_{\mathfrak{n}}^{1})$, $\mathfrak{b}_{\mathfrak{n}}^{1} = (1 - \mathfrak{a}_{\mathfrak{n}}^{3})$ *in the n_v iteration, we obtain the Abbas and Nazir iterative scheme [1].*

($\mathcal{B}_6$) $\mathfrak{a}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{b}_{\mathfrak{n}}^{0} = \mathfrak{b}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{2} = \mathfrak{b}_{\mathfrak{n}}^{3} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$, $\mathfrak{a}_{\mathfrak{n}}^{0} = (1 - \mathfrak{a}_{\mathfrak{n}}^{1})$, $\mathfrak{b}_{\mathfrak{n}}^{2} = (1 - \mathfrak{a}_{\mathfrak{n}}^{3})$ *and* $\sigma_{\eta}^{2} = (1 - \mathfrak{b}_{\mathfrak{n}}^{3})$ *in the n_v iteration, we obtain the P iterative scheme [24].*

($\mathcal{B}_7$) $\mathfrak{a}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{b}_{\mathfrak{n}}^{0} = \mathfrak{b}_{\mathfrak{n}}^{2} = \mathfrak{c}_{\mathfrak{n}}^{1} = \mathfrak{c}_{\mathfrak{n}}^{2} = \mathfrak{c}_{\mathfrak{n}}^{0} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$, $\mathfrak{a}_{\mathfrak{n}}^{0} = (1 - \mathfrak{a}_{\mathfrak{n}}^{1})$, $\mathfrak{b}_{\mathfrak{n}}^{1} = (1 - \mathfrak{a}_{\mathfrak{n}}^{3})$ *and* $\sigma_{\eta}^{2} = (1 - \mathfrak{b}_{\mathfrak{n}}^{3})$ *in the n_v iteration, we obtain the D iterative scheme [9].*

($\mathcal{B}_8$) $\mathfrak{a}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{a}_{\mathfrak{n}}^{0} = \mathfrak{a}_{\mathfrak{n}}^{1} = \mathfrak{b}_{\mathfrak{n}}^{2} = \mathfrak{a}_{\mathfrak{n}}^{3} = \mathfrak{b}_{\mathfrak{n}}^{0} = \mathfrak{b}_{\mathfrak{n}}^{1} = \mathfrak{b}_{\mathfrak{n}}^{3} = \mathfrak{c}_{\mathfrak{n}}^{2} = \mathfrak{c}_{\mathfrak{n}}^{4} = 0$, $\mathfrak{c}_{\mathfrak{n}}^{0} = (1 - \mathfrak{c}_{\mathfrak{n}}^{1})$ *in the n_v iteration, we obtain the Mann iterative scheme [19].*

($\mathcal{B}_9$) $\mathfrak{a}_\mathfrak{n}^2 = \mathfrak{a}_\mathfrak{n}^3 = \mathfrak{a}_\mathfrak{n}^o = \mathfrak{a}_\mathfrak{n}^1 = \mathfrak{b}_\mathfrak{n}^2 = \mathfrak{a}_\mathfrak{n}^3 = \mathfrak{c}_\mathfrak{n}^1 = \mathfrak{c}_\mathfrak{n}^2 = \mathfrak{c}_\mathfrak{n}^4 = \mathfrak{c}_\mathfrak{n}^4 = 0$, $\mathfrak{b}_\mathfrak{n}^o = (1 - \mathfrak{b}_\mathfrak{n}^1)$ *and* $\mathfrak{c}_\mathfrak{n}^o = (1 - \mathfrak{b}_\mathfrak{n}^3)$ *in the* n_v *iteration, we obtain the Ishikawa iterative scheme [12].*

The following Lemma is a consequence of Lemma 2.9 of [17] which will be used to prove our main results.

Lemma 1.10. *Let* $\mathfrak{C}$ *be a complete* $CAT(0)$ *space and let* $\mathfrak{m}^* \in \mathfrak{C}$. *Suppose* $\{\phi_\mathfrak{n}\}$ *is a sequence in* $[s, t]$ *for some* $s, t \in (0, 1)$ *and*$\{\mathfrak{m}_\mathfrak{n}\}$, $\{\mathfrak{j}_\mathfrak{n}\}$ *are sequences in* $\mathfrak{C}$ *such that* $\limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) \leq r$, $\limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) \leq r$, *and* $\lim_{\mathfrak{n}\to\infty} \mathfrak{d}((1-\phi_\mathfrak{n})\mathfrak{m}_\mathfrak{n} \oplus \phi_\mathfrak{n}\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) = r$ *for some* $r \geq 0$. *Then*

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{j}_\mathfrak{n}) = 0.$$

In this paper, we prove that the sequence $\{\mathfrak{m}_\mathfrak{n}\}$ in Standard three-step iteration scheme described by (1) Δ-converges to a fixed point of $\wp$. This result is an analog of a result on weak and strong convergence theorem for asymptotically nonexpansive mapping in $CAT(0)$ spaces. In this process, the results of Panyanak [7], Nanjaras and Panyanak [20], Xu and Noor [28] and many others are extended and improved.

2. $\Delta-$ Convergence Theorems

We initially write the lemma in the context of $CAT(0)$ spaces before proving our main results:

Lemma 2.1. [29] *Let* $\{\phi_\mathfrak{n}\}$ *and* $\{\varphi_\mathfrak{n}\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$\phi_{\mathfrak{n}+1} \leq (1 + \varphi_\mathfrak{n})\phi_\mathfrak{n}, n \geq 1.$$

If $\sum_{n=1}^{\infty} \varphi_\mathfrak{n} < +\infty$, *then* $\lim_{\mathfrak{n}\to\infty} \phi_\mathfrak{n}$ *exists.*

Lemma 2.2. *Let* $\mathfrak{C}_1$ *be a nonempty closed bounded convex subset of a complete* $CAT(0)$ *space* $\mathfrak{C}$ *and let* $\wp : \mathfrak{C}_1 \to \mathfrak{C}_1$ *be an asymptotically nonexpansive mapping with* $\{\kappa_\mathfrak{n}\}$ *satisfying* $\kappa_\mathfrak{n} \geq 1$ *and* $\sum_{n=1}^{\infty} \Bigg(-\kappa_\mathfrak{n} - \frac{1}{2} \Bigg) < +\infty$. *Let* $\{\mathfrak{m}_\mathfrak{n}\}$ *be a sequence generated by three-step iteration scheme. For a given* $\mathfrak{m}_1 \in \mathfrak{C}$, *consider the sequence* $\{\mathfrak{m}_\mathfrak{n}\}, \{\mathfrak{j}_\mathfrak{n}\}$ *and* $\{\mathfrak{l}_\mathfrak{n}\}$ *defined by*

$$\begin{cases} \mathfrak{l}_\mathfrak{n} &= \mathfrak{a}_\mathfrak{n}^0\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{a}_\mathfrak{n}^1\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{a}_\mathfrak{n}^2\mathfrak{j}_\mathfrak{n} \oplus \mathfrak{a}_\mathfrak{n}^3\wp^\mathfrak{n}\mathfrak{j}_\mathfrak{n}; \\ \mathfrak{j}_\mathfrak{n} &= \mathfrak{b}_\mathfrak{n}^0\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{b}_\mathfrak{n}^1\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{b}_\mathfrak{n}^2\mathfrak{l}_\mathfrak{n} \oplus \mathfrak{b}_\mathfrak{n}^3\wp^\mathfrak{n}\mathfrak{l}_\mathfrak{n}; \\ \mathfrak{m}_{\mathfrak{n}+1} &= \mathfrak{c}_\mathfrak{n}^0\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^1\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^2\mathfrak{j}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^3\wp^\mathfrak{n}\mathfrak{j}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^4\mathfrak{l}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^5\wp^\mathfrak{n}\mathfrak{l}_\mathfrak{n}, \end{cases} \tag{2.1}$$

where $\sum_{i=0}^{3} \mathfrak{a}_\mathfrak{n}^\mathfrak{i} \leq 1, \sum_{i=0}^{3} \mathfrak{b}_\mathfrak{n}^\mathfrak{i} \leq 1, \sum_{i=0}^{5} \mathfrak{c}_\mathfrak{n}^\mathfrak{i} \leq 1$. *Then* $\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*)$ *exists for all*

$\mathfrak{m}^* \in \mathfrak{P}_\wp$.

Proof. We first note that $\mathfrak{P}_\wp$ is nonempty. It is given that $\wp$ is asymptotically nonexpansive mapping then by Theorem 1.2. For each $\mathfrak{m}^* \in f_\wp$, we have

$$\begin{aligned}\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*) &= \mathfrak{d}(\mathfrak{a}_\mathfrak{n}^0\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{a}_\mathfrak{n}^1\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{a}_\mathfrak{n}^2\mathfrak{j}_\mathfrak{n} \oplus \mathfrak{a}_\mathfrak{n}^3\wp^\mathfrak{n}\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq \mathfrak{a}_\mathfrak{n}^0\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{a}_\mathfrak{n}^1\mathfrak{d}(\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{a}_\mathfrak{n}^2\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{a}_\mathfrak{n}^3\mathfrak{d}(\wp^\mathfrak{n}\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq \mathfrak{a}_\mathfrak{n}^0\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{a}_\mathfrak{n}^1\kappa_\mathfrak{n}\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{a}_\mathfrak{n}^2\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{a}_\mathfrak{n}^3\kappa_\mathfrak{n}\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq (1+\kappa_\mathfrak{n})(\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*)).\end{aligned}$$

Also,

$$\begin{aligned}\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) &= \mathfrak{d}(\mathfrak{b}_\mathfrak{n}^0\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{b}_\mathfrak{n}^1\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{b}_\mathfrak{n}^2\mathfrak{l}_\mathfrak{n} \oplus \mathfrak{b}_\mathfrak{n}^3\wp^\mathfrak{n}\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq \mathfrak{b}_\mathfrak{n}^0\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{b}_\mathfrak{n}^1\mathfrak{d}(\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{b}_\mathfrak{n}^2\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*) + d_\mathfrak{n}^1\mathfrak{d}(\wp^\mathfrak{n}\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq \mathfrak{b}_\mathfrak{n}^0\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{b}_\mathfrak{n}^1\kappa_\mathfrak{n}\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{b}_\mathfrak{n}^2\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{b}_\mathfrak{n}^3\kappa_\mathfrak{n}\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq (1+\kappa_\mathfrak{n})(\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)).\end{aligned}$$

Using the value of $\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)$, we have

$$\begin{aligned}\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) &\leq (1+\kappa_\mathfrak{n})\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + (1+\kappa_\mathfrak{n})^2(\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*))\\
&\leq (1+\kappa_\mathfrak{n} + (1+\kappa_\mathfrak{n})^2)\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + (1+\kappa_\mathfrak{n})^2\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq (2+3\kappa_\mathfrak{n}+\kappa_\mathfrak{n}^2)\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + (1+2\kappa_\mathfrak{n}+\kappa_\mathfrak{n}^2)\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*)\\
\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) &\leq \left[\frac{\kappa_\mathfrak{n}+1}{-\kappa_\mathfrak{n}}\right]\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*).\end{aligned}$$

Now,

$$\begin{aligned}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \mathfrak{m}^*) &= (\mathfrak{c}_\mathfrak{n}^0\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^1\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^2\mathfrak{j}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^3\wp^\mathfrak{n}\mathfrak{j}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^4\mathfrak{l}_\mathfrak{n} \oplus \mathfrak{c}_\mathfrak{n}^5\wp^\mathfrak{n}\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq \mathfrak{c}_\mathfrak{n}^0\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{c}_\mathfrak{n}^1\mathfrak{d}(\wp^\mathfrak{n}\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{c}_\mathfrak{n}^2\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{c}_\mathfrak{n}^3\mathfrak{d}(\wp^\mathfrak{n}\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{c}_\mathfrak{n}^4\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)\\
&+ \mathfrak{c}_\mathfrak{n}^5\mathfrak{d}(\wp^\mathfrak{n}\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)\\
&\leq (\mathfrak{c}_\mathfrak{n}^0 + \mathfrak{c}_\mathfrak{n}^1\kappa_\mathfrak{n})\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + (\mathfrak{c}_\mathfrak{n}^2 + \mathfrak{c}_\mathfrak{n}^3\kappa_\mathfrak{n})\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) + (\mathfrak{c}_\mathfrak{n}^4 + \mathfrak{c}_\mathfrak{n}^5\kappa_\mathfrak{n})\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*).\\
&\leq (1+\kappa_\mathfrak{n})\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + (1+\kappa_\mathfrak{n})\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) + (1+\kappa_\mathfrak{n})\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*).\\
&\leq (1+\kappa_\mathfrak{n})(\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) + \mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*)).\end{aligned}$$

Since,

$$\mathfrak{d}(\mathfrak{j}_\mathfrak{n}, \mathfrak{m}^*) \leq \left[\frac{\kappa_\mathfrak{n}+1}{-\kappa_\mathfrak{n}}\right]\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*)$$

and

$$\mathfrak{d}(\mathfrak{l}_\mathfrak{n}, \mathfrak{m}^*) \leq \left[\frac{\kappa_\mathfrak{n}+1}{-\kappa_\mathfrak{n}}\right]\mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*)$$

we have

$$\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \mathfrak{m}^*) \leq \left(1 + \left[\frac{-2\kappa_{\mathfrak{n}} - 1}{\kappa_{\mathfrak{n}}}\right]\right)\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}^*)$$

Since $\{\kappa_{\mathfrak{n}}\}$ is bounded, there exists $\mathcal{M} > 0$ such that

$$\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \mathfrak{m}^*) \leq (1 + \mathcal{M}(-\kappa_{\mathfrak{n}} - \frac{1}{2}))\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}^*).$$

Lemma 2.3. *Let* $\mathfrak{C}_s, \mathfrak{C}, \wp, \{\kappa_{\mathfrak{n}}, \{\mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}}\}, \{\mathfrak{b}^{\mathfrak{i}}_{\mathfrak{n}}\}, \{\mathfrak{c}^{\mathfrak{i}}_{\mathfrak{n}}\}, \{\mathfrak{m}_{\mathfrak{n}}\}, \{\mathfrak{j}_{\mathfrak{n}}\}, \{\mathfrak{l}_{\mathfrak{n}}\}$ *are as in Lemma 2.2*

(i) *If* $0 < \liminf_{\mathfrak{n}\to\infty} \mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}} \leq \limsup_{\mathfrak{n}\to\infty} \mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}} < 1,$ *then*

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) = 0.$$

(ii) *If* $0 < \liminf_{\mathfrak{n}\to\infty} \mathfrak{c}^{\mathfrak{i}}_{\mathfrak{n}} \leq \limsup_{\mathfrak{n}\to\infty} \mathfrak{c}^{\mathfrak{i}}_{\mathfrak{n}} < 1,$ *and* $\liminf_{\mathfrak{n}\to\infty} \mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}} > 0$ *then*

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) = 0.$$

Proof. We can achieve the desired result using Lemma 1.9 and the proof of Lemma 2.2 in [28] with $\rho = 2$ and $\omega(\gamma) = \gamma(1-\gamma)$ for $\gamma \in [0, 1]$.

Lemma 2.4. *Let* $\mathfrak{C}_1$ *be a nonempty closed bounded convex subset of a complete* $CAT(0)$ *space* $\mathfrak{C}$ *and let* $\wp : \mathfrak{C}_1 \to \mathfrak{C}_1$ *be an asymptotically nonexpansive mapping with* $\{\kappa_{\mathfrak{n}}\}$ *satisfying* $\kappa_{\mathfrak{n}} \geq 1$ *and* $\sum_{n=1}^{\infty}\left(- \kappa_{\mathfrak{n}} - \frac{1}{2}\right) < +\infty$ *and* $\frac{\kappa_n \mathfrak{b}^2_{\mathfrak{n}}\mathfrak{a}^2_{\mathfrak{n}}}{(1-\kappa_{\mathfrak{n}}\mathfrak{b}^1_{\mathfrak{n}})}$ *and* $(1 + \kappa_{\mathfrak{n}})(\mathfrak{c}^4_{\mathfrak{n}} + \mathfrak{c}^2_{\mathfrak{n}}\mathfrak{b}^2_{\mathfrak{n}})$ *are null sequences. Let* $\{\mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}}\}$, $\{\mathfrak{b}^{\mathfrak{i}}_{\mathfrak{n}}\}$ *for* $i = 0, 1, 2, 3$ *and* $\{\mathfrak{c}^{\mathfrak{i}}_{\mathfrak{n}}\}$ *for* $i = 0, 1, 2, 3, 4, 5$ *be real sequences in* $[0, 1]$ *satisfying*

(i) *If* $0 < \liminf_{\mathfrak{n}\to\infty} \mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}} \leq \limsup_{\mathfrak{n}\to\infty} \mathfrak{a}^{\mathfrak{i}}_{\mathfrak{n}} < 1.$

(ii) *If* $0 < \liminf_{\mathfrak{n}\to\infty} \mathfrak{b}^{\mathfrak{i}}_{\mathfrak{n}} \leq \limsup_{\mathfrak{n}\to\infty} \mathfrak{b}^{\mathfrak{i}}_{\mathfrak{n}} < 1.$

For a given $\mathfrak{m}_1 \in \mathfrak{C}$, *consider the sequence* $\{\mathfrak{m}_{\mathfrak{n}}\}, \{\mathfrak{j}_{\mathfrak{n}}\}$ *and* $\{\mathfrak{l}_{\mathfrak{n}}\}$ *defined by*

$$\begin{cases} \mathfrak{l}_{\mathfrak{n}} & = \mathfrak{a}^0_{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{a}^1_{\mathfrak{n}}\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{a}^2_{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{a}^3_{\mathfrak{n}}\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}; \\ \mathfrak{j}_{\mathfrak{n}} & = \mathfrak{b}^0_{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{b}^1_{\mathfrak{n}}\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{b}^2_{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}} \oplus \mathfrak{b}^3_{\mathfrak{n}}\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}; \\ \mathfrak{m}_{\mathfrak{n}+1} & = \mathfrak{c}^0_{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{c}^1_{\mathfrak{n}}\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{c}^2_{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{c}^3_{\mathfrak{n}}\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{c}^4_{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}} \oplus \mathfrak{c}^5_{\mathfrak{n}}\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}, \end{cases} \tag{2.2}$$

where $\sum_{i=0}^{3} \mathfrak{a}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1, \sum_{i=0}^{3} \mathfrak{b}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1, \sum_{i=0}^{5} \mathfrak{c}_{\mathfrak{n}}^{\mathfrak{i}} \leq 1$. *Then* $\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\wp\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) = 0$.

Proof. From Lemma 2.3, we have

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) = 0$$

and

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) = 0.$$

Thus

$$\begin{aligned}
\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) &\leq \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
&\leq \kappa_n \mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{j}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
&= \kappa_n \mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{b}_{\mathfrak{n}}^{0}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{b}_{\mathfrak{n}}^{1}\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{l}_{\mathfrak{n}} \oplus \mathfrak{b}_{\mathfrak{n}}^{3}\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) &= \kappa_n \mathfrak{b}_{\mathfrak{n}}^{1}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}) + \kappa_n \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{d}(\mathfrak{m}_{n}, \mathfrak{l}_{\mathfrak{n}}) + \kappa_n \mathfrak{b}_{\mathfrak{n}}^{3}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}), \\
&= \kappa_n \mathfrak{b}_{\mathfrak{n}}^{1}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}) + \kappa_n \mathfrak{b}_{\mathfrak{n}}^{2}\left[\mathfrak{d}(\mathfrak{m}_{n}, \mathfrak{a}_{\mathfrak{n}}^{0}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{a}_{\mathfrak{n}}^{1}\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{a}_{\mathfrak{n}}^{2}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{a}_{\mathfrak{n}}^{3}\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}})\right] \\
&+ \kappa_n \mathfrak{b}_{\mathfrak{n}}^{3}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
&= \kappa_n \mathfrak{b}_{\mathfrak{n}}^{1}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}) + \kappa_n \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{a}_{\mathfrak{n}}^{1}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}) + \kappa_n \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{a}_{\mathfrak{n}}^{2}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{j}_{\mathfrak{n}}) \\
&+ \kappa_n \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{a}_{\mathfrak{n}}^{3}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}) + \kappa_n \mathfrak{b}_{\mathfrak{n}}^{3}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}})
\end{aligned}$$

for $\mathfrak{n} \to +\infty$, we have

$$\begin{aligned}
\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) &= \kappa_n \mathfrak{b}_{\mathfrak{n}}^{1}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}) + \kappa_n \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{a}_{\mathfrak{n}}^{2}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{j}_{\mathfrak{n}}) \\
\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) &= \frac{\kappa_n \mathfrak{b}_{\mathfrak{n}}^{2}\mathfrak{a}_{\mathfrak{n}}^{2}}{(1 - \kappa_{\mathfrak{n}}\mathfrak{b}_{\mathfrak{n}}^{1})}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{j}_{\mathfrak{n}}) \to 0 \quad as \quad n \to +\infty.
\end{aligned}$$

so as

$$\begin{aligned}
\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}+1}) &\leq \mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \mathfrak{m}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}+1}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
&\leq \mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \mathfrak{m}_{\mathfrak{n}}) + \kappa_{\mathfrak{n}}\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \mathfrak{m}_{\mathfrak{n}}) + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
&\leq (1 + \kappa_{\mathfrak{n}})\mathfrak{d}(\mathfrak{c}_{\mathfrak{n}}^{0}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{1}\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{2}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{3}\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{4}\mathfrak{l}_{\mathfrak{n}} \oplus \mathfrak{c}_{\mathfrak{n}}^{5}\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
&+ \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) \\
&\leq (1 + \kappa_{\mathfrak{n}})\left[\mathfrak{c}_{\mathfrak{n}}^{1}\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) + \mathfrak{c}_{\mathfrak{n}}^{2}\mathfrak{d}(\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) + \mathfrak{c}_{\mathfrak{n}}^{3}\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) + \mathfrak{c}_{\mathfrak{n}}^{4}\mathfrak{d}(\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}})\right. \\
&\left.+ \mathfrak{c}_{\mathfrak{n}}^{5}\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}})\right] + \mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}})
\end{aligned}$$

for $\mathfrak{n} \to +\infty$, we have

$$\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}+1}) = (1+\kappa_{\mathfrak{n}})\Big[\mathfrak{c}_{\mathfrak{n}}^2(\mathfrak{d}(\mathfrak{j}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) + \mathfrak{c}_{\mathfrak{n}}^4\mathfrak{d}(\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}})\Big]$$
$$= (1+\kappa_{\mathfrak{n}})\Big[\mathfrak{c}_{\mathfrak{n}}^2\mathfrak{b}_{\mathfrak{n}}^1\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}}) + \mathfrak{b}_{\mathfrak{n}}^3\mathfrak{d}(\wp^{\mathfrak{n}}\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}})) + (\mathfrak{c}_{\mathfrak{n}}^4 + \mathfrak{c}_{\mathfrak{n}}^2\mathfrak{b}_{\mathfrak{n}}^2)\mathfrak{d}(\mathfrak{l}_{\mathfrak{n}}, \mathfrak{m}_{\mathfrak{n}})\Big]$$

for $\mathfrak{n} \to +\infty$, $\mathfrak{d}(\mathfrak{m}_{\mathfrak{n}+1}, \wp^{\mathfrak{n}}\mathfrak{m}_{\mathfrak{n}+1}) \to 0$.

Theorem 2.5. *Let $\mathfrak{C}_1$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathfrak{C}$ and Let $\wp : \mathfrak{C}_1 \to \mathfrak{C}_1$ be an asymptotically nonexpansive mapping with $\{\kappa_{\mathfrak{n}}\}$ satisfying $\kappa_{\mathfrak{n}} \geq 1$. Also, $\sum_{n=1}^{\infty}\left(-\kappa_{\mathfrak{n}} - \frac{1}{2}\right) < +\infty$ and $\frac{\kappa_n \mathfrak{b}_{\mathfrak{n}}^2\mathfrak{a}_{\mathfrak{n}}^2}{(1-\kappa_n\mathfrak{b}_{\mathfrak{n}}^1)}$ and $(1+\kappa_{\mathfrak{n}})(\mathfrak{c}_{\mathfrak{n}}^4 + \mathfrak{c}_{\mathfrak{n}}^2\mathfrak{b}_{\mathfrak{n}}^2)$ are null sequences. The sequences $\{\mathfrak{a}_{\mathfrak{n}}^{\mathfrak{i}}\}$, $\{\mathfrak{b}_{\mathfrak{n}}^{\mathfrak{i}}\}$ and $\{\mathfrak{c}_{\mathfrak{n}}^{\mathfrak{i}}\}$ real sequences satisfies the conditions defined in Lemma 2.3 and Lemma 2.4. Then the sequence $\{\mathfrak{m}_{\mathfrak{n}}\}$ Δ-converges to a fixed point of $\wp$.*

Proof. Lemma 2.4 guarantees that $\{\mathfrak{m}_{\mathfrak{n}}\}$ is a bounded sequence and

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \wp\mathfrak{m}_{\mathfrak{n}}) = 0.$$

Let $W_\omega(\{\mathfrak{m}_{\mathfrak{n}}\}) =: \cup A(\{\mathfrak{m}'_{\mathfrak{n}}\})$, is union of all subsequence $\{\mathfrak{m}'_{\mathfrak{n}}\}$ over $\{\mathfrak{m}_{\mathfrak{n}}\}$. To show the Δ-convergence of $\{\mathfrak{m}_{\mathfrak{n}}\}$ to a fixed point of $\wp$, we show that $W_\omega(\{(\mathfrak{m}_{\mathfrak{n}}\}) \subset \mathfrak{P}_\wp$ and $W_\omega(\{\mathfrak{m}_{\mathfrak{n}}\})$ is a singleton set. To show that $W_\omega(\{(\mathfrak{m}_{\mathfrak{n}}\}) \subset \mathfrak{P}_\wp$ let $\mathfrak{m}' \in W_\omega(\{(\mathfrak{m}_{\mathfrak{n}}\})$. Then, there exists a subsequence $\{\mathfrak{m}'_{\mathfrak{n}}\}$ of $\{\mathfrak{m}_{\mathfrak{n}}\}$ such that $A(\{\mathfrak{m}_{\mathfrak{n}}\}) = \mathfrak{m}'$. By Lemma 1.4 and Lemma 1.5, there exists a subsequence $\{\mathfrak{m}''_{\mathfrak{n}}\}$ of $\{\mathfrak{m}'_{\mathfrak{n}}\}$ such that $\Delta - \lim_{\mathfrak{n}\to\infty} \mathfrak{m}''_{\mathfrak{n}} = \mathfrak{m}'' \in \mathfrak{C}$. Since $\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}''_{\mathfrak{n}}, \wp\mathfrak{m}''_{\mathfrak{n}}) = 0$, then $\mathfrak{m}'' \in \mathfrak{P}_\wp$. Lets claim that $\mathfrak{m}'' = \mathfrak{m}'$. On contrary, since $\mathfrak{m}'' \in \mathfrak{P}_\wp$, by Lemma 2.2 $\lim_{\mathfrak{n}\to\infty} \mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}''$ exists. Due to the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}''_{\mathfrak{n}}, \mathfrak{m}'') &< \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}''_{\mathfrak{n}}, \mathfrak{m}') \\
&\leq \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}'_{\mathfrak{n}}, \mathfrak{m}') \\
&< \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}'_{\mathfrak{n}}, \mathfrak{m}'') \\
&= \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_{\mathfrak{n}}, \mathfrak{m}'') \\
&= \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}''_{\mathfrak{n}}, \mathfrak{m}''),
\end{aligned}$$

which is a contradiction. Hence $\mathfrak{m}'' = \mathfrak{m}'$. To assert that $W_\omega(\{(\mathfrak{m}_{\mathfrak{n}}\})$ is a singleton set, let $\{\mathfrak{m}'_{\mathfrak{n}}\}$ be a subsequence of $\{\mathfrak{m}_{\mathfrak{n}}\}$. Owing to Lemmas 1.4 and 1.5, a subsequence $\{\mathfrak{m}''_{\mathfrak{n}}\}$ of $\{\mathfrak{m}'_{\mathfrak{n}}\}$ exists there such that $\Delta - \lim_{n\to\infty} \mathfrak{m}''_{\mathfrak{n}} = \mathfrak{m}''$. Let $A(\{\mathfrak{m}'_{\mathfrak{n}}\}) = \mathfrak{m}'$

and $A(\{\mathfrak{m}_\mathfrak{n}\}) = \mathfrak{m}$. Previously, we have shown that $\mathfrak{m}'' = \mathfrak{m}'$. So it is enough to show $\mathfrak{m}'' = \mathfrak{m}$. If $\mathfrak{m}'' \neq \mathfrak{m}$. Taking into consideration of Lemma 2.2, $\lim_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}'')$ exists, then by uniqueness of asymptotic centers

$$\begin{aligned}
\limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}''_\mathfrak{n}, \mathfrak{m}'') &< \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}''_\mathfrak{n}, \mathfrak{m}) \\
&\leq \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}) \\
&< \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}'') \\
&= \limsup_{\mathfrak{n}\to\infty} \mathfrak{d}(\mathfrak{m}''_\mathfrak{n}, \mathfrak{m}'')
\end{aligned}$$

which is a contradiction. Hence $\mathfrak{P}_\wp$ is nonempty and $\mathfrak{m}''_\mathfrak{n}$ Δ- converges to a fixed point of $\wp$ so that conclusion is drawn.
We have the following corollary of the previous theorem in light of Theorem 3.1. We may also obtain strong convergence theorems for completely continuous asymptotically nonexpansive mappings by employing the same concepts and methods. So without providing any proof, we might claim the following conclusive results.

Theorem 2.6. *Let $\mathfrak{C}$ be a complete $CAT(0)$ space and $\mathfrak{C}_1$ be its nonempty closed convex subset. A mapping $\wp : \mathfrak{C}_1 \to \mathfrak{C}_1$ be an a completely continuous asymptotically nonexpansive mapping with $\{\kappa_\mathfrak{n}\}$ satisfying $\kappa_\mathfrak{n} \geq 1$. Also, $\sum_{n=1}^{\infty}\left(-\kappa_\mathfrak{n} - \frac{1}{2}\right) < +\infty$ and $\frac{\kappa_n \mathfrak{b}_\mathfrak{n}^2 \mathfrak{a}_\mathfrak{n}^2}{(1-\kappa_n \mathfrak{b}_\mathfrak{n}^1)}$ and $(1+\kappa_\mathfrak{n})(\mathfrak{c}_\mathfrak{n}^4 + \mathfrak{c}_\mathfrak{n}^2 \mathfrak{b}_\mathfrak{n}^2)$ are null sequences. The sequences $\{\mathfrak{a}_\mathfrak{n}^\mathfrak{i}\}$, $\{\mathfrak{b}_\mathfrak{n}^\mathfrak{i}\}$ and $\{\mathfrak{c}_\mathfrak{n}^\mathfrak{i}\}$ real sequences satisfies the conditions defined in Lemma 2.3 and Lemma 2.4. Then the sequence $\{\mathfrak{m}_\mathfrak{n}\}$ converges strongly to a fixed point of $\wp$.*

3. Conclusion

It is obvious that the recently adopted general three-step iteration scheme is an unified version of numerous existing iteration schemes. We reexamined standard three step iteration scheme [25] in geodesic $CAT(0)$ for the purpose of extending and improving the results given by Dhompongsa and Panyanak [7], Nanjaras and Panyanak [20], Xu and Noor [28].

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