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CARTAN SPACES WITH SLOPE METRIC UNDER h-METRICAL d-CONNECTION

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Abstract: This paper studies Cartan space with Matsumoto metric or slope metric under the effect of h-metrical d-connection. Then we deduce the conditions under which the Cartan space with slope metric becomes locally Minkowski and conformally flat.

Keywords and Phrases: Finsler space, h-metrical d-connection, Cartan space, Conformal flatness, Slope metric.

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1. Introduction

In 1933, E. Cartan [2] proposed the theory of an space known as Cartan space. This space is considered as dual of Finsler space [7]. That is, if we look for dual of any general Finsler space, that dual is nothing but a Cartan space. The affinity between these two spaces, Finsler space and Cartan space, has been studied by F. Brickell [1], H. Rund [10] and others. Igrasi ([3], [4]) was great geometer who first realized the need of (α, β)-metric in Cartan space, i.e., in dual Finsler space. He obtained the metric tensors and invariants, which characterize the special class of Cartan spaces with (α, β)-metric. G. Shankar ([12], [13], [14]), H. G. Nagaraja [8]

and M. Rafee [9] also have made significant contribution to theory of Cartan spaces with (α, β)-metric. Motivated by these results we have calculated the conditions under which the manifold becomes locally Minkowski and conformally flat. The paper is organized as follows:
In section 2, we give basic definitions and results required for subsequent sections. In section 3, we deal with Cartan space with slope metric under the influence of h-metrical d-connection. In section 4, we apply the conformal theory of Finsler space to Cartan space with slope metric and deduce some important results.

2. Preliminaries

Consider a manifold M and its associated tangent bundle $TM = \bigcup_{x \in M} T_xM$, where T_xM is a tangent space at a point $x \in M$. The tangent bundle is devised with a natural projection map $\pi : TM \to M$ defined by $\pi(x, y) = x$, which maps every vector $y \in T_xM$ to a point $x \in M$ at which it is tangent. In the same way, the disjoint union $T^*M = \bigcup_{x \in M} T^*_xM$, where T^*_xM is a cotangent space at a point $x \in M$, is called cotangent bundle of M. The cotangent bundle is also devised with a natural projection map $\pi : T^*M \to M$ defined by $\pi(x, \omega) = x$, which maps every covector or differential one form $\omega \in T^*_xM$ to a point $x \in M$ at which it is a cotangent.
Now, we define a real valued function $K : T^*M \backslash \{0\} \to R$ as follows:

Definition 2.1. Finsler Metric over cotangent bundle.
Suppose M is a differentiable manifold and T^*M is its cotangent bundle. An smooth function $K : T^*M \backslash \{0\} \to R$ is called Finsler metric over the cotangent bundle T^*M, if $K(x, \omega)$ satisfies the following conditions:

(a) *Positivity: $K(x, \omega) \geq 0$ for all ω in $T^*_pM \backslash \{0\}$.*

(b) *Positive Homogeneity: $K(x, \omega)$ is +ve 1-homogeneous on the fibers of the cotangent bundle T^*M, i.e., $K(x, \lambda\omega) = \lambda K(x, \omega)$, $\forall \lambda > 0$; for any x in M and ω in $T^*_xM \backslash \{0\}$.*

(c) *Strict Convexity of $K(x, \omega)$: The hessian matrix defined by $g^{ij}(x, \omega) = \frac{1}{2} \frac{\partial^2 K^2}{\partial \omega^i \partial \omega^j}(x, \omega)$ is positive definite for all (x, ω) in $T^*M \backslash \{0\}$.*

Definition 2.2. Cartan Space.
A differentiable manifold M endowed with a Finsler metric $K(x, \omega)$ defined over the slit cotangent bundle $T^*M \backslash \{0\}$ is called a Cartan space.

An n-dimensional Cartan space is denoted by $C^n = (M, K(x, \omega))$, where $K(x, \omega)$ represents norm of covector $\omega \in T^*_xM \backslash \{0\}$ based at any point $x \in M$ of thr Manifold M. The function $K(x, \omega)$ is called the fundamental function and

$g^{ij}(x,\omega) = \frac{1}{2}\frac{\partial^2 K^2}{\partial\omega_i\partial\omega_j}(x,\omega)$ is called the fundamental metric tensor of the Cartan space C^n. The metric tensor $g^{ij}(x,\omega)$ has $g_{ij}(x,\omega)$ as its reciprocal metric tensor which is characterized by $g_{ij}(x,\omega)g^{jk}(x,\omega) = \delta_i^k$, where $g_{ij}(x,\omega)$ and $g^{ij}(x,\omega)$ satisfy symmetry as well as zero degree homogeneity conditions in one form $\omega_j \in T^*M$. In Cartan space the metric $K : T^*M\backslash\{0\} \to R$ is defined from slit cotangent bundle T^*M to non-negative real numbers, so at a point $x \in M$, $K(x,-)$ eats one-form $\omega \in T_p^*M\backslash\{0\}$ and spits non-negative reals, amounts to saying that Cartan space is constructed on the cotangent bundle T^*M in the same way a Finsler space $(M, F(x,y))$, where $F : TM \to R$, is constructed on the tangent bundle TM.

Definition 2.3. [8] *If the fundamental function $K(x,\omega)$ of a Cartan space $C^n = (M, K(x,\omega))$ is a function of variable $\alpha(x,\omega) = (a^{ij}\omega_i\omega_j)^{\frac{1}{2}}$, $\beta(x,\omega) = \omega_i b^i(x)$, where $a^{ij}(x)$ is a Riemannian metric and $b^i(x)$ is a vector field depending only on x, then C^n is called Cartan space with (α,β)-metric.*

In the above definition, it is to be remarked that $K(x,\omega)$ satisfy all the stipulations set in definition Cartan space.

Definition 2.4. *The metric given by*

$$K = \frac{\alpha^2(x,\omega)}{\alpha(x,\omega) - \beta(x,\omega)}, \alpha - \beta > 0$$

is known as Matsumoto or slope metric. Then the structure $\left(M, K = \frac{\alpha^2(x,\omega)}{\alpha(x,\omega)-\beta(x,\omega)}\right)$ determined with Matsumoto or slope metric, is called Matsumoto space. This metric was first introduced by M. Matsumoto [5] *during investigating the model of a Finsler space.*

Definition 2.5. *Let $C^n = (M, K(\alpha(x,\omega), \beta(x,\omega)))$ be a Cartan space with a (α,β)-metric. Then the space constructed over the same manifold M along with Riemannian metric $\alpha(x,\omega)$, i.e., $(M, \alpha(x,\omega))$ is called associated Riemannian manifold.*

Let us consider a Cartan space $C^n = (M, K(x,\omega))$ with a (α,β)-metric, known as Matsumoto metric or slope metric, $K(x,\omega) = \frac{\alpha^2}{\alpha-\beta}$, where $\alpha = (a^{ij}(x,\omega)\omega_i\omega_j)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$.

The [3] fundamental tensor $g^{lm}(x,\omega)$ and its reciprocal tensor $g_{lm}(x,\omega)$ of the Cartan space $C^n = (M, K(\alpha,\beta))$ are given by

$$g^{lm} = \rho a^{lm} + \rho_0 b^l b^m + \rho_{-1}(b^l\omega^m + b^m\omega^l) + \rho_{-2}\omega^l\omega^m, \tag{1}$$

where ρ, ρ_0, ρ_{-1} and ρ_{-2} are invariants, given by

$$\rho = \frac{1}{2\alpha}K_\alpha = \frac{\alpha - 2\beta}{2(\alpha-\beta)^2}, \rho_0 = \frac{1}{2}K_{\beta\beta} = \frac{\alpha^2}{(\alpha-\beta)^3},$$
$$\rho_{-1} = \frac{1}{2\alpha}K_{\alpha\beta} = -\frac{\beta}{(\alpha-\beta)^3}, \rho_{-2} = \frac{1}{2\alpha^2}\left(K_{\alpha\alpha} - \frac{1}{\alpha}K_\alpha\right) = \frac{3\beta-\alpha}{2\alpha(\alpha-\beta)^3}$$

and

$$g_{mn} = \sigma a_{mn} - \sigma_0 b_m b_n + \sigma_{-1}(b_m\omega_n + b_n\omega_m) + \sigma_{-2}\omega_m\omega_n, \tag{2}$$

where

$$\sigma = \frac{1}{\rho}, \sigma_0 = \frac{\rho_0}{\rho\tau},$$
$$\tau = \sigma + \sigma_0 B^2 + \rho_{-1}\beta, \sigma_{-1} = \frac{\rho_{-1}}{\rho\tau}, \sigma_{-2} = \frac{\rho_{-2}}{\rho\tau},$$

where $B^2 = b^i b_j$. Here B represents the norm of the differential form $\beta(x,\omega) = \omega_i b^i(x)$.
The Cartan torsion tensor C^{lmn} [6] is given by

$$C^{lmn} = -\frac{1}{2}\left[r_{-1}b^l b^m b^n + \{\rho_{-1}a^{lm}b^n + \rho_{-2}a^{lm}\omega^n + r_{-2}b^l b^m \omega^n + r_{-3}b^l\omega^m\omega^n + l|m|n\} + r_{-4}\omega^l\omega^m\omega^n\right], \tag{3}$$

where

$$r_{-1} = \frac{1}{2}K_{\beta\beta\beta} = \frac{3\alpha^2}{(\alpha-\beta)^4}, r_{-2} = \frac{1}{2\alpha}K_{\alpha\beta\beta} = \frac{-\alpha-2\beta}{(\alpha-\beta)^4},$$
$$r_{-3} = \frac{1}{2\alpha^2}\left(K_{\alpha\alpha\beta} - \frac{1}{\alpha}K_{\alpha\beta}\right) = \frac{3\beta}{\alpha(\alpha-\beta)^4},$$
$$r_{-4} = \frac{1}{2\alpha^3}\left(K_{\alpha\alpha\alpha} - \frac{3}{\alpha}K_{\alpha\alpha} + \frac{3}{\alpha^2}K_\alpha\right) = \frac{3(\alpha^2+\beta^2-4\alpha\beta)}{2\alpha^3(\alpha-\beta)^4}$$

and $l|m|n$ denotes the cyclic sum in the indices l, m, n.
Let γ^i_{jk} be Christoffel symbols which is defined using the metric a_{ij}. Whenever we talk about Christoffel symbols γ^i_{jk} defined from a_{ij}, we mean
$\gamma^i_{jk} = \frac{1}{2}a^{li}\left(\frac{\partial a_{kl}}{\partial x^j} + \frac{\partial a_{lj}}{\partial x^k} - \frac{\partial a_{jk}}{\partial x^l}\right)$.
Throughout the paper we use the symbol ':' to indicate covariant derivative with

regard to γ^i_{jk}. Since $\omega_{i:k} = 0$ and from Ricci's theorem of tensor calculus [11] we have, $a^{ij}_{:k} = 0$. Also, let $\Gamma^i_{jk}(p) = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$ be the Christoffel symbols constructed from fundamental metric tensor $g_{ij}(x,\omega)$ of the Cartan space $(M, K(x,\omega))$. Now, for the Cartan space $(M, K(x,\omega))$, we state canonical d-connection

$$D\Gamma = (N_{jk}, H^i_{jk}, C^{jk}_i),$$

where

$$N_{ij} = \Gamma^k_{ij}\omega_k - \frac{1}{2}\Gamma^k_{hr}\omega_k\omega^r\dot{\partial}^h g_{ij} \tag{4}$$

$$H^i_{jk} = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}) \tag{5}$$

$$C^{jk}_i(x,\omega) = -\frac{1}{2}g_{ir}(x,\omega)\frac{\partial g^{jk}(x,\omega)}{\partial\omega^r} = g_{ir}(x,\omega)C^{rjk}(x,\omega). \tag{6}$$

are respectively called canonical N-connection, Christoffel symbols and d-tensor field of type (2,1).

Throughout the paper, we use the symbol $'|'_h$ to indicate h-covariant derivative with regard to $D\Gamma$. Now, we utilize the following definition in the section that follows.

Definition 2.6. [8] *A d-connection $D\Gamma$ over a Cartan space $C^n = (M, K(\alpha(x,\omega),$ $\beta(\omega))$ with (α,β)-metric is said to be h-metrical, if following properties is satisfied:*

(i) h-deflection tensor $D_{ij}(=\omega_{i|j}) = 0$,

(ii) $a^{ij}_{|h} = 0$,

(iii) $g^{ij}_{|h} = 0$.

It is to noted that "HMDC" is abbreviation of h-metrical d-connection in the section that follows.

Definition 2.7. *Suppose M is an n-dimensional differentiable manifold equipped with two different Finsler metrics $\tilde{K}(x,\omega)$ and $K(x,\omega)$. Then $\tilde{K}(x,\omega)$ is said to be conformal to $K(x,\omega)$ if $\exists$ a position dependent function $\sigma(x)$ such that $\tilde{K}(x,\omega) = e^{\sigma(x)}K(x,\omega)$.*

3. Cartan Manifold with Slope Metric under h-metrical d-connection

In this section we impose a condition on d-connection $D\Gamma$ of the Cartan space with slope metric to be h-metrical and in consequence we assess what shapes the

corresponding Cartan space assumes.
First we take the h-covariant derivative of slope metric:

$$\begin{aligned} K(x,\omega) &= \frac{\alpha^2}{\alpha-\beta} \\ (g^{ij}\omega_i\omega_j)_{|h} &= \frac{\alpha^2}{\alpha-\beta} \\ g^{ij}\times(\omega_i\omega_j)_{|h}+\omega_i\omega_j\times g^{ij}_{|h} &= \frac{(\alpha-\beta)\times 2\alpha\alpha_{|h}-\alpha^2\times(\alpha_{|h}-\beta_{|h})}{(\alpha-\beta)^2} \\ g^{ij}(\omega_i\omega_{j|h}+\omega_j\omega_{i|h})+\omega_i\omega_j g^{ij}_{|h} &= \frac{(\alpha-\beta)\times 2\alpha\alpha_{|h}-\alpha^2\times(\alpha_{|h}-\beta_{|h})}{(\alpha-\beta)^2} \end{aligned}$$

As we have stipulated the d-connection $D\Gamma$ of associated Cartan space is h-metrical, therefore by Definition 2.6, we have

$$\omega_{j|h}=0, \omega_{i|h}=0, \alpha_{|h}=0, g^{ij}_{|h}=0$$

Using these values in above expression, we get

$$\begin{aligned} \beta_{|h} &= 0 \quad (\because \alpha\neq 0, \beta\neq 0) \qquad (7) \\ (\omega_i b^i(x))_{|h} &= 0 \quad (\because \beta(x,\omega)=\omega_i b^i(x)) \\ \omega_i\times b^i(x)_{|h}+b^i(x)\times\omega_{i|h} &= 0 \end{aligned}$$

Since the d-connection $D\Gamma$ of the Cartan space is h-metrical, therefore by Definition 2.6, we have

$$\omega_{i|h}=0,$$

Using these values in above expression, we get

$$\begin{aligned} \omega_i\times b^i(x)_{|h}+b^i(x)\times 0 &= 0 \\ \omega_i\times b^i(x)_{|h} &= 0 \\ b^i(x)_{|h} &= 0 \qquad (8) \end{aligned}$$

Now, we find h-covariant derivatives of the coefficients of metric tensor g^{ij} and then use conditions of HMDC of Cartan space as follows:

$$\because \rho = \frac{1}{2\alpha}K_\alpha = \frac{\alpha - 2\beta}{2(\alpha-\beta)^2}$$

$$\therefore \rho_{|h} = \frac{1}{2}\frac{(\alpha-\beta)^2 \times (\alpha_{|h} - 2\beta_{|h}) - (\alpha - 2\beta) \times 2(\alpha-\beta)(\alpha_{|h} - \beta_{|h})}{(\alpha-\beta)^4}$$

$$\rho_{|h} = 0. \tag{9}$$

$$\because \rho_0 = \frac{\alpha^2}{(\alpha-\beta)^3}$$

$$\rho_{0|h} = \frac{(\alpha-\beta)^3 \times 2\alpha \times \alpha_{|h} - \alpha^2 \times 3(\alpha-\beta)^2\alpha_{|h} - \beta_{|h})}{(\alpha-\beta)^6}$$

$$\rho_{0|h} = 0. \tag{10}$$

$$\because \rho_{-1} = -\frac{\beta}{(\alpha-\beta)^3}$$

$$\therefore \rho_{-1|h} = -\frac{(\alpha-\beta)^3 \times \beta_{|h} - \beta \times 3(\alpha-\beta)^2(\alpha_{|h} - \beta_{|h})}{(\alpha-\beta)^6}$$

$$\rho_{-1|h} = 0. \tag{11}$$

$$\because \rho_{-2} = \frac{3\beta - \alpha}{2\alpha(\alpha-\beta)^3}$$

$$\therefore \rho_{-2|h} = \frac{1}{2}\frac{\alpha(\alpha-\beta)^3(3\beta_{|h} - \alpha_{|h}) - (3\beta-\alpha)(\alpha(\alpha-\beta)^3)_{|h}}{\alpha^2(\alpha-\beta)^6}$$

$$\therefore \rho_{-2|h} = -\frac{1}{2}\frac{(3\beta-\alpha)\left(\alpha(\alpha-\beta)^3\right)_{|h}}{\alpha^2(\alpha-\beta)^6}$$

$$\rho_{-2|h} = 0. \tag{12}$$

Differentiating Equation (1), we get

$$g^{ij}_{|h} = \rho a^{ij}_{|h} + a^{ij}\rho_{|h} + \rho_0(b^ib^j)_{|h} + b^ib^j\rho_0 + \rho_{-1}(b^i\omega^j + b^j\omega^i)_{|h} + (b^i\omega^j + b^j\omega^i)\rho_{-1|h} + \rho_{-2}(\omega^i\omega^j)_{|h} + \omega^i\omega^j\rho_{-2|h}$$

$$g^{ij}_{|h} = \rho a^{ij}_{|h} + a^{ij}\rho_{|h} + \rho_0\left(b^ib^j_{|h} + b^jb^i_{|h}\right) + b^ib^j\rho_{0|h} + \rho_{-1}\left(b^i\omega^j_{|h} + \omega^ib^i_{|h} + b^j\omega^i_{|h} + \omega^ib^j_h\right) \rho_{-1|h}\left(b^i\omega^j + b^j\omega^i\right) + \rho_{-2|h}\left(\omega^i\omega^j_{|h} + \omega^j\omega^i_{|h}\right) + \omega^i\omega^j\rho_{-2|h}$$

Using the conditions of HMDC of Cartan space and Equations (8), (9), (10), (11) and (12), above equation reduces to

$$g^{ij}_{|h} = 0.$$

Thus, allowing d-connection $D\Gamma$ of Cartan space to be h-metrical, it gives two important quantities namely $a^{ij}_{|h} = 0$ (by definition of h-metrical d-connection) and $g^{ij}_{|h} = 0$, i.e., h-covariant derivatives of fundamental metric tensors of associated Riemannian space and Cartan space vanishes.

Now, since $a^{ij}_{|h} = 0$ and $g^{ij}_{|h} = 0$, therefore there corresponding Chritoffel symbols will also be same, i.e., $H^i_{jh} = \gamma^i_{jh}$ and its equivalent condition, from Theorem 35 of [6], is given by

$$b^i_{:k} = 0 \tag{13}$$

Now, since $H^i_{jh} = \gamma^i_{jh}$ therefore both the d-connection $D\Gamma$ and the Riemannian connection $R\Gamma = (\gamma^i_{jk}, \gamma^i_{jk} y_i, 0)$ has same curvature tensors, i.e.,

$$D^i_{hjk} = R^i_{hjk}$$

If Riemannian curvature tensor vanishes, i.e., $R^i_{hjk} = 0$, then curvature tensor of d-connection also vanishes, i.e., $D^i_{hjk} = 0$. The whole discussion can be summarized as follows:

Proposition 3.1. *Suppose $(M, K = \frac{\alpha^2}{\alpha-\beta})$ is a Cartan manifold under the effect of HMDC. It is said to be flat in local environment if and only if associated Riemannian manifold is also flat in their local environment.*

Now, we find h-covariant derivatives of the coefficients of Cartan torsion tensor C^{ijk} and then use conditions of HMDC of Cartan space and Equation (7) as follows:

$$\because r_{-1} = \frac{3\alpha^2}{(\alpha-\beta)^4}$$

$$\therefore r_{-1|h} = \frac{(\alpha-\beta)^4 \times 6\alpha\alpha_{|h} - 12\alpha^2(\alpha-\beta)^3(\alpha_{|h} - \beta_{|h})}{(\alpha-\beta)^8}$$

$$r_{-1|h} = 0 \tag{14}$$

$$\because r_{-2} = -\frac{\alpha + 2\beta}{(\alpha-\beta)^4}$$

$$\therefore r_{-2|h} = -\frac{(\alpha-\beta)^4 \times (\alpha_{|h} + 2\beta_{|h}) - (\alpha+2\beta) \times 4(\alpha-\beta)^3(\alpha_{|h} - \beta_{|h})}{(\alpha-\beta)^8}$$

$$r_{-2|h} = 0 \tag{15}$$

$$\because r_{-3} = \frac{3\beta}{\alpha(\alpha-\beta)^4}$$

$$\therefore r_{-3|h} = \frac{\alpha(\alpha-\beta)^4 \times 3\beta_{|h} - 3\beta \times \left(\alpha \times 4(\alpha-\beta)^3(\alpha_{|h} - \beta_{|h})\right)}{\alpha^2(\alpha-\beta)^8}$$

$$r_{-3|h} = 0 \tag{16}$$

$$\because r_{-4} = \frac{3}{2}\frac{\alpha^2+\beta^2-4\alpha\beta}{\alpha^3(\alpha-\beta)^4}$$

Similarly, it can be shown that

$$r_{-4|h} = 0 \tag{17}$$

Thus, we have h-covariant derivatives of the coefficients of Cartan torsion tensor C^{ijk} under the HMDC of Cartan space vanishes.

Now we calculate the value of h-covariant derivative of d-tensor field C_i^{jk} of type (2,1) under the assumption of HMDC as follows:

$$\begin{aligned}
\because C_k^{ij} &= g_{kr}C^{rij} \\
\therefore C_{k|h}^{ij} &= \left(g_{kr}C^{rij}\right)_{|h} \\
&= g_{kr} \times C_{|h}^{rij} + C^{rij} \times g_{kr|h} \\
&= g_{kr}C_{|h}^{rij} \\
&= -g_{kr}\frac{1}{2}[r_{-1}b^rb^ib^j + r_{-2}b^rb^i\omega^j + r_{-3}b^r\omega^i\omega^j + r_{-4}\omega^r\omega^i\omega^j + \rho_{-1}a^{ri}b^j + \\
&\quad \rho_{-2}a^{ri}\omega^j + r|i|j]_{|h} \\
&= -g_{kr}\frac{1}{2}[r_{-1} \times (b^rb^ib^j)_{|h} + b^rb^ib^j \times r_{-1|h} + r_{-2} \times (b^rb^i\omega^j)_{|h} + b^rb^i\omega^j \times \\
&\quad r_{-2|h} + r_{-3} \times (b^r\omega^i\omega^j)_{|h} + b^r\omega^i\omega^j \times r_{-3|h} + r_{-4} \times (\omega^r\omega^i\omega^j)_{|h} + \omega^r\omega^i\omega^j \times \\
&\quad r_{-4|h} + \rho_{-1} \times (a^{ri}b^j)_{|h} + a^{ri}b^j \times \rho_{-1|h} + \rho_{-2} \times (a^{ri}\omega^j)_{|h} + a^{ri}\omega^j \times \\
&\quad \rho_{-2|h} + (r|i|j)_{|h}] \\
&= -g_{kr}\frac{1}{2}[r_{-1}(b^rb^ib^j)_{|h} + r_{-2}(b^rb^i\omega^j)_{|h} + r_{-3}(b^r\omega^i\omega^j)_{|h} + r_{-4}(\omega^r\omega^i\omega^j)_{|h} + \\
&\quad \rho_{-1}(a^{ri}b^j)_{|h} + \rho_{-2}(a^{ri}\omega^j)_{|h} + (r|i|j)_{|h}] \\
&= -g_{kr}\frac{1}{2}[r_{-1}(b^rb^ib^j_{|h} + b^rb^jb^i_{|h} + b^ib^jb^r_{|h}) + r_{-2}(b^rb^i\omega^j_{|h} + b^r\omega^jb^i_{|h} + b^i\omega^jb^r_{|h}) + \\
&\quad r_{-3}(b^r\omega^i\omega^j_{|h} + b^r\omega^j\omega^i_{|h} + \omega^i\omega^jb^r_{|h}) + r_{-4}(\omega^r\omega^i\omega^j_{|h} + \omega^r\omega^j\omega^i_{|h} + \omega^i\omega^j\omega^r_{|h}) + \\
&\quad \rho_{-1}(a^{ri}b^j_{|h} + b^ja^{ri}_{|h}) + \rho_{-2}(a^{ri}\omega^j_{|h} + \omega^ja^{ri}_{|h}) + (r|i|j)_{|h}] \\
C_{k|h}^{ij} &= 0
\end{aligned} \tag{18}$$

One knows that ([14], [18]) h-covariant derivative of d-tensor field, C^{ij}_{k}, vanishes, i.e., $C^{ij}_{k|h} = 0$, iff associated Cartan manifold becomes affinely-connected space or Berwald space. So, from Equation (18), we reach at the following conclusion:

Proposition 3.2. *Suppose* $(M, K = \frac{\alpha^2}{\alpha-\beta})$ *is a Cartan manifold. It becomes a Berwald manifold provided that* $(M, K = \frac{\alpha^2}{\alpha-\beta})$ *is under the effect of HMDC.*

In [14], it has been deduced that a Berwald manifold becomes locally Minkowski, provided that its curvature tensor discards. Hence, from the Propositions 3.1 and 3.2, we come to an interesting result as follows:

Theorem 3.3. *Suppose* $(M, K = \frac{\alpha^2}{\alpha-\beta})$ *is a Cartan manifold under the effect of HMDC. Then the Cartan manifold is locally Minkowski if and only if associated Riemannian manifold* $(M, \alpha(x, \omega))$ *bears local flatness.*

4. Conformal Transformation of Cartan Space with Slope Metric

In this section our aim is to conformally transform a Cartan manifold $(M, K(x, \omega))$ to another Cartan manifold $(M, \tilde{K}(x, \omega))$ and then to determine the nature of curvature tensor $\tilde{D}^{i}_{hjk}$ in the conformally transformed space $(M, \tilde{K}(x, \omega))$ under the influence of h-metrical d-connection on the original Cartan space $(M, K(x, \omega))$. That is, we are going to determine the shape of conformally transformed space $(M, \tilde{K}(x, \omega))$ under the condition of h-metrical d-connection on $(M, K(x, \omega))$.
For that, we take into account an specific n-dimensional Cartan manifold $(M, K = \frac{\alpha^2}{\alpha-\beta})$, where $\alpha = (a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$. By a conformal change $\sigma : K \to \tilde{K}$ such that $\tilde{K}(\tilde{\alpha}, \tilde{\beta}) = e^{\sigma} K(\alpha, \beta)$, we have the another Cartan space $\tilde{C}^n = (M, \tilde{K}(\tilde{\alpha}, \tilde{\beta}))$, where $\tilde{\alpha} = e^{\sigma}\alpha$ and $\tilde{\beta} = e^{\sigma}\beta$.
Putting $\alpha = (a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$ in the above relations, we get

$$\begin{aligned}
\tilde{\alpha} &= e^{\sigma}\alpha \\
\tilde{\alpha} &= e^{\sigma}(a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}} \\
\tilde{\alpha} &= (\underline{e^{2\sigma}a^{ij}(x, \omega)}\omega_i\omega_j)^{\frac{1}{2}} \\
\tilde{\alpha} &= (\underline{\tilde{a}^{ij}}\omega_i\omega_j)^{\frac{1}{2}} \\
\Longrightarrow \tilde{a}^{ij} &= e^{2\sigma}a^{ij}(x, \omega)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\beta} &= e^{\sigma}\beta \\
\tilde{\beta} &= e^{\sigma}\omega_i b^i(x) \\
\tilde{\beta} &= \omega_i \underline{e^{\sigma}b^i(x)}
\end{aligned}$$

$$\tilde{\beta} = \omega_i \underline{\tilde{b^i}}$$
$$\tilde{b^i} = e^{\sigma} b^i(x)$$

Now we calculate the Christoffel symbols $\tilde{\gamma}^p_{rk}$ in conformally transformed space $(M, \tilde{K}(x,\omega))$ as follows:
We know from Riemannian geometry that Christoffel symbols of second kind γ^p_{rk} from fundamental metric tensor $a^{pq}(x,\omega)$ can be defined as

$$\gamma^p_{qk} = \frac{1}{2} a^{lp} \left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l} \right)$$

Similarly, we can also define the Christoffel symbols $\tilde{\gamma}^p_{rk}$ in conformally transformed space $(M, \tilde{K}(x,\omega))$ as

$$\begin{aligned}
\tilde{\gamma}^p_{qk} &= \frac{1}{2} \tilde{a}^{lp} \left(\frac{\partial \tilde{a}_{kl}}{\partial x^q} + \frac{\partial \tilde{a}_{lq}}{\partial x^k} - \frac{\partial \tilde{a}_{qk}}{\partial x^l} \right) \\
&= \frac{1}{2} e^{2\sigma} a^{lp}(x,\omega) \left(\frac{\partial e^{2\sigma} a_{kl}(x,\omega)}{\partial x^q} + \frac{\partial e^{2\sigma} a_{lq}(x,\omega)}{\partial x^k} - \frac{\partial e^{2\sigma} a_{qk}(x,\omega)}{\partial x^l} \right) \\
&= \frac{1}{2} e^{2\sigma} a^{lp} \left[\left(e^{2\sigma} \frac{\partial a_{kl}}{\partial x^q} + a_{kl} \frac{\partial e^{2\sigma}}{\partial x^q} \right) + \left(e^{2\sigma} \frac{\partial a_{lq}}{\partial x^k} + a_{lq} \frac{\partial e^{2\sigma}}{\partial x^k} \right) - \left(e^{2\sigma} \frac{\partial a_{jq}}{\partial x^l} + a_{qk} \frac{\partial e^{2\sigma}}{\partial x^l} \right) \right] \\
&= \frac{1}{2} e^{2\sigma} a^{lp} \left[\left(e^{2\sigma} \frac{\partial a_{kl}}{\partial x^q} + 2e^{2\sigma} a_{kl} \frac{\partial \sigma}{\partial x^q} \right) + \left(e^{2\sigma} \frac{\partial a_{lq}}{\partial x^k} + 2e^{2\sigma} a_{lq} \frac{\partial \sigma}{\partial x^k} \right) - \right. \\
&\quad \left. \left(e^{2\sigma} \frac{\partial a_{qk}}{\partial x^l} + 2e^{2\sigma} a_{qk} \frac{\partial \sigma}{\partial x^l} \right) \right] \\
&= \frac{1}{2} e^{4\sigma} a^{lp} \left[\left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l} \right) + \left(2a_{kl} \frac{\partial \sigma}{\partial x^q} + 2a_{lq} \frac{\partial \sigma}{\partial x^k} - 2a_{qk} \frac{\partial \sigma}{\partial x^l} \right) \right] \\
&= e^{4\sigma} \left[\frac{1}{2} a^{lp} \left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l} \right) + \left(a^{lp} a_{kl} \sigma_q + a^{lp} a_{lq} \sigma_k - a^{lp} a_{qk} \sigma_l \right) \right] \\
&= e^{4\sigma} \left[\gamma^p_{qk} + \left(\delta^p_k \sigma_q + \delta^p_q \sigma_k - a_{qk} \sigma^i \right) \right]
\end{aligned}$$

Hence, the components of Christoffel symbols $\tilde{\gamma}^p_{qk}$, constructed from $\tilde{a}^{pq}$, in conformally transformed space are given by

$$\tilde{\gamma}^p_{qk} = \gamma^p_{qk} + B^p_{qk}, \tag{19}$$

where $B^p_{qk} = \sigma_k \delta^p_q + \sigma_q \delta^p_k - a_{kq} \sigma^p$, $\sigma^p = \sigma_q a^{pq}$.
Now, differentiating covariantly $\tilde{b}^p$ w.r.t. $\tilde{\gamma}^p_{rk}$, yields

$$\tilde{b}^p_{:k} = e^{\sigma} \left(b^p_{:k} + 2\sigma_k b^p + b^r \sigma_r \delta^p_k - \sigma_p b^r a_{rk} \right). \tag{20}$$

Transvecting the Equation (20) by $\tilde{b}^k$, and putting

$$M^p = \frac{1}{B^2}\left(b^k b^p_{:k} - \frac{b^r_{:r} b^p}{n+4}\right), \tag{21}$$

we have $\sigma^p = \tilde{M}^p - M^p$, from which we get $\sigma_p = \tilde{M}^p - M_p$. Substituting the values of σ_p and σ^p in Equation (19) and using $D^p_{hq} = \gamma^p_{hq} + \delta^p_h M_q + \delta^p_h M_q + \delta^p_q M_h - M^p a_{hq}$, we find

$$\tilde{D}^p_{hq} = D^p_{hq}. \tag{22}$$

In the above equation, D^p_{hq} and $\tilde{D}^p_{hq}$ are respectively linear connections in the Cartan manifold $(M, K = \frac{\alpha^2}{\alpha-\beta})$ and conformally transformed Cartan manifold $(M, \tilde{K} = \frac{\tilde{\alpha}^2}{\tilde{\alpha}-\tilde{\beta}})$. Further, equality of these linear connections is called conformal invariance of linear connection D^p_{hq} over the manifold M.

The whole discussion can be summarized by a proposition as follows:

Proposition 4.1. *Suppose* $(M, K = \frac{\alpha^2}{\alpha-\beta})$ *is a Cartan manifold. Then, there exists a conformally invariant symmetric linear connection* D^p_{qk} *on* M*.*

Next, if we denote the curvature tensors of D^p_{qk} by D^p_{hqk} and $\tilde{D}^p_{hq}$ by $\tilde{D}^p_{hqk}$ then from the Equation (22), we get

$$\tilde{D}^p_{hqk} = D^p_{hqk}. \tag{23}$$

From Equation (13), we have $b^p_{:k} = 0$. Put the value of $b^p_{:k}$ in Equation (21), we get $M^i = 0$. Hence, we deduce that $D^p_{qk} = \gamma^p_{qk}$ and $D^p_{hqk} = R^p_{hqk}$.
Thus we have the following proposition.

Proposition 4.2. *Suppose* $(M, K = \frac{\alpha^2}{\alpha-\beta})$ *is a Cartan manifold admitting HMDC. Then*

1. *linear connections of Cartan manifold and associated Riemannian manifold will coincide, i.e.,* $D^p_{qk} = \gamma^p_{qk}$*.*

2. *curvatures of Cartan manifold and associated Riemannian manifold will coincide, i.e.,* $D^p_{hqk} = R^p_{hqk}$*.*

Next, if we impose the condition of local flatness on the associated Riemannian manifold $(M, \alpha(x, \omega))$, that is, $R^p_{hqk} = 0$, then from Proposition 4.2 and Equation (23), we deduce that $\tilde{D}^p_{hqk} = 0$, that is, the space C^n is conformally flat.

Hence, in the light of above calculations, we arrive at the following result:

Theorem 4.3. *Suppose* $(M, K = \frac{\alpha^2}{\alpha-\beta})$ *is a Cartan manifold admitting HMDC.*

Then the Cartan manifold $(M, K = \frac{\alpha^2}{\alpha-\beta})$ *will be conformally flat if and only if the associated Riemannian manifold* $(M, \alpha(x, \omega))$ *is locally flat.*

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