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AUTOMATA RINGS

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Abstract: Automaton is a system that spontaneously gives an output from an input. The input may be energy, information, materials, etc. The system works without the intervention of man. Simply automaton (plural: automata or automatons) is a self–operating machine. Its synonym is ROBOT. In this paper, the authors study the theory of finite automata rings. Finite automata rings extend the notion of finite automata and lead to the study of various properties of rings obtained by using finite automata. Also, commutative automata rings, zero-divisors of an automata ring, automata integral domain, sub-automata rings, ideal of an automata ring and related substructures along with automata ring homomorphism are defined. Some results of rings in terms of automata are given in this paper together with their proofs. Besides, some properties of ring homomorphism are derived in terms of automata. Finally, we provide certain examples as well as non-examples of automata rings.

Keywords and Phrases: Automata rings, Sub-automata rings, Ideal of an automata ring, Automata ring homomorphism.

2020 Mathematics Subject Classification: 13A99, 03D05, 20M35, 68Q70.

1. Introduction

Abstract machines and computational problems are mainly studied in automata theory. These abstract machines are known as automata. In 1936, British Mathematician and Logician Alan Turing presented a model, when there were no computers, the abstract machine was called the Turing machine which could execute all the computational operations. The term automation was invented by an engineer D. S. Harder, in the automobile industry in about 1946 to describe the increased use of automatic devices and controls in mechanized production lines. The term is used widely in a manufacturing context. It is also used in which there is a significant substitution of mechanical, electrical, or computerized action for human effort and intelligence. Finite state automata are significant and play a vital role in different areas, including Electrical Engineering, Linguistics, Computer Science, Philosophy, Biology, Mathematics, and Logic [7], [8].

The theory of automata is a mathematical theory that studies abstract computing devices. The abstract computing devices are also called machines, which are used to accept input or strings by the transition rules. There are many types of automata that have been discovered. One type of automata is the finite automata which are first studied by Stephen Kleene in 1950 [4]. The theory of automata has a close association with theoretical computer science and has a wide range of applications including in machine learning and artificial intelligence.

The study of algebraic structures using automata theory has become an exciting research topic in the last few years. There are many articles on assigning an automaton to automata groups. The mathematicians like F. W. Heng, G. Y. Siang, N. H. Sarmin, S. Turaev, et. al. have done considerable work on various aspects of automata for subgroups, permutation groups in automata diagram, automata representation for abelian groups, etc. [2]. The researcher K. Muthukumaran, S. Shanmugavadivoo, K. Thiagarajan, S. Jeya Bharathi, A. Jeyanthi, and J. Padmashree, etc. have done extensive work on finite abelian group, associative and commutative finite binary automata, quotient finite group automata, isomorphic finite group automata, homomorphism on finite group automata, finite state automaton group, etc. [6], [11].

In this paper, automata rings, commutative automata rings, zero-divisors of an automata ring, automata integral domain, sub-automata rings, ideal of an automata ring, automata ring homomorphism are defined and some of their properties are investigated.

In the next section, some basic concepts and notations that are used in this paper are presented.

2. Preliminaries

To relate automata theory to rings, some definitions, properties, and results of automata and ring theory are discussed in the following section.

A finite-state automaton consists of a finite set of states and a set of transitions from state to state that occurs on input symbols from a set of alphabets. An alphabet is a finite, non-empty set of symbols denoted by A, e.g. $A = \{0, 1\}$, the set of binary alphabet. A string (or word) is a finite sequence of symbols chosen from the set A, e.g. $01101, 01, 1, 0$ are some strings over the alphabet set $A = \{0, 1\}$. An automaton is a system of the type $\Sigma = (Q, A, B, F, G)$, where Q is a set of states, A is a set of inputs, B is a set of outputs, $F : Q \times A \to Q$ and $G : Q \times A \to B$ are functions usually known as state transition function and output function respectively [3], [5], [9].

A system $(R, +, \bullet)$ where R is a non-empty set, '+' and '$\bullet$' are two binary operations defined on the set R, is called a Ring if it satisfies the following postulates. For any $a, b, c \in R, (i)(R, +)$ is an abelian group, $(ii)(R, \bullet)$ is a semi group, (iii) Two distributive law hold. A ring R in which $ab = ba, \forall a, b \in R$ is called a commutative ring. An element e of a ring R is called a unity (or an identity of R if $ae = ea = a, \forall a \in R$. An element $a \in R$ is called a left zero-divisor if $ab = 0$ for some non-zero $b \in R$. An element $a \in R$ is called a right zero-divisors if there exists a non-zero $b \in R$ such that $ba = 0$. A commutative ring R is called an integral domain if $\forall a, b \in R, ab = 0 \implies a = 0$ or $b = 0$. An element a of a ring R with unity 1 is said to be unit (or invertible) if there exists an element b of R such that $ab = ba = 1$. A non-empty subset S of a ring R is called a subring of R if (i) for any $a, b \in S, a + b \in S, ab \in S.(ii)S$ itself is a ring under the induced addition and multiplication operations. A non-empty subset I of a ring R is called an Ideal if $a, b \in I \implies a - b \in I, a \in I, r \in R \implies ar \in I, ra \in I$. A mapping f of a ring R to a ring R_1 is called a ring homomorphism if it satisfies the following properties for all $a, b \in R, (i)f(a + b) = f(a) + f(b), (ii)f(ab) = f(a)f(b)$. Let $f : R \to R_1$ be a ring homomorphism. Kernel of f is defined as the set $\{r \in R \ : f(r) = 0\}$. We shall denote Kernel of f by $Ker f$. [1], [10].

Theorem 2.1. $\forall a, b \in R$,
$(i)a0 = 0a = 0, (ii)a(-b) = (-a)b = -ab, (iii)(-a)(-b) = ab.$

Theorem 2.2. *In a ring R with unity 1, units form a subgroup of the semi group $(R, \bullet)$.*

Theorem 2.3. *A ring R is without zero-divisors $\Leftrightarrow \forall a, b, c \in R, ab = ac$ and $a \neq 0 \Rightarrow b = c$.*

Theorem 2.4. *A non-empty subset S of a ring R is a subring of $R \Leftrightarrow a-b \in S, ab \in S$.*

Theorem 2.5. *Intersection of two right (left) ideals of a ring R is a right (left) ideal of R.*

Theorem 2.6. *Let $f : R \to R_1$ be a ring homomorphism, Then* $(i) f(0)=0, (ii) \forall a \in R, f(-a)=-f(a)$.

Theorem 2.7. *The kernel of a homomorphism from a ring R to a ring R_1 is an ideal of R.* [1], [10].

3. Main Works

Now an attempt is made to develop automata theory in rings. In this section, a finite automata ring (T,μ,ν) is defined and an example is presented.

Definition 3.1. *Automata Rings*

Let us consider the automata $\sum = (Q, A, B, F, G)$ with $o(Q)=n$ and $o(A)=m$. Let $T = \{F(q_a, x_i) : q_a \in Q, x_i \in A, a, i \in \mathbb{N} \cup \{0\}\}$. Let us define the maps $\mu : T \times T \to T$ as $\mu(F(q_a,x_i), F(q_b,x_j)) = F\left(q_{a \bigoplus_n b}, x_{i \bigoplus_m j}\right)$ and $\nu : T \times T \to T$ as $\nu(F(q_a,x_i), F(q_b,x_j)) = F\left(q_{a \bigotimes_n b}, x_{i \bigotimes_m j}\right)$ where $\bigoplus_n$ denotes addition modulo n and $\bigotimes_n$ denotes multiplication modulo n. Then the triple (T,μ,ν) is an automata ring if it satisfies the following axioms

$(a)\,(T,\mu)$ is an abelian group, i.e.

(i) For any state transition function $F(q_a,x_i), F(q_b,x_j) \in T$ implies $\mu(F(q_a,x_i), F(q_b,x_j)) \in T$,

(ii) For any state transition function $F(q_a,x_i), F(q_b,x_j), F(q_c,x_k) \in T$ implies $\mu(F(q_a,x_i), \mu(F(q_b,x_j), F(q_c,x_k))) = \mu(\mu(F(q_a,x_i), F(q_b,x_j)), F(q_c,x_k))$,

(iii) There exists a state transition $F(q_e,x_e) \in T$ such that $\mu(F(q_a,x_i), F(q_e,x_e)) = F(q_a,x_i) = \mu(F(q_e,x_e), F(q_a,x_i))$,

(iv) For every state transition function $F(q_a,x_i) \in T$ there exists $F\left(q_a^{-1}, x_i^{-1}\right) \in T$ such that $\mu\left(F(q_a,x_i), F\left(q_a^{-1}, x_i^{-1}\right)\right) = F(q_e,x_e) = \mu\left(F\left(q_a^{-1}, x_i^{-1}\right), F(q_a,x_i)\right)$

$(b)\,(T,\nu)$ is a semi group, i.e.

(i) For any state transition function $F(q_a,x_i), F(q_b,x_j) \in T$ implies $\nu(F(q_a,x_i), F(q_b,x_j)) \in T$,

(ii) For any state transition function $F(q_a,x_i), F(q_b,x_j), F(q_c,x_k) \in T$ implies $\nu(F(q_a,x_i), \nu(F(q_b,x_j), F(q_c,x_k))) = \nu(\nu(F(q_a,x_i), F(q_b,x_j)), F(q_c,x_k))$,

(iii) For any state transition function $F(q_a,x_i), F(q_b,x_j), F(q_c,x_k) \in T$ implies
$\nu(F(q_a,x_i), \mu(F(q_b,x_j), F(q_c,x_k))) = \mu(\nu(F(q_a,x_i), F(q_b,x_j)), \nu(F(q_a,x_i), F(q_c,x_k)))$
$\nu(\mu(F(q_a,x_i), F(q_b,x_j)), F(q_c,x_k)) = \mu(\nu(F(q_a,x_i), F(q_c,x_k)), \nu(F(q_b,x_j), F(q_c,x_k)))$.

Example 3.1. A finite automata which is an automata ring.

Let us consider finite state automata $\sum = (Q, A, B, F, G)$ with $Q = \{q_0, q_1, q_2\}$, $A = \{x_0, x_1\}, B = \{y_0, y_1\}$ and the next state function F is defined by table 1 and state diagram of $\sum$ in figure 1.

Table 1: State Transition table of $\sum$

F	x_0	x_1
q_0	q_2	q_1
q_1	q_0	q_2
q_2	q_0	q_1

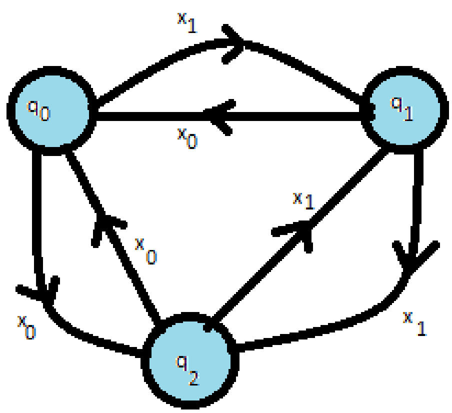

Figure 1: State diagram of $\sum$

Let $T = \{F(q_0, x_0), F(q_0, x_1), F(q_1, x_0), F(q_1, x_1), F(q_2, x_0), F(q_2, x_1)\}$. Two maps μ and ν on T are defined as follows $\mu : T \times T \to T$ as
$\mu(F(q_0, x_0), F(q_0, x_0)) = F(q_0, x_0), \mu(F(q_0, x_0), F(q_0, x_1)) = F(q_0, x_1),$
$\mu(F(q_0, x_0), F(q_1, x_0)) = F(q_1, x_0), \mu(F(q_0, x_0), F(q_1, x_1)) = F(q_1, x_1),$
$\mu(F(q_0, x_0), F(q_2, x_0)) = F(q_2, x_0), \mu(F(q_0, x_0), F(q_2, x_1)) = F(q_2, x_1),$
$\mu(F(q_0, x_1), F(q_0, x_0)) = F(q_0, x_1), \mu(F(q_0, x_1), F(q_0, x_1)) = F(q_0, x_0),$
$\mu(F(q_0, x_1), F(q_1, x_0)) = F(q_1, x_1), \mu(F(q_0, x_1), F(q_1, x_1)) = F(q_1, x_0),$
$\mu(F(q_0, x_1), F(q_2, x_0)) = F(q_2, x_1), \mu(F(q_0, x_1), F(q_2, x_1)) = F(q_2, x_0),$
$\mu(F(q_1, x_0), F(q_0, x_0)) = F(q_1, x_0), \mu(F(q_1, x_0), F(q_0, x_1)) = F(q_1, x_1),$
$\mu(F(q_1, x_0), F(q_1, x_0)) = F(q_2, x_0), \mu(F(q_1, x_0), F(q_1, x_1)) = F(q_2, x_1),$
$\mu(F(q_1, x_0), F(q_2, x_0)) = F(q_0, x_0), \mu(F(q_1, x_0), F(q_2, x_1)) = F(q_0, x_1),$
$\mu(F(q_1, x_1), F(q_0, x_0)) = F(q_1, x_1), \mu(F(q_1, x_1), F(q_0, x_1)) = F(q_1, x_0),$
$\mu(F(q_1, x_1), F(q_1, x_0)) = F(q_2, x_1), \mu(F(q_1, x_1), F(q_1, x_1)) = F(q_2, x_0),$

$\mu(F(q_1,x_1),F(q_2,x_0)) = F(q_0,x_1), \mu(F(q_1,x_1),F(q_2,x_1)) = F(q_0,x_0),$
$\mu(F(q_2,x_0),F(q_0,x_0)) = F(q_2,x_0), \mu(F(q_2,x_0),F(q_0,x_1)) = F(q_2,x_1),$
$\mu(F(q_2,x_0),F(q_1,x_0)) = F(q_0,x_0), \mu(F(q_2,x_0),F(q_1,x_1)) = F(q_0,x_1),$
$\mu(F(q_2,x_0),F(q_2,x_0)) = F(q_1,x_0), \mu(F(q_2,x_0),F(q_2,x_1)) = F(q_1,x_1),$
$\mu(F(q_2,x_1),F(q_0,x_0)) = F(q_2,x_1), \mu(F(q_2,x_1),F(q_0,x_1)) = F(q_2,x_0),$
$\mu(F(q_2,x_1),F(q_1,x_0)) = F(q_0,x_1), \mu(F(q_2,x_1),F(q_1,x_1)) = F(q_0,x_0),$
$\mu(F(q_2,x_1),F(q_2,x_0)) = F(q_1,x_1), \mu(F(q_2,x_1),F(q_2,x_1)) = F(q_1,x_0),$
With the binary map μ defined above we can show that (T,μ) is an abelian group with $F(q_0,x_0)$ as its identity and $\nu : T \times T \to T$ as
$\nu(F(q_0,x_0),F(q_0,x_0)) = F(q_0,x_0), \nu(F(q_0,x_0),F(q_0,x_1)) = F(q_0,x_0),$
$\nu(F(q_0,x_0),F(q_1,x_0)) = F(q_0,x_0), \nu(F(q_0,x_0),F(q_1,x_1)) = F(q_0,x_0),$
$\nu(F(q_0,x_0),F(q_2,x_0)) = F(q_0,x_0), \nu(F(q_0,x_0),F(q_2,x_1)) = F(q_0,x_0),$
$\nu(F(q_0,x_1),F(q_0,x_0)) = F(q_0,x_0), \nu(F(q_0,x_1),F(q_0,x_1)) = F(q_0,x_1),$
$\nu(F(q_0,x_1),F(q_1,x_0)) = F(q_0,x_0), \nu(F(q_0,x_1),F(q_1,x_1)) = F(q_0,x_1),$
$\nu(F(q_0,x_1),F(q_2,x_0)) = F(q_0,x_0), \nu(F(q_0,x_1),F(q_2,x_1)) = F(q_0,x_1),$
$\nu(F(q_1,x_0),F(q_0,x_0)) = F(q_0,x_0), \nu(F(q_1,x_0),F(q_0,x_1)) = F(q_0,x_0),$
$\nu(F(q_1,x_0),F(q_1,x_0)) = F(q_1,x_0), \nu(F(q_1,x_0),F(q_1,x_1)) = F(q_1,x_0),$
$\nu(F(q_1,x_0),F(q_2,x_0)) = F(q_2,x_0), \nu(F(q_1,x_0),F(q_2,x_1)) = F(q_2,x_0),$
$\nu(F(q_1,x_1),F(q_0,x_0)) = F(q_0,x_0), \nu(F(q_1,x_1),F(q_0,x_1)) = F(q_0,x_1),$
$\nu(F(q_1,x_1),F(q_1,x_0)) = F(q_1,x_0), \nu(F(q_1,x_1),F(q_1,x_1)) = F(q_1,x_1),$
$\nu(F(q_1,x_1),F(q_2,x_0)) = F(q_2,x_0), \nu(F(q_1,x_1),F(q_2,x_1)) = F(q_2,x_1),$
$\nu(F(q_2,x_0),F(q_0,x_0)) = F(q_0,x_0), \nu(F(q_2,x_0),F(q_0,x_1)) = F(q_0,x_0),$
$\nu(F(q_2,x_0),F(q_1,x_0)) = F(q_2,x_0), \nu(F(q_2,x_0),F(q_1,x_1)) = F(q_2,x_0),$
$\nu(F(q_2,x_0),F(q_2,x_0)) = F(q_1,x_0), \nu(F(q_2,x_0),F(q_2,x_1)) = F(q_1,x_0),$
$\nu(F(q_2,x_1),F(q_0,x_0)) = F(q_0,x_0), \nu(F(q_2,x_1),F(q_0,x_1)) = F(q_0,x_1),$
$\nu(F(q_2,x_1),F(q_1,x_0)) = F(q_2,x_0), \nu(F(q_2,x_1),F(q_1,x_1)) = F(q_2,x_1),$
$\nu(F(q_2,x_1),F(q_2,x_0)) = F(q_1,x_0), \nu(F(q_2,x_1),F(q_2,x_1)) = F(q_1,x_1),$
With the binary map ν defined above we can show that (T,ν) is a semi group.
Also, $\nu(F(q_2,x_0),\mu(F(q_1,x_1),F(q_1,x_0))) = \nu(F(q_2,x_0),F(q_2,x_1)) = F(q_1,x_0),$
$\mu(\nu(F(q_2,x_0),F(q_1,x_1)),\nu(F(q_2,x_0),F(q_1,x_0))) = \mu(F(q_2,x_0),F(q_2,x_0)) = F(q_1,x_0).$
So, $\nu(F(q_2,x_0),\mu(F(q_1,x_1),F(q_1,x_0))) = \mu(\nu(F(q_2,x_0),F(q_1,x_1)),\nu(F(q_2,x_0),F(q_1,x_0))),$
$\nu(\mu(F(q_1,x_0),F(q_2,x_1)),F(q_0,x_1)) = \nu(F(q_0,x_1),F(q_0,x_1)) = F(q_0,x_1),$
$\mu(\nu(F(q_1,x_0),F(q_0,x_1)),\nu(F(q_2,x_1),F(q_0,x_1))) = \mu(F(q_0,x_0),F(q_0,x_1)) = F(q_0,x_1).$
So, $\nu(\mu(F(q_1,x_0),F(q_2,x_1)),F(q_0,x_1)) = \mu(\nu(F(q_1,x_0),F(q_0,x_1)),\nu(F(q_2,x_1),F(q_0,x_1))).$
Thus ν is distributive with respect to μ. Hence we can say that (T,μ,ν) is an automata Ring.

Definition 3.2. *Commutative Automata Rings*
Automata ring (T,μ,ν) is a commutative ring as
$\nu(F(q_a,x_i),F(q_b,x_j)) = \nu(F(q_b,x_j),F(q_a,x_i)).$

Clearly (T, μ, ν) as shown in the above example is a commutative ring.

Theorem 3.1. *For all state transition function* $F(q_a, x_i), F(q_b, x_j) \in T$ *in an automata ring* (T, μ, ν)
(i) $\nu(F(q_a, x_i), F(q_e, x_e)) = \nu(F(q_e, x_e), F(q_a, x_i)) = F(q_e, x_e)$,
(ii) $\nu(F(q_a, x_i), F(q_b^{-1}, x_j^{-1})) = \nu(F(q_a^{-1}, x_i^{-1}), F(q_b, x_j)) = \nu(F(q_a, x_i)^{-1}, F(q_b, x_j)^{-1})$,
(iii) $\nu\left(F\left(q_a^{-1}, x_i^{-1}\right), F\left(q_b^{-1}, x_j^{-1}\right)\right) = \nu(F(q_a, x_i), F(q_b, x_j))$.

Proof. (i) $\forall F(q_a, x_i) \in T, \nu(F(q_a, x_i), F(q_e, x_e)) = \nu(F(q_a, x_i), \mu(F(q_e, x_e), F(q_e, x_e)))$
$= \mu(\nu(F(q_a, x_i), F(q_e, x_e)), \nu(F(q_a, x_i), F(q_e, x_e)))$.
Again, $\mu\left(\nu(F(q_a, x_i), F(q_e, x_e)), \nu\left(F(q_a, x_i)^{-1}, F(q_e, x_e)^{-1}\right)\right)$
$= \mu\left(\mu(\nu(F(q_a, x_i), F(q_e, x_e)), \nu(F(q_a, x_i), F(q_e, x_e))), \nu\left(F(q_a, x_i)^{-1}, F(q_e, x_e)^{-1}\right)\right)$
$\Rightarrow F(q_e, x_e) = \mu(\nu(F(q_a, x_i), F(q_e, x_e)), \mu(\nu(F(q_a, x_i), F(q_e, x_e)), \nu(F(q_a, x_i)^{-1}, F(q_e, x_e)^{-1})))$
$= \mu(\nu(F(q_a, x_i), F(q_e, x_e)), F(q_e, x_e)) \Rightarrow F(q_e, x_e) = \nu(F(q_a, x_i), F(q_e, x_e))$.
Similarly, $\nu(F(q_e, x_e), F(q_a, x_i)) = F(q_e, x_e)$.
Hence $\nu(F(q_a, x_i), F(q_e, x_e)) = \nu(F(q_e, x_e), F(q_a, x_i)) = F(q_e, x_e), \forall F(q_a, x_i) \in T$.
(ii) $\nu\left(F(q_a, x_i), \mu\left(F(q_b, x_j), F\left(q_b^{-1}, x_j^{-1}\right)\right)\right) = \nu(F(q_a, x_i), F(q_e, x_e)) = F(q_e, x_e)$,
(by (i)) $\Rightarrow \mu\left(\nu(F(q_a, x_i), F(q_b, x_j)), \nu\left(F(q_a, x_i), F\left(q_b^{-1}, x_j^{-1}\right)\right)\right) = F(q_e, x_e)$,
$\Rightarrow \nu\left(F(q_a, x_i), F\left(q_b^{-1}, x_j^{-1}\right)\right) = \nu\left(F(q_a, x_i)^{-1}, F(q_b, x_j)^{-1}\right)$.
Similarly, we can prove that $\nu\left(F\left(q_a^{-1}, x_i^{-1}\right), F(q_b, x_j)\right) = \nu\left(F(q_a, x_i)^{-1}, F(q_b, x_j)^{-1}\right)$.
(iii) $\nu\left(F\left(q_a^{-1}, x_i^{-1}\right), F\left(q_b^{-1}, x_j^{-1}\right)\right) = \nu\left(F(q_a, x_i)^{-1}, F\left(q_b^{-1}, x_j^{-1}\right)^{-1}\right)$, (using (ii))
$= \nu\left((F(q_a, x_i)^{-1})^{-1}, \left(F(q_b, x_j)^{-1}\right)^{-1}\right) = \nu(F(q_a, x_i), F(q_b, x_j))$.

Example 3.2. A finite automata which is not an automata ring.
Let us consider finite state automata $\sum = (Q, A, B, F, G)$ with $Q = \{q_0, q_1, q_2\}, A = \{x_0, x_1\}, B = \{y_0, y_1\}$ and the next state function F is defined by table 2 and state diagram of $\sum$ in figure 2.

Table 2: State Transition table of $\sum$

F	x_0	x_1
q_0	q_1	q_2
q_1	-	q_0
q_2	q_0	q_1

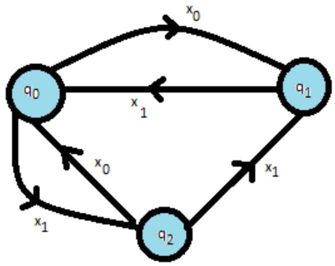

Figure 2: State diagram of $\sum$

Let $T = \{F(q_0, x_0), F(q_0, x_1), F(q_1, x_1), F(q_2, x_0), F(q_2, x_1)\}$.
But $\mu(F(q_0, x_1), F(q_1, x_1)) = F(q_1, x_0) \notin T$. It doesn't satisfy the closure law. So (T, μ, ν) is not an automata ring.

Definition 3.3. *Zero divisors of an automata ring*

The state transition function $F(q_a, x_i) \in T$ is called a zero divisor if
$\nu(F(q_a, x_i), F(q_b, x_j)) = F(q_e, x_e) = \nu(F(q_b, x_j), F(q_a, x_i))$
for some non-zero state transition function $F(q_b, x_j) \in T$.

In the above example 3.1, $Z_T = \{F(q_0, x_1), F(q_2, x_0), F(q_1, x_0)\}$ is a set of zero divisors of automata ring T.

Definition 3.4. *Automata integral domain*

A commutative automata ring T is called an automata integral domain if
$\forall F(q_a, x_i), F(q_b, x_j) \in T, \nu(F(q_a, x_i), F(q_b, x_j)) = F(q_e, x_e)$
$\Rightarrow F(q_a, x_i) = F(q_e, x_e) \, or F(q_b, x_j) = F(q_e, x_e)$.

The above Example is not an example of automata integral domain because,
$\nu(F(q_1, x_0), F(q_0, x_1)) = F(q_0, x_0) \nRightarrow F(q_1, x_0) = F(q_0, x_0)$ or $F(q_0, x_1) = F(q_0, x_0)$.

Definition 3.5.

An element $F(q_e^{'}, x_e^{'})$ of an automata ring T is called a unity (or an identity) of T if
$\nu\left(F(q_a, x_i), F(q_e^{'}, x_e^{'})\right) = \nu\left(F(q_e^{'}, x_e^{'}), F(q_a, x_i)\right) = F(q_a, x_i), \forall F(q_a, x_i) \in T$

In the above Example 3.1, $F(q_1, x_1)$ is the identity element of an automata ring T.

Definition 3.6.

An element $F(q_a, x_i)$ of an automata ring T with unity $F(q_e^{'}, x_e^{'})$ is said to be unit (or invertible) if $\exists$ an element $F(q_b, x_j)$ of T such that $\nu(F(q_a, x_i), F(q_b, x_j)) = \nu(F(q_b, x_j), F(q_a, x_i)) = F(q_e^{'}, x_e^{'})$, In the above Example 3.1, $\nu(F(q_1, x_1), F(q_1, x_1)) = F(q_1, x_1), \nu(F(q_2, x_1), F(q_2, x_1)) = F(q_1, x_1)$. So $\{F(q_1, x_1), F(q_2, x_1)\}$ are units of T.

Theorem 3.2. *In an automata ring T with unity $F(q_e^{'}, x_e^{'})$, units form a subgroup of the semi group (T, ν).*

Proof. Let S be the set of units. Clearly, $\nu(F(q_e^{'}, x_e^{'}), F(q_e^{'}, x_e^{'})) = F(q_e^{'}, x_e^{'})$ $\Rightarrow F(q_e^{'}, x_e^{'}) \in S$. Consider, $F(q_a, x_i), F(q_b, x_j) \in S$, Then $F(q_a, x_i), F(q_b, x_j)$ are units, so $\exists F(q_c, x_k), F(q_d, x_l) \in S$ such that $\nu(F(q_a, x_i), F(q_c, x_k)) = F(q_e^{'}, x_e^{'})$ $= \nu(F(q_c, x_k), F(q_a, x_i))$ and $\nu(F(q_b, x_j), F(q_d, x_l)) = F(q_e^{'}, x_e^{'}) = \nu(F(q_d, x_l), F(q_b, x_j))$. Then $\nu(\nu(F(q_a, x_i), F(q_b, x_j)), \nu(F(q_d, x_l), F(q_c, x_k)))$
$= \nu(\nu(F(q_a, x_i), \nu(F(q_b, x_j), F(q_d, x_l))), F(q_c, x_k))$
$= \nu\left(\nu\left(F(q_a, x_i), F(q_e^{'}, x_e^{'})\right), F(q_c, x_k)\right) = \nu(F(q_a, x_i), F(q_c, x_k)) = F(q_e^{'}, x_e^{'})$.
Similarly, $\nu(\nu(F(q_d, x_l), F(q_c, x_k)), \nu(F(q_a, x_i), F(q_b, x_j))) = F(q_e^{'}, x_e^{'})$.
Thus $\nu(F(q_a, x_i), F(q_b, x_j))$ is a unit.$\Rightarrow \nu(F(q_a, x_i), F(q_b, x_j)) \in S$. Clearly $F(q_c, x_k) = [F(q_a, x_i)]^{-1}$ is a unit. Thus $[F(q_a, x_i)]^{-1} \in S$. Hence S is a subgroup of (T, ν).

In the above Example 3.1, the unit set $\{F(q_1, x_1), F(q_2, x_1)\}$ forms a subgroup of (T, ν).

Theorem 3.3. *An automata ring T is without zero-divisors iff for all* $F(q_a, x_i), F(q_b, x_j), F(q_c, x_k) \in T, \nu(F(q_a, x_i), F(q_b, x_j)) = \nu(F(q_a, x_i), F(q_c, x_k))$ *and* $F(q_a, x_i) \neq F(q_e, x_e) \Rightarrow F(q_b, x_j) = F(q_c, x_k)$.

Proof. Let T be without zero divisors and let $F(q_a, x_i), F(q_b, x_j), F(q_c, x_k) \in T$ be such that $\nu(F(q_a, x_i), F(q_b, x_j)) = \nu(F(q_a, x_i), F(q_c, x_k))$ and $F(q_a, x_i) \neq F(q_e, x_e)$.
Now $\nu(F(q_a, x_i), F(q_b, x_j)) = \nu(F(q_a, x_i), F(q_c, x_k))$
$\Rightarrow \nu\left(F(q_a, x_i), \mu\left(F(q_b, x_j), F\left(q_c^{-1}, x_k^{-1}\right)\right)\right) = F(q_e, x_e)$. But $F(q_a, x_i) \neq F(q_e, x_e)$ and T has no zero divisors. So $\mu\left(F(q_b, x_j), F\left(q_c^{-1}, x_k^{-1}\right)\right) = F(q_e, x_e)$
$\Rightarrow F(q_b, x_j) = F(q_c, x_k)$.
Conversely, suppose that T satisfies the given condition and
let $\forall F(q_a, x_i), F(q_b, x_j) \in T$ be such that $\nu(F(q_a, x_i), F(q_b, x_j)) = F(q_e, x_e)$ and $F(q_a, x_i) \neq F(q_e, x_e)$.
Now $F(q_e, x_e) = \nu(F(q_a, x_i), F(q_e, x_e))$
$\Rightarrow \nu(F(q_a, x_i), F(q_b, x_j)) = \nu(F(q_a, x_i), F(q_e, x_e))$ with $F(q_a, x_i) \neq F(q_e, x_e)$
This by our hypothesis implies $F(q_b, x_j) = F(q_e, x_e)$.
Hence T is an automata ring without zero divisors.

Example 3.3. Let T be a system satisfying all the postulates for an automata ring with the possible exception of $\mu(F(q_a, x_i), F(q_b, x_j)) = \mu(F(q_b, x_j), F(q_a, x_i))$. If there exists one element $\forall F(q_c, x_k) \in T$, such that $\nu(F(q_a, x_i), F(q_c, x_k)) = \nu(F(q_b, x_j), F(q_c, x_k))$ $\Rightarrow F(q_a, x_i) = F(q_b, x_j), \forall F(q_a, x_i), F(q_b, x_j) \in T$. Then T is an automata ring.
Since $\nu(\mu(F(q_a, x_i), F(q_b, x_j)), \mu(F(q_c, x_k), F(q_c, x_k)))$
$= \mu(\nu(F(q_a, x_i), \mu(F(q_c, x_k), F(q_c, x_k))), \nu(F(q_b, x_j), \mu(F(q_c, x_k), F(q_c, x_k)))$
$= \mu(\nu(F(q_a, x_i), F(q_c, x_k)), \mu(\nu(F(q_a, x_i)F(q_c, x_k)), \nu(F(q_b, x_j), F(q_c, x_k))),$
$\nu((F(q_b, x_j), F(q_c, x_k))))$.
Again, $\nu(\mu(F(q_a, x_i), F(q_b, x_j)), \mu(F(q_c, x_k), F(q_c, x_k)))$
$= \mu(\nu(\mu(F(q_a, x_i), F(q_b, x_j)), F(q_c, x_k)), \nu(\mu(F(q_a, x_i), F(q_b, x_j)), F(q_c, x_k)))$
$= \mu(\nu(F(q_a, x_i), F(q_c, x_k)), \mu(\nu(F(q_b, x_j), F(q_c, x_k)),$

$\nu(F(q_a, x_i), F(q_c, x_k))), \nu(F(q_b, x_j), F(q_c, x_k))),$
This gives $\mu\left(\nu\left(F\left(q_a, x_i\right), F\left(q_c, x_k\right)\right), \nu\left(F\left(q_b, x_j\right), F\left(q_c, x_k\right)\right)\right)$
$= \mu\left(\nu\left(F\left(q_b, x_j\right), F\left(q_c, x_k\right)\right), \nu\left(F\left(q_a, x_i\right), F\left(q_c, x_k\right)\right)\right)$
$\Rightarrow \nu\left(\mu\left(F\left(q_a, x_i\right), F\left(q_b, x_j\right)\right), F\left(q_c, x_k\right)\right) = \nu\left(\mu\left(F\left(q_b, x_j\right), F\left(q_a, x_i\right)\right), F\left(q_c, x_k\right)\right)$
$\Rightarrow \mu\left(F\left(q_a, x_i\right), F\left(q_b, x_j\right)\right) = \mu\left(F\left(q_b, x_j\right), F\left(q_a, x_i\right)\right)$, (Using given property).
So (T, μ, ν) is an Automata Ring.

Definition 3.7. *Sub-Automata ring*

A non-empty subset S_T of T is said to be a sub-automata ring if (S_T, μ, ν) itself is a ring under the operations μ and ν.

Criteria for sub-automata ring: A non-empty subset S_T of T is said to be sub-automata ring if $\mu\left(F\left(q_a^{'}, x_i\right), F\left(q_b^{'}, x_j\right)\right) \in S_T, \nu\left(F\left(q_a^{'}, x_i\right), F\left(q_b^{'}, x_j\right)\right) \in S_T$, $F\left(q_a^{-1}, x_i^{-1}\right) \in S_T, \forall F\left(q_a^{'}, x_i\right), F\left(q_b^{'}, x_j\right) \in S_T$.

Example 3.4. For the finite state automata $\sum = (Q, A, B, F, G)$ with $Q = \{q_0, q_1, q_2\}$, $A = \{x_0, x_1\}, B = \{y_0, y_1\}$ and the next state function $F$is defined by table 3

Table 3: State Transition table of $\sum$

F	x_0	x_1
q_0	q_2	q_1
q_1	q_0	q_2
q_2	q_0	q_1

contain some non sub-automata ring.
$T = \{F\left(q_0, x_0\right), F\left(q_0, x_1\right), F\left(q_1, x_0\right), F\left(q_1, x_1\right), F\left(q_2, x_0\right), F\left(q_2, x_1\right)\}$ forms a ring under the operations μ and ν. Then $S_T = \{F\left(q_0, x_0\right), F\left(q_1, x_1\right)\}$,
$S_T^{'} = \{F\left(q_0, x_0\right), F\left(q_0, x_1\right), F\left(q_1, x_0\right), F\left(q_1, x_1\right)\}, S_T^{''} = \{F\left(q_0, x_0\right), F\left(q_2, x_0\right), F\left(q_2, x_1\right)\}$ etc. are non sub-automata ring of (T, μ, ν).

Definition 3.8. *Ideal of an automata ring*

A non-empty subset I_T of T is said to be an ideal if $\mu\left(F\left(q_a^{'}, x_i\right), F\left(q_b^{'}, x_j\right)\right) \in I_T$, $\nu\left(F\left(q_r, x_r\right), F\left(q_a^{'}, x_i\right)\right) \in I_T, \nu\left(F\left(q_a^{'}, x_i\right), F\left(q_r, x_r\right)\right) \in I_T$, $\forall F\left(q_a^{'}, x_i\right), F\left(q_b^{'}, x_i\right) \in I_T, F\left(q_r, x_r\right) \in T$.

Theorem 3.4. *Intersection of two right (left) ideals of an automata ring T is a right (left) ideal of T.*

Proof. Let I_A and I_B are two right ideals of an automata ring T. Since $F\left(q_e, x_e\right) \in I_A, F\left(q_e, x_e\right) \in I_B \Rightarrow F\left(q_e, x_e\right) \in I_A \cap I_B, I_A \cap I_B \neq \phi$,
Let $F\left(q_a, x_i\right), F\left(q_b, x_j\right) \in I_A \cap I_B, F\left(q_r, x_r\right) \in T$.

Then $F(q_a, x_i), F(q_b, x_j) \in I_A \cap I_B \Rightarrow F(q_a, x_i) \in I_A, F(q_a, x_i) \in I_B$
$F(q_b, x_j) \in I_B, F(q_b, x_j) \in I_B$. As I_A is a right ideal of T.
So $\mu\left(F(q_a, x_i), F\left(q_b^{-1}, x_j^{-1}\right)\right) \in I_A$ and $\nu(F(q_a, x_i), F(q_r, x_r)) \in I_A$.
For the same reason, $\mu\left(F(q_a, x_i), F\left(q_b^{-1}, x_j^{-1}\right)\right) \in I_B$ and $\nu(F(q_a, x_i), F(q_r, x_r)) \in I_B$.
Thus $F(q_a, x_i), F(q_b, x_j)) \in I_A \cap I_B \Rightarrow \mu\left(F(q_a, x_i), F\left(q_b^{-1}, x_j^{-1}\right)\right) \in I_A \cap I_B$ and
$\nu(F(q_a, x_i), F(q_r, x_r)) \in I_A \cap I_B, \forall F(q_r, x_r) \in T$.
So $I_A \cap I_B$ is a right ideal of T. Similarly $I_A \cap I_B$ is a left ideal of T.

Definition 3.9. *Automata ring homomorphism*

Let (T, μ, ν) and (T', μ, ν) be two automata rings. A mapping $f : T \to T'$ is called automata ring homomorphism $\forall F(q_a, x_i), F(q_b, x_j) \in T$.
$(i)\ f(\mu(F(q_a, x_i), F(q_b, x_j))) = \mu(f(F(q_a, x_i)), f(F(q_b, x_j)))$,
$(ii)\ f(\nu(F(q_a, x_i), F(q_b, x_j))) = \nu(f(F(q_a, x_i)), f(F(q_b, x_j)))$,
$(iii)\ f(F(q_e, x_e)) = F\left(q_e', x_e'\right)$.

Example 3.5. Consider two finite automata $\sum_1 = (Q_1, A_1, B_1, F_1, G_1)$ with $Q_1 = \{q_0, q_1, q_2\}, A_1 = \{x_0, x_1\}, B_1 = \{y_0, y_1\}$ and the next state function $F_1$is defined by table 4 and state diagram of $\sum_1$ in figure 3.

Table 4: State Transition table of $\sum_1$

F_1	x_0	x_1
q_0	q_2	q_1
q_1	q_0	q_2
q_2	q_0	q_1

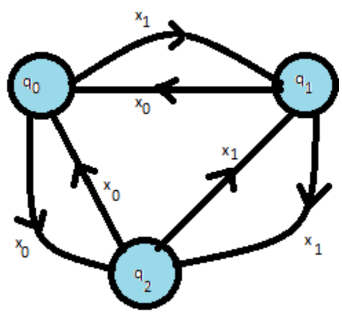

Figure 3: State diagram of $\sum_1$

The set $T = \{F_1(q_0, x_0), F_1(q_0, x_1), F_1(q_1, x_0), F_1(q_1, x_1), F_1(q_2, x_0), F_1(q_2, x_1)\}$ forms a ring under the binary operations μ and ν. Consider $\sum_2 = (Q_2, A_2, B_2, F_2, G_2)$ with

$Q_2 = \{q_0, q_1, q_2\}, A_2 = \{x_0, x_1\}, B_2 = \{y_0, y_1\}$ and the next state function F_2 is defined by table 5 and state diagram in figure 4.

Table 5: State Transition table of $\sum_2$

F_2	x_0	x_1
q_0	q_1	q_0
q_1	q_1	q_2
q_2	q_0	q_2

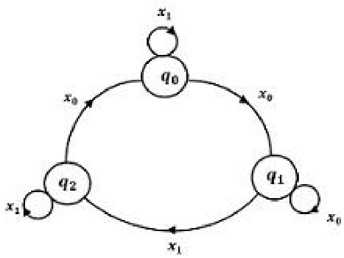

Figure 4: State diagram of $\sum_2$

The set $T^{'} = \{F_2(q_0, x_0), F_2(q_0, x_1), F_2(q_1, x_0), F_2(q_1, x_1), F_2(q_2, x_0), F_2(q_2, x_1)\}$ forms a ring under the binary operations μ and ν
Now, we define a function $\phi : T \to T^{'}$ defined by $\phi(F_1(q_i, x_j) = F_2(q_i, x_j)$,
check ϕ is an automata ring homomorphism i.e.
(i) To show that $\phi(\mu(F_1(q_i, x_j), F_1(q_k, x_l))) = \mu(\phi(F_1(q_i, x_j)), \phi(F_1(q_k, x_l)))$.
Consider, $F_1(q_0, x_0), F_1(q_2, x_1) \in T, F_2(q_0, x_0), F_2(q_2, x_1) \in T^{'}$,
$\phi(\mu(F_1(q_0, x_0), F_1(q_2, x_1))) = \phi(F_1(q_2, x_1) = F_2(q_2, x_1)$,
$\mu(\phi(F_1(q_0, x_0)), \phi(F_1(q_2, x_1))) = \mu(F_2(q_0, x_0), F_2(q_2, x_1)) = F_2(q_2, x_1)$.
So $\phi(\mu(F_1(q_0, x_0), F_1(q_2, x_1))) = \mu(\phi(F_1(q_0, x_0)), \phi(F_1(q_2, x_1)))$.
(ii) To show that $\phi(\nu(F_1(q_i, x_j), F_1(q_k, x_l))) = \nu(\phi(F_1(q_i, x_j)), \phi(F_1(q_k, x_l)))$.
Consider $\phi(\nu(F_1(q_0, x_0), F_1(q_2, x_1))) = \phi(F_1(q_0, x_0) = F_2(q_0, x_0)$,
$\nu(\phi(F_1(q_0, x_0)), \phi(F_1(q_2, x_1))) = \nu(F_2(q_0, x_0), F_2(q_2, x_1)) = F_2(q_0, x_0)$.
So $\phi(\nu(F_1(q_0, x_0), F_1(q_2, x_1))) = \nu(\phi(F_1(q_0, x_0)), \phi(F_1(q_2, x_1)))$.
Also $\phi(F_1(q_0, x_0)) = F_2(q_0, x_0)$.
Hence ϕ is a ring homomorphism from (T, μ, ν) to $(T^{'}, \mu, \nu)$.

Remark. *Clearly (i) shows that automata ring homomorphism of the additive group (T, μ) and it preserves the product. An automata ring homomorphism $f : T \to T^{'}$ is called an automata ring epimorphism if f is onto. It is called an automata ring monomorphism if it is one-one and isomorphism if it is both one-one and onto. A homomorphism f of* *an automata ring (T, μ, ν) into itself is called an automata ring endomorphism. An automata ring endomorphism is called an automata ring automorphism if it is an automata ring isomorphism. Homomorphism, which preserves the algebraic structures play an important role in analyzing the properties of the algebraic structures.*

Theorem 3.5. *Let $f : T \to T^{'}$ is an automata ring homomorphism, then*
(i) $f\left(F\left(q_e, x_e\right)\right) = F\left(q_e^{'}, x_e^{'}\right)$,
(ii) $\forall F\left(q_a, x_i\right) \in T, f\left(F\left(q_a^{-1}, x_i^{-1}\right)\right) = f\left(F\left(q_a, x_i\right)\right)^{-1}$.
Proof. *(i)* $\forall F\left(q_a, x_i\right) \in T, f\left(F\left(q_a, x_i\right)\right) = f\left(\mu\left(F\left(q_a, x_i\right), F\left(q_e, x_e\right)\right)\right)$
$= \mu\left(f\left(F\left(q_a, x_i\right)\right), f\left(F\left(q_e, x_e\right)\right)\right)$.
As $f\left(F\left(q_a, x_i\right)\right) \in T^{'}$, we also have $f\left(F\left(q_a, x_i\right)\right) = \mu\left(f\left(F\left(q_a, x_i\right)\right), F\left(q_e^{'}, x_e^{'}\right)\right)$.
Then $\mu\left(f\left(F\left(q_a, x_i\right)\right), f\left(F\left(q_e, x_e\right)\right)\right) = \mu\left(f\left(F\left(q_a, x_i\right)\right), F\left(q_e^{'}, x_e^{'}\right)\right)$.
Hence, by cancellation laws w.r.t. addition, $f\left(F\left(q_e, x_e\right)\right) = F\left(q_e^{'}, x_e^{'}\right)$
(ii) Now, $\mu\left(F\left(q_a, x_i\right), F\left(q_a^{-1}, x_i^{-1}\right)\right) = F\left(q_e, x_e\right) \Rightarrow f\left(\mu\left(F\left(q_a, x_i\right), F\left(q_a^{-1}, x_i^{-1}\right)\right)\right)$
$= f\left(F\left(q_e, x_e\right)\right) = F\left(q_e^{'}, x_e^{'}\right)$ (Using (i))
$\Rightarrow \mu\left(f\left(F\left(q_a, x_i\right)\right), f\left(F\left(q_a^{-1}, x_i^{-1}\right)\right)\right) = F\left(q_e^{'}, x_e^{'}\right) \Rightarrow f(F\left(q_a^{-1}, x_i^{-1}\right) = f\left(F\left(q_a, x_i\right)\right)^{-1}$.

Definition 3.10. *Let $f : T \to T^{'}$ is an automata ring homomorphism, kernel of f is defined as the set $\{F\left(q_a, x_i\right) \in T : f\left(F\left(q_a, x_i\right)\right) = F\left(q_e^{'}, x_e^{'}\right)\}$. We shall denote kernel of f by $kerf$.*

Theorem 3.6. *The kernel of an automata ring homomorphism from an automata ring T to an automata ring $T^{'}$ is an ideal of T.*
Proof. Let $f : T \to T^{'}$ is an automata ring homomorphism.
$f\left(F\left(q_e, x_e\right)\right) = F\left(q_e^{'}, x_e^{'}\right)$. So $F\left(q_e, x_e\right) \in kerf \Rightarrow kerf \neq \phi$.
Let $F(q_a, x_i), F(q_b, x_j) \in kerf \Rightarrow f(F(q_a, x_i)) = F(q_e^{'}, x_e^{'})$ and $f(F(q_b, x_j)) = F(q_e^{'}, x_e^{'})$.
Thus, $f(\mu(F(q_a, x_i), F(q_b^{-1}, x_j^{-1}))) = \mu(f(F(q_a, x_i)), f(F(q_b^{-1}, x_j^{-1}))) = F(q_e^{'}, x_e^{'})$.
So $\mu\left(F\left(q_a, x_i\right), F\left(q_b^{-1}, x_j^{-1}\right)\right) \in kerf$. Again, $F\left(q_r, x_r\right) \in T, F\left(q_a, x_i\right) \in kerf$
$\Rightarrow f\left(\nu\left(F\left(q_a, x_i\right), F\left(q_r, x_r\right)\right)\right) = \nu\left(f\left(F\left(q_a, x_i\right)\right), f\left(F\left(q_r, x_r\right)\right)\right) = F\left(q_e^{'}, x_e^{'}\right)$,
since $f\left(F\left(q_a, x_i\right)\right) = F\left(q_e^{'}, x_e^{'}\right)$.
Similarly, $f\left(\nu\left(F\left(q_r, x_r\right), F\left(q_a, x_i\right)\right)\right) = F\left(q_e^{'}, x_e^{'}\right), \Rightarrow \nu\left(F\left(q_a, x_i\right), F\left(q_r, x_r\right)\right) \in kerf$ and $\nu\left(F\left(q_r, x_r\right), F\left(q_a, x_i\right)\right) \in kerf$. Hence $kerf$ is an ideal of T.

4. Conclusion

In this paper, the theory of finite automata rings are studied. Finite automata rings extend the notion of finite automata and lead to the study of various properties of rings obtained by using finite automata. The concepts like commutative automata rings, zero-divisors of an automata ring, automata integral domain, sub- automata rings, ideal of an automata ring, and related substructures along with automata ring homomorphism are defined and studied. Some results of rings in terms of automata are established in this paper. Also, some properties of ring homomorphism in terms of automata are derived. Some examples of the finite automata which form automata rings and non automata rings are also presented. Different applications of finite automata to ring theory along with some generalizations to the wider context of groups can be investigated.

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