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PAIRWISE BICOMPACTNESS IN BISPACES AND PRODUCT OF BISPACES

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Abstract: In this paper we have studied the idea of K-pairwise bicompactness and FHP pairwise bicompactness in a bispace. Also we have investigated few results in the product of bispaces.

Keywords and Phrases: σ-space, bispace, FHP pairwise bicompactness, K-pairwise bicompactness, product bispace.

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1. Introduction

One of the important generalizations of the notion of a topological space is that of Alexandroff space [1] or σ-space or simply space where only countable union of open sets were taken to be open. The idea of a bitopological space was introduced by J. C. Kelly [4]in 1963. Later many works on topological properties were done in the setting of a bitopological space ([8, 9, 11] etc.). In 1968 Y. W Kim [5] introduced a special type of compactness called K-pairwise compactness in a bitopological space. The concept of compactness for bitopological space was also studied by Fletcher, Hoyle and Patty [3] which is known as FHP pairwise compactness. But the two definitions are not the same.

Here we have studied the ideas of K-pairwise compactness and FHP pairwise compactness in a bispace and their mutual relations. We have also investigated the validity of several results which are very much true in a bitopological space. Also we have studied the product of bispaces in a similar fashion as that of a bitopological space and have investigated some of its important properties.

2. Preliminaries

Definition 2.1. [1] *A set X is called an Alexandroff space or σ- space or simply space if it is chosen a system $\mathcal{F}$ of subsets of X, satisfying the following axioms*
(i) The intersection of countable number sets in $\mathcal{F}$ is a set in $\mathcal{F}$.
(ii) The union of finite number of sets from $\mathcal{F}$ is a set in $\mathcal{F}$.
(iii) The void set and X is a set in $\mathcal{F}$.

The members of $\mathcal{F}$ are called closed sets. A subset of X is called open if its complement is closed. So one may also rewrite the definition of a space in terms of open set axioms where the countable union of open sets and finite intersection of open sets are open together with the condition that X and the void set are open. We denote the collection of such open sets by $\mathcal{P}$ and the space by $(X, \mathcal{P})$. It is noted that $\mathcal{P}$ is not a topology in general as can be seen by taking $X = \mathbb{R}$, the set of real numbers and τ as the collection of all F_σ sets in $\mathbb{R}$.

Definition 2.2. [1] *To every set M we correlate its closure $\overline{M}$= the intersection of all closed sets containing M.*

Generally the closure of a set in a σ-space is not a closed set. We denote the closure of a set M in a space $(X, \mathcal{P})$ by $\mathcal{P}$-cl(M) or cl(M) or simply $\overline{M}$ when there is no confusion about $\mathcal{P}$. The idea of limit points, derived set, interior of a set etc. in a space are similar as in the case of a topological space which have been thoroughly discussed in [6].

Definition 2.3. [2] *Let $(X, \mathcal{P})$ be a space. A family of open sets B is said to form a base (open) for $\mathcal{P}$ if and only if every open set can be expressed as countable union of members of B.*

Theorem 2.1. [2] *A collection of subsets B of a set X forms an open base of a suitable space structure $\mathcal{P}$ of X if and only if*
1) the null set $\phi \in B$
2) X is the countable union of some sets belonging to B.
3) intersection of any two sets belonging to B is expressible as countable union of some sets belonging to B.

Definition 2.4. [7] *Let X be a non-empty set. If $\mathcal{P}$ and $\mathcal{Q}$ be two collection of subsets of X such that $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ are two spaces, then X is called a bispace.*

Definition 2.5. [7] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise T_1 if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

Definition 2.6. [7] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise Hausdorff if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U$, $y \in V$, $U \cap V = \phi$.*

Definition 2.7. [7] *In a bispace $(X, \mathcal{P}, \mathcal{Q})$, $\mathcal{P}$ is said to be regular with respect to $\mathcal{Q}$ if for any $x \in X$ and a $\mathcal{P}$-closed set F not containing x, there exist $U \in \mathcal{P}$, $V \in \mathcal{Q}$ such that $x \in U$, $F \subset V$, $U \cap V = \phi$. $(X, \mathcal{P}, \mathcal{Q})$ is said to be pairwise regular if $\mathcal{P}$ and $\mathcal{Q}$ are regular with respect to each other.*

Definition 2.8. [7] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be pairwise normal if for any $\mathcal{P}$-closed set F_1 and $\mathcal{Q}$-closed set F_2 satisfying $F_1 \cap F_2 = \phi$, there exist $G_1 \in \mathcal{P}$, $G_2 \in \mathcal{Q}$ such that $F_1 \subset G_2$, $F_2 \subset G_1$, $G_1 \cap G_2 = \phi$*

Definition 2.9. [2] *A function f mapping a bispace $(X, \mathcal{P}, \mathcal{Q})$ into a bispace $(X, \mathcal{P}^*, \mathcal{Q}^*)$ is said to be continuous if and only if induced mappings $f_1 : (X, \mathcal{P}) \longrightarrow (X, \mathcal{P}^*)$ and $f_2 : (X, \mathcal{Q}) \longrightarrow (X, \mathcal{Q}^*)$ are continuous.*

3. Pairwise Bicompactness

Definition 3.1. [7] *A space (or a set) is called bicompact if every open cover of it has a finite subcover.*

Definition 3.2. [7] *A cover B of $(X, \mathcal{P}, \mathcal{Q})$ is said to be pairwise open if $B \subset \mathcal{P} \cup \mathcal{Q}$ and B contains at least one nonempty member from each of $\mathcal{P}$ and $\mathcal{Q}$.*

Definition 3.3. [7] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be FHP pairwise bicompact if every pairwise open cover of it has a finite subcover.*

The idea of K-pairwise compactness was given by Kim [5] in a bitopological space. Here we use it in a bispace and discuss on some important results.

Definition 3.4. [5] *Let $(X, \mathcal{P}, \mathcal{Q})$ be a bispace and A and B be nonempty members of $\mathcal{Q}$ and $\mathcal{P}$ respectively. Now let us define*

$$\mathcal{P}(A) = \{\emptyset, X\} \cup \{U \cup A : U \in \mathcal{P}\}$$

$$\mathcal{Q}(B) = \{\emptyset, X\} \cup \{V \cup B : V \in \mathcal{Q}\}$$

It is easy to verify that $(X, \mathcal{P}(A))$ and $(X, \mathcal{Q}(B))$ are spaces.

The bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be K-pairwise bicompact if $(X, \mathcal{P}(A))$ and $(X, \mathcal{Q}(B))$ are bicompact for every non-empty members A and B of $\mathcal{Q}$ and $\mathcal{P}$ respectively.

Example 3.1. Let $X = \mathbb{R}$, the set of real numbers. We now consider the collections $\mathcal{P}$ and $\mathcal{Q}$ of subsets of X as follows:
$\mathcal{P} = \{\phi, X\} \cup \{$countable sub sets of irrational numbers$\}$, $\mathcal{Q} = \{\phi, X\} \cup \{$countable sub sets of X$\}$. Clearly $(X, \mathcal{P}, \mathcal{Q})$ is a bispace. It is easy to examine that $(X, \mathcal{P}(A))$ and $(X, \mathcal{Q}(B))$ are spaces which are not topological spaces for any non-empty members $A(\neq X)$ and $B(\neq X)$ of $\mathcal{Q}$ and $\mathcal{P}$ respectively.

Theorem 3.1. *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is K-pairwise bicompact if and only if each $\mathcal{P}$-closed set $C(\neq X)$ is $\mathcal{Q}$-bicompact and each $\mathcal{Q}$-closed set $E(\neq X)$ is $\mathcal{P}$-bicompact.*
Proof. First suppose that $(X, \mathcal{P}, \mathcal{Q})$ is K-pairwise bicompact. Since $C(\neq X)$ is $\mathcal{P}$-closed set, $X \setminus C$ is $\mathcal{P}$-open. So $Q(X \setminus C) = \{\emptyset, X\} \cup \{V \cup (X \setminus C) : V \in \mathcal{Q}\}$. Let $\mathcal{B}$ be a $\mathcal{Q}$-open cover of C. Then $\{V \cup (X \setminus C) : V \in \mathcal{B}\}$ is an $\mathcal{Q}(X \setminus C)$ open cover for X. Since $(X, \mathcal{P}, \mathcal{Q})$ is K-pairwise bicompact, there exists a finite sub-cover $\{V_1 \cup (X \setminus C), V_2 \cup (X \setminus C), \ldots, V_n \cup (X \setminus C)\}$ *(say)* of this open cover. So $\bigcup_{i=1}^{n}(V_i \cup (X \setminus C)) = (\bigcup_{i=1}^{n} V_i) \cup (X \setminus C) = X$. Hence we have $C \subset \bigcup_{i=1}^{n} V_i$. Therefore C is $\mathcal{Q}$-bicompact. Similarly we can show that every $\mathcal{Q}$-closed set is $\mathcal{P}$-bicompact.

Conversely, let each $\mathcal{P}$-closed set is $\mathcal{Q}$-bicompact and $\mathcal{Q}$-closed set is $\mathcal{P}$-bicompact. We show that $(X, \mathcal{P}, \mathcal{Q})$ is K-pairwise bicompact. Now let A be a $\mathcal{Q}$-open set and $A \neq \emptyset$. Consider the space $(X, \mathcal{P}(A))$. Since $X \setminus A$ is $\mathcal{Q}$-closed, it is $\mathcal{P}$-bicompact. Let $\{V_i \cup A\}_{i\in\Lambda}$ be a $\mathcal{P}(A)$-open cover for X where $V_i \in \mathcal{P}$. So we have $\bigcup_{i\in\Lambda}(V_i \cup A) = X$ i.e., $(\bigcup_{i\in\Lambda} V_i) \cup A = X$. Therefore $X \setminus A \subset \bigcup_{i\in\Lambda} V_i$. Now $X \setminus A$ is $\mathcal{P}$-bicompact and $\{V_i\}_{i\in\Lambda}$ is a $\mathcal{P}$-open cover for $X \setminus A$. So there exists a finite sub-cover $\{V_1, V_2, \ldots, V_n\}$(say) of this open cover. Therefore $\bigcup_{i=1}^{n}(V_i \cup A) = X$. So we see that $(X, \mathcal{P}(A))$ is bicompact. Similarly we can show $(X, \mathcal{Q}(B))$ is bicompact for any non-empty $\mathcal{P}$-open set B. Therefore $(X, \mathcal{P}, \mathcal{Q})$ is K-pairwise bicompact.

Note 3.1. *It follows that if $(X, \mathcal{P}, \mathcal{Q})$ is FHP pairwise bicompact then each $\mathcal{P}$-closed set $C(\neq X)$ is $\mathcal{Q}$-bicompact and each $\mathcal{Q}$-closed set $E(\neq X)$ is $\mathcal{P}$-bicompact. So it follows from the above theorem that if $(X, \mathcal{P}, \mathcal{Q})$ is FHP pairwise bicompact then it is K-pairwise bicompact. But the converse may not be true as shown in the following example.*

Example 3.2. Example of a K-pairwise bicompact bispace which is not FHP pairwise bicompact.
Let $X =$ the set of all irrational numbers. Let $\mathcal{P} = \{X, \emptyset,$ countable subsets of negative irrational numbers$\}$, $\mathcal{Q} = \{X, \emptyset,$ countable subsets of positive irrational numbers$\}$. Now consider the bispace $(X, \mathcal{P}, \mathcal{Q})$. We see that $(X, \mathcal{P}, \mathcal{Q})$ is K-pairwise bicompact as every $\mathcal{P}$-closed set is $\mathcal{Q}$-bicompact and every $\mathcal{Q}$-closed set is $\mathcal{P}$-bicompact. But $(X, \mathcal{P}, \mathcal{Q})$ is not FHP pairwise bicompact because the pair-

wise open cover $\{\{x\} : x \in \mathbb{R}\}$ of X has no finite subcover.

Definition 3.5. [8] *Two non empty subsets A and B in $(X, \mathcal{P}, \mathcal{Q})$ are said to be pairwise separated if there exists a $\mathcal{P}$-open set U and a $\mathcal{Q}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = B \cap U = \phi$ or there exists a $\mathcal{Q}$-open set U and a $\mathcal{P}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = B \cap U = \phi$.*

Definition 3.6. [8] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be connected if and only if X can not be expressed as the union of two non empty pairwise separated sets.*

Definition 3.7. [11] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be totally disconnected if for any two distinct points x and y there exists a disconnection $X = A|B$ with $x \in A$ and $y \in B$, where A is $\mathcal{P}$-open and B is $\mathcal{Q}$-open.*

Theorem 3.2. *Let $(X, \mathcal{P}, \mathcal{Q})$ be K-pairwise bicompact and totally disconnected (hence pairwise Hausdorff). Then $\mathcal{P}$ has a base whose members are $\mathcal{Q}$-closed and $\mathcal{Q}$ has a base whose members are $\mathcal{P}$-closed.*

Proof. We show that $\mathcal{P}$ has a base whose members are $\mathcal{Q}$-closed. Let $x \in X$ and G be a $\mathcal{P}$-open set containing x. We now find a $\mathcal{P}$-open set B which is $\mathcal{Q}$-closed such that $x \in B \subset G$. Now $X \setminus G$ is a $\mathcal{P}$-closed set. By K-pairwise bicompactness $X \setminus G$ is $\mathcal{Q}$-bicompact. Since $(X, \mathcal{P}, \mathcal{Q})$ is totally disconnected we can find for each $y \in X \setminus G$, a $\mathcal{Q}$-open $\mathcal{P}$-closed set E_y containing y but not containing x. Now varying the point y over $X \setminus G$ we can obtain a $\mathcal{Q}$-open cover $\{E_y : y \in X \setminus G\}$ of $X \setminus G$. Hence there exists a finite subcover of this open cover $\{E_{y_1}, E_{y_2}, \ldots, E_{y_n}\}$ (say) such that $X \setminus G \subset \bigcup_{i=1}^{n}\{E_{y_i}\} = E$ (say). Since each E_{y_i} is $\mathcal{Q}$-open and $\mathcal{P}$-closed so also is E. Now let us consider $B = X \setminus E$. It is clear to us that B is $\mathcal{P}$-open and $\mathcal{Q}$-closed set such that $x \in B \subset G$.

We now give the definition of bicompactness in the following alternative way.

Definition 3.8. [11] *Let $(X, \mathcal{P}, \mathcal{Q})$ be a bispace. A cover $\mathcal{U}$ for X is called $\mathcal{PQ}$-open cover if $\mathcal{U} \subset \mathcal{P} \cup \mathcal{Q}$.*
A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be $\mathcal{PQ}$ pairwise bicompact if every $\mathcal{PQ}$-open cover of X has a finite subcover.

Remark 3.1. *It is clear immediately that if $(X, \mathcal{P}, \mathcal{Q})$ is pairwise bicompact in the above sense, it is FHP pairwise bicompact and hence K-pairwise bicompact. Furthermore it follows that if $(X, \mathcal{P}, \mathcal{Q})$ is pairwise bicompact in the above sense, then $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ are both bicompact.*
If $(X, \mathcal{P}, \mathcal{Q})$ is pairwise Hausdorff and $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ are bicompact, we can say from the example 5 of Lahiri and Das [7] that it may not imply $\mathcal{P} = \mathcal{Q}$ in a bispace. In our next theorem we give an additional condition on $(X, \mathcal{P}, \mathcal{Q})$ for which it would imply $\mathcal{P} = \mathcal{Q}$.

Theorem 3.3. *Let the bispace $(X, \mathcal{P}, \mathcal{Q})$ be pairwise Hausdorff and $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ be bicompact. Also suppose that $(X, \mathcal{P}, \mathcal{Q})$ has the property that every $\mathcal{P}$-open set is $\mathcal{Q}$-bicompact and $\mathcal{Q}$-open set is $\mathcal{P}$-bicompact. Then $\mathcal{P} = \mathcal{Q}$.*

Proof. Suppose $\mathcal{P}$ is not a subset of $\mathcal{Q}$ and U be a $\mathcal{P}$-open set which is not $\mathcal{Q}$-open. Since $(X, \mathcal{P})$ is bicompact $X \setminus U$ is $\mathcal{P}$-bicompact. Now we claim that there exists a point $p \in U$ which is a $\mathcal{Q}$-limit point of $X \setminus U$. For, if we suppose there does not exist any such point then for every point $x \in U$ there is a $\mathcal{Q}$-open set V_x such that $V_x \cap (X \setminus U) = \emptyset$. Now consider the collection $\mathcal{B} = \{V_x : x \in U$ and $V_x \cap (X \setminus U) = \emptyset\}$. Then it is clear that $\mathcal{B}$ is a $\mathcal{Q}$-open cover for U and since U is $\mathcal{Q}$-bicompact, there exists a finite sub-cover say $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$. Now $U = \bigcup_{i=1}^{n} V_{x_i}$. Hence U becomes a $\mathcal{Q}$-open set, which is a contradiction to our supposition.

Since $(X, \mathcal{P}, \mathcal{Q})$ is pairwise Hausdorff, for each $x \in X \setminus U$ there is a $\mathcal{P}$-open set U_x and $\mathcal{Q}$-open set W_x such that $x \in U_x$, $p \in W_x$ and $U_x \cap W_x = \emptyset$. Now $\mathcal{U} = \{U_x : x \in X \setminus U\}$ is a $\mathcal{P}$-open cover of $X \setminus U$. So there exists a finite subcover of this $\mathcal{P}$-open cover say $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$. Let $W_{x_1}, W_{x_2}, \ldots, W_{x_n}$ be corresponding $\mathcal{Q}$-open sets such that $x_i \in U_{x_i}$, $p \in W_{x_i}$ and $U_{x_i} \cap W_{x_i} = \phi$ $i = 1, 2, 3, \ldots, n$. Now $W = \bigcap_{i=1}^{n} W_{x_i}$ is $\mathcal{Q}$-open, $p \in W$ and $W \cap (X \setminus U) = \emptyset$. But this is impossible because p is a $\mathcal{Q}$-limit point of $X \setminus U$. Therefore $\mathcal{P} \subset \mathcal{Q}$. Similarly we have $\mathcal{Q} \subset \mathcal{P}$. Hence $\mathcal{P} = \mathcal{Q}$.

Definition 3.9. [7] *A space $(X, \mathcal{P})$ is called locally bicompact if each point $x \in X$ has a bicompact neighborhood.*

The one point compactification of a space can be done in a similar fashion given bellow as in the case of a topological space.

Theorem 3.4. *Let $(X, \mathcal{P})$ be a space and $X^* = X \cup \{\infty\}$, where ∞ is an element not belonging to X. Now consider the collection $\mathcal{T}$ of subsets of X^* as follows:*
$U \in \mathcal{T}$ if and only if i) $U \cap X \in \mathcal{P}$ and
ii) Whenever $\infty \in U$, $(X \setminus U)$ is bicompact in $(X, \mathcal{P})$.
Then clearly $(X^*, \mathcal{T})$ is a space and $(X^*, \mathcal{T})$ is Hausdorff if and only if $(X, \mathcal{P})$ is locally bicompact Hausdorff space.

The proof is parallel as in the case of a topological space and so is omitted.

Definition 3.10. *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be embedded as a subspace in a bispace $(X^*, \mathcal{P}^*, \mathcal{Q}^*)$ if $(X, \mathcal{P}, \mathcal{Q})$ is homeomorphic to a subspace of $(X^*, \mathcal{P}^*, \mathcal{Q}^*)$.*

Theorem 3.5. *Every bispace $(X, \mathcal{P}, \mathcal{Q})$ can be embedded as a subspace in a $\mathcal{PQ}$ pairwise bicompact bispace $(X_\infty, \mathcal{P}_\infty, \mathcal{Q}_\infty)$ where $X_\infty = X \cup \{\infty\}, \infty \notin X$ and $\mathcal{P}_\infty$ and $\mathcal{Q}_\infty$ are defined as follows:*
$U \in \mathcal{P}_\infty$ if (i) $U \in \mathcal{P}$

or (ii) $U = V \cup \{\infty\}$, $X \setminus V$ *is bicompact and* $\mathcal{P}$*-closed.*
Similarly for $\mathcal{Q}_\infty$.

The proof is similar as in the case of a bitopological space [11] and so is omitted.

4. Product Bispaces

Let $(X, \mathcal{P})$ be a space. A family of subsets S of X is said to form a subbase of a space structure $\mathcal{P}$ if the collection of subsets obtained as the intersection of all finite sub-collections of S constitute a base for $\mathcal{P}$.

A collection of subsets S of a given set X forms a subbase of a suitable space structure of X if and only if
1)either $\phi \in S$ or ϕ is the intersection of a finite number of subsets belonging to S.
2) X is the countable union of subsets belonging to S.
Let $\{(X_i, \mathcal{P}_i) : i \in I\}$ be a family of spaces and let X denote the Cartesian product of sets $X_i, i \in I$. Let S be a family of subsets of X defined by
$S = \left\{p_i^{-1}(U_i) : U_i \in \mathcal{P}_i, i \in I\right\}$ where $p_i : X \longrightarrow X_i$ is the *i-th* projection mapping for each $i \in I$. Then as in the case of a topological space it can be easily checked that S forms a sub base of a space structure $\mathcal{P}$ on X. The space $(X, \mathcal{P})$ is called the product space of the given family of spaces.

Let $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$ be any family of bispaces. We construct two spaces $(\prod X_i, \mathcal{P})$ and $(\prod X_i, \mathcal{Q})$, where $(\prod X_i, \mathcal{P})$ is the Cartesian product of spaces $(X_i, \mathcal{P}_i)$'s determined by the subbase generated by the family of all sets of the form $p_i^{-1}(G)$, where i is any index and $G \in \mathcal{P}_i$, p_i is the *i-th* projection mapping.

Similarly $(\prod X_i, \mathcal{Q})$ is the Cartesian product of spaces $(X_i, \mathcal{Q}_i)$'s. The bispace $(\prod X_i, \mathcal{P}, \mathcal{Q})$ is called the product bispace generated by the family of bispaces $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$.

Theorem 4.1. *Let* $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$ *be an arbitrary family of nonempty bispaces. Then for each fixed i, the projection map* $p_k : (\prod X_i, \mathcal{P}, \mathcal{Q}) \longrightarrow (X_k, \mathcal{P}_k, \mathcal{Q}_k)$ *is a continuous surjection.*
Proof. Let $S_\mathcal{P}$ and $S_\mathcal{Q}$ be the subbases for $\mathcal{P}$ and $\mathcal{Q}$ respectively. Since $p_k^{-1}(U_k) \in S_\mathcal{P}$, for each $U_k \in \mathcal{P}_k$ and $p_k^{-1}(V_k) \in S_\mathcal{Q}$, for each $V_k \in \mathcal{Q}_k$, the proof follows directly from definition of continuous function.

Theorem 4.2. *Let* $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$ *be any nonempty family of bispaces and let*

$$f : (Y, \tau_1, \tau_2) \longrightarrow (\prod X_i, \mathcal{P}, \mathcal{Q})$$

be any map from an arbitrary bispace (Y, τ_1, τ_2) *to the product bispace* $(\prod X_i, \mathcal{P}, \mathcal{Q})$*. Then* f *is continuous if and only if* $p_k o f$ *is continuous for each index* k*.*
Proof. First suppose f is continuous. Now since composition of two continuous

functions is again continuous so it is obvious that $p_k of$ is continuous.

Conversely, suppose $p_k of : (Y, \tau_1, \tau_2) \longrightarrow (X_k, \mathcal{P}_k, \mathcal{Q}_k)$ is continuous. So the induced functions $p_k of_1 : (Y, \tau_1) \longrightarrow (X_k, \mathcal{P}_k)$ and $p_k of_2 : (Y, \tau_2) \longrightarrow (X_k, \mathcal{Q}_k)$ are continuous. We shall show that f is continuous. Now consider the induced function $f_1 : (Y, \tau_1) \longrightarrow (\prod X_i, \mathcal{P})$ and recall that a subbase of the product space $(\prod X_i, \mathcal{P})$ is defined by $\mathcal{B} = \{p_k^{-1}(G_\alpha) : k \in \Lambda, G_\alpha \in \mathcal{P}_k\}$.
Now if $B \in \mathcal{B}$, $f_1^{-1}(B) = f_1^{-1}[p_k^{-1}(G_\alpha)] = f_1^{-1} o p_k^{-1}(G_\alpha) = (p_k of_1)^{-1}(G_\alpha)$, where $G_\alpha \in \mathcal{P}_k$. Hence $f_1^{-1}(B)$ is open as $p_k of_1$ is continuous. Similarly we can show $f_2 : (Y, \tau_2) \longrightarrow (\prod X_i, \mathcal{Q})$ is continuous. Therefore $f : (Y, \tau_1, \tau_2) \longrightarrow (\prod X_i, \mathcal{P}, \mathcal{Q})$ is continuous.

Corollary: *Let (Y, τ_1, τ_2) be any given bispace and let $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$ be any family of bispaces. Suppose for each i there is a map $f_i : (Y, \tau_1, \tau_2) \longrightarrow (X_i, \mathcal{P}_i, \mathcal{Q}_i)$. Then f is continuous if and only if each f_i is continuous where $f : (Y, \tau_1, \tau_2) \longrightarrow (\prod X_i, \mathcal{P}, \mathcal{Q})$ is defined by $f(y) = \{f_i(y)\}$.*
Proof. We see that for all $y \in Y$, $(p_i of)(y) = f_i(y)$. Therefore $p_i of \equiv f_i$. So by the above theorem the result follows.

Theorem 4.3. *Let $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$ be a family of nonempty bispaces. Then $(\prod X_i, \mathcal{P}, \mathcal{Q})$ is pairwise Hausdorff if and only if $(X_i, \mathcal{P}_i, \mathcal{Q}_i)$ is pairwise Hausdorff for each i.*
The proof is parallel as in the case of a bitopological space [11] and so is omitted.

Theorem 4.4. *If $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$ is a family of nonempty bispaces such that $(\prod X_i, \mathcal{P}, \mathcal{Q})$ is K- pairwise bicompact, then each $(X_i, \mathcal{P}_i, \mathcal{Q}_i)$ is K- pairwise bicompact.*
Proof. The projection map $p_k : (\prod X_i, \mathcal{P}, \mathcal{Q}) \longrightarrow (X_k, \mathcal{P}_k, \mathcal{Q}_k)$ is a continuous surjection. Let $(\prod X_i, \mathcal{P}, \mathcal{Q})$ be K- pairwise bicompact. Now consider a $\mathcal{P}_k$-closed set G in $(X_k, \mathcal{P}_k, \mathcal{Q}_k)$. Therefore $X_k \setminus G$ is $\mathcal{P}_k$-open set. Now $p_k^{-1}(X_k \setminus G) = \prod X_i \setminus p_k^{-1}(G)$. Therefore $p_k^{-1}(G)$ is $\mathcal{P}$-closed and hence $\mathcal{Q}$-bicompact. Now let $\{A_i\}$ be a $\mathcal{Q}_k$-open cover for G. Then $\{p_k^{-1}(A_i)\}$ be a $\mathcal{Q}$-open cover for $p_k^{-1}(G)$. Since $p_k^{-1}(G)$ is $\mathcal{Q}$-bicompact so we have a finite sub-cover $\{p_k^{-1}(A_1), p_k^{-1}(A_2), \ldots, p_k^{-1}(A_n)\}$ (say). Therefore $p_k^{-1}(G) \subset \bigcup_{i=1}^{n} p_k^{-1}(A_i)$. Now $A_1, A_2 \ldots, A_n$ becomes a finite subcover for G. Hence G is $\mathcal{Q}_k$-bicompact.
Similarly in $(X_k, \mathcal{P}_k, \mathcal{Q}_k)$ we can show every $\mathcal{Q}_k$-closed set is $\mathcal{P}_k$-bicompact. Hence $(X_k, \mathcal{P}_k, \mathcal{Q}_k)$ is K- pairwise bicompact.

Theorem 4.5. *If $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i)\}$ is a family of nonempty bispaces such that $(\prod X_i, \mathcal{P}, \mathcal{Q})$ is FHP pairwise bicompact, then each $(X_i, \mathcal{P}_i, \mathcal{Q}_i)$ is FHP pairwise bicompact.*
The proof is parallel as in the case of a bitopological space [11] and so is omitted.

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