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MAPPING PROPERTIES OF HYPERGEOMETRIC TRANSFORMS ON CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract: The purpose of the present paper is to obtain certain sufficient condition for analytic function defined by a generalized Hypergeometric operator for the class $UCD(\beta)$. Relevant connection of the results presented here with various well-known results are briefly indicated.

Keywords and Phrases: Analytic functions, Univalent functions, Hypergeometric functions.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$.

As usual, we denote by S the subclass of A consisting of functions which are also univalent in U. A function $f \in A$ is said to be starlike of order $\alpha(0 \le \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U).$$

The class of this function is denoted by $S^*(\alpha)$. We also write $S^*(0) = S^*$, where S^* denote the class of functions $f \in A$ that are starlike in U with respect to the origin. A function $f \in A$ is said to be convex of order $\alpha(0 \le \alpha < 1)$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in U).$$

This class is dented by $K(\alpha)$. Further, $K = K(0)$, the well known standard class of convex functions. It is an established fact that

$$f \in K(\alpha) \Leftrightarrow zf' \in S^*(\alpha).$$

A function $f \in A$ is said to be in the class UCV of uniformly convex functions in U if and only if it has property that, for every circular arc γ contained in unit disk U, with center ζ also in U, the image curve $f(\gamma)$ is convex arc. The function class UCV was introduced by Goodman [6]. Furthermore, we denote by $k-UCV$ and $k-ST$, $(0 \le k < \infty)$, two interesting subclasses of S consisting respectively of functions which are $k-$uniformly convex and k-starlike in U. Namely, we have for $(0 \le k < \infty)$

$$k - UCV = \left\{ f \in S : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, (z \in U)\right\}$$

and

$$k - ST = \left\{ f \in S : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, (z \in U)\right\}.$$

The class $k - UCV$ was introduced by Kanas and Wisniowska [8], where its geometric definition and connections with conic domains were considered. The class $k - ST$ was investigated in [9]. In fact, it is related to the class $k - UCV$ by by means of the well-known Alexzander equivalence between the usual classes of convex and starlike functions (see also the work of Kanas and Srivastava [10] for further developments involving each of the classes $k - UCV$ and $k - ST$). In particular, when $k = 1$, we obtain $1 - UCV = UCV$ and $1 - ST = SP$, where UCV and SP are the familiar classes of uniformly convex functions and parabolic starlike functions in U, respectively (see for detail study [6]). In fact, by making use of a certain fractional calculus operator, Srivastava and Mishra [22] presented a systematic and unified study of the classes UCV and SP. A function $f \in A$ is said to be in the class $UCD(\beta), \beta \in R$, if

$$\operatorname{Re}\left(f^{'}(z)\right) \geqslant \left|zf^{''}(z)\right| (z \in U).$$

The class $UCD(\beta)$ is introduced Breaz [1]. A function $f \in A$ is said to be in the class $R^{\tau}(A, B), (\tau \in C \setminus 0, -1 \leq B < A \leq 1)$, if it satisfied the inequality

$$\left|\frac{f^{'}(z) - 1}{(A - B)\tau - B\left[f'(z) - 1\right]}\right| < 1 \, (z \in U).$$

The class $R^{\tau}(A, B)$ was introduced earlier by Dixit and Pal [4]. Two of the many interesting subclasses of the class $R^{\tau}(A, B)$ are worthy of mention here. First of all, by setting

$$\tau = e^{i\eta} \cos\cos\eta \left(-\frac{\pi}{2} < \eta < \frac{\pi}{2}\right), A = 1 - 2\beta \, (0 \leqslant \beta < 1)$$

and $B = -1$, the class $R^{\tau}(A, B)$ reduces essentially to the class $R_{\eta}(\beta)$ introduced and studied by Ponnusamy and Ronning [13], where

$$R_{\eta}(\beta) = \left\{f \in \mathcal{A} : \operatorname{Re}\left(e^{i\eta}\left(f^{'}(z) - \beta\right)\right) > 0 \left(z \in U; -\frac{\pi}{2} < \eta < \frac{\pi}{2}, 0 \leqslant \beta < 1\right)\right\}.$$

Second, if we put $\tau = 1, A = \beta$ and $B = -\beta \, (0 < \beta \leqslant 1)$, we obtain the class of functions $f \in A$ satisfying the inequality

$$\left|\frac{f^{'}(z) - 1}{f^{'}(z) + 1}\right| < \beta \, (z \in U; 0 < \beta \leqslant 1)$$

Which was studied by (among others) Padmanabham [12] and Caplinger and Causey [2], (see the work in [16] also). The Gaussian hypergeometric function $F(a, b; c; z)$ given by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \, (z \in U) \tag{1.2}$$

is the solution of the homogeneous hypergeometric differential equation

$$z(1-z)w''(z)+[c-(a+b+1)z]w'(z)-abw(z)=0$$

and has rich applications in various fields such as conformal mappings, quasiconformal theory, continued fractions and so on.

Here, a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \ldots\ldots..$, $(a)_0 = 1$ for $a \neq 0$, and for each positive integer n $(a)_n = a(a+1)(a+2)\ldots.(a+n-1)$ is the Pochhammer symbol. In the case of $c = -k, k = 0, 1, 2, \ldots, F(a, b; c; z)$ is defined if $a = -j orb = -j$ where $j \leq k$. In this situation, $F(a, b; c; z)$ becomes a polynomial of degree j with respect to z result regarding $F(a, b; c; z)$ when $\Re(c-a-b)$ is positive, zero or negative are abundant in the literature. In particular when $\Re(c-a-b) > 0$, the function is bounded. This, and zero balanced case $\Re(c-a-b) = 0$ are discussed in detail by many authors (see [14]). The hypergeometric function $F(a, b; c; z)$ has been studied extensively by various authors and plays an important role in Geometric Function Theory. It is useful in unifying various functions by giving appropriate values to the parameters a, b and c. We refer to [3, 5, 11, 13] [15]-[21] and references therein for some important results.

For function $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U. \tag{1.3}$$

For $f \in A$, we recall the operator $I_(a, b, c)(f)$ of Hohlov [7] which maps A into itself defined by means of the Hadamard product as

$$I_{a,b,c}(f)(z) = I(a, b; c; \lambda, \mu, z) * f(z). \tag{1.4}$$

Therefore, for a function defined by (1.1), we have In present paper we introduce a operator $I(a, b; c; \lambda, \mu, z)$ as follows

$$I(a, b; c; \lambda, \mu, z) = \lambda z F(a, b; c; z) + \mu z(zF(a, b; c; z))' + (1-\lambda-\mu)\int\limits_0^z F(a, b; c; t)\,dt$$

using the series representation,

$$\begin{aligned} I(a, b; c; \lambda, \mu, z) &= z + \sum_{n=2}^{\infty}\left(\lambda + \mu n + \frac{1-\lambda-\mu}{n}\right)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n \\ &= {}_2I_1(a, b; c; \lambda, \mu, z). \end{aligned} \tag{1.5}$$

If we put $\mu = 1 - \lambda$ then it reduces to

$$I(a,b;c;\lambda,\mu,z) = \lambda z F(a,b;c;z) + (1-\lambda)(zF(a,b;c;z))',$$

Studied by Sharma *et al.* [18]. In the present paper, we obtain certain inclusion relation involving the classes $k-UCV$, $k-ST$, UCV, SP. Particularly $\lambda = 1, \mu = 0$ we obtain the corresponding result of Shivasubramanian and Sokol [22].

Lemma 1.1. ([5]) *A function $f \in A$ is in the class $UCD(\beta)$ if*

$$\sum_{n=2}^{\infty} n(1+\beta(n-1))|a_n| \leqslant 1. \tag{1.6}$$

Lemma 1.2. ([6]) *If $f \in R^{\tau}(A,B)$ is of the form (1.1), then*

$$|a_n| \leqslant (A-B)\frac{|\tau|}{n}, n \in N\backslash\{1\}. \tag{1.7}$$

The result is sharp. Let us denote (see [8], [9]*)*

$$P_1(k) = \begin{cases} \frac{8(arc\cos\cos k)^2}{\pi^2(1-k^2)}, & for 0 \leqslant k < 1 \\ \frac{8}{\pi^2}, & for\ k = 1 \\ \frac{\pi^2}{4\sqrt{t}(1+t)(k^2-1)K^2(t)} & for\ k > 1, \end{cases} \tag{1.8}$$

where $t \in (0,1)$ is determined by $k = \cosh\cosh(\pi K'(t)/[4K(t)])$, K is Legender's complete elliptic integral of the first kind

The domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic when $k = 1$ and a right half plane when $k = 0$. If p is an analytic function with $p(0) = 1$ which maps the unit disc U conformally onto the region Ω_k, then $P_1(k) = p'(0).P_1(k)$ is a strictly decreasing function of the variable k and its values are included in the interval (0,2]. Let $f \in A$ be of the form (1.1). If $f \in k-ST$, then the following coefficient inequalities hold true (cf. [9]):

$$|a_n| \leqslant \frac{(P_1(k))_{n-1}}{(n-1)!}, n \in N\backslash\{1\}. \tag{1.9}$$

Let us also denote ${}_3I_2(a,b,d;c,e;\lambda,\mu,z)$ by

$${}_3I_2(a,b,d;c,e;\lambda,\mu,z) = z + \sum_{n=2}^{\infty}\left(\lambda + \mu n + \frac{1-\lambda-\mu}{n}\right)\frac{(a)_{n-1}(b)_{n-1}(d)_{n-1}}{(c)_{n-1}(e)_{n-1}(1)_{n-1}} z^n.$$

In this paper, we estimate certain inclusion relation involving the class $k-ST$ and $UCD(\beta)$.

2. Main Results

We begin deriving a sufficient condition for the functions belonging to the class $G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$. This result is contained in the following.

Theorem 2.1. *Let $a,b\in C\backslash 0$. Also, let c be a real number such that $c>|a|+|b|+2$. Let $f\in A$ and of the form (1.1). If the hypergeometric inequality*

$$\frac{\Gamma(c)\,\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(c-|a|-|b|-3)(c-|a|-|b|-2)(c-|a|-|b|-1)\right.$$
$$+(\lambda+\beta+2\mu+\beta\lambda+3\mu\beta)\,|ab|\,(c-|a|-|b|-3)(c-|a|-|b|-2)$$
$$+(\mu+\beta\lambda+5\mu\beta)\,|ab|\,(1+|a|)(1+|b|)(c-|a|-|b|-3)$$
$$\left.+\mu\beta\,|ab|\,(1+|a|)(1+|b|)(2+|a|)(2+|b|)\right]\leq 2$$

is satisfied, then $I(a,b;c;\lambda,\mu,z)\in UCD(\beta)$.

Proof. The function $I(a,b;c;\lambda,\mu,z)$ has the series representation given by

$$I(a,b;c;\lambda,\mu,z)=z+\sum_{n=2}^{\infty}\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n.$$

In view of Lemma 1.1, it suffices to show that

$$S(a,b;c;\lambda,\mu,\beta)=\sum_{n=2}^{\infty}n\,(1+\beta\,(n-1))\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|\leqslant 1 \tag{2.1}$$

From the fact that $|(a)_n|\leqslant(|a|)_n$, we observe that, since c is real and positive, under the hypothesis

$$S(a,b;c;\lambda,\mu,\beta)\leqslant\sum_{n=2}^{\infty}n\,(1+\beta\,(n-1))\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$=\sum_{n=2}^{\infty}(1+\beta\,(n-1))\left(\lambda n+\mu n^2+1-\lambda-\mu\right)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$=\sum_{n=2}^{\infty}\left\{\mu\beta n^3+(\beta\lambda+\mu-\mu\beta)\,n^2+(\lambda+\beta-\beta\lambda-\mu\beta)\,n\right.$$
$$\left.+(1-\lambda-\mu-\beta+\beta\lambda+\mu\beta)\right\}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}.$$

Writing $\{\mu\beta n^3 + (\beta\lambda + \mu - \mu\beta) n^2 + (\lambda + \beta - \beta\lambda - \mu\beta) n + (1 - \lambda - \mu - \beta + \beta\lambda + \mu\beta)\}$ as,
$[1 + (\lambda + 2\mu + \beta + \beta\lambda + 3\mu\beta)(n-1) + (\mu + \beta\lambda + 5\mu\beta)(n-1)(n-2) + \mu\beta(n-1)(n-2)(n-3)]$ we get

$$\begin{aligned} S(a,b;c;\lambda,\mu,\beta) \leqslant & \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ & + (\lambda + 2\mu + \beta + \beta\lambda + 3\mu\beta) \sum_{n=2}^{\infty} (n-1) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ & + (\mu + \beta\lambda + 5\mu\beta) \sum_{n=2}^{\infty} (n-1)(n-2) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ & + \mu\beta \sum_{n=2}^{\infty} (n-1)(n-2)(n-3) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \end{aligned}$$

Using the fact that

$$(a)_n = a(a+1)_{n-1}, \tag{2.2}$$

It is so easy to see that,

$$\begin{aligned} S(a,b;c;\lambda,\mu,\beta) \leqslant & \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ & + (\lambda + 2\mu + \beta + \beta\lambda + 3\mu\beta) \frac{|ab|}{c} \sum_{n=2}^{\infty} \frac{(1+|a|)_{n-2}(1+|b|)_{n-2}}{(1+c)_{n-2}(1)_{n-2}} \\ & + (\mu + \beta\lambda + 5\mu\beta) \frac{|ab|(1+|a|)(1+|b|)}{c(1+c)} \sum_{n=3}^{\infty} \frac{(2+|a|)_{n-3}(2+|b|)_{n-3}}{(2+c)_{n-3}(1)_{n-3}} \\ & + \mu\beta \frac{|ab|(1+|a|)(1+|b|)(2+|a|)(2+|b|)}{c(1+c)(2+c)} \sum_{n=4}^{\infty} \frac{(3+|a|)_{n-4}(3+|b|)_{n-4}}{(3+c)_{n-4}(1)_{n-4}}. \end{aligned}$$

From (1.2), we have

$$\begin{aligned} S(a,b;c;\lambda,\mu,\beta) \leqslant & I(|a|,|b|;c;\lambda,\mu,1) - 1 \\ & + (\lambda + 2\mu + \beta + \beta\lambda + 3\mu\beta) \frac{|ab|}{c} I(1+|a|,1+|b|;1+c;\lambda,\mu,1) \\ & + (\mu + \beta\lambda + 5\mu\beta) \frac{|ab|(1+|a|)(1+|b|)}{c(1+c)} I(2+|a|,2+|b|;2+c;\lambda,\mu,1) \\ & + \mu\beta \frac{|ab|(1+|a|)(1+|b|)(2+|a|)(2+|b|)}{c(1+c)(2+c)} I(3+|a|,3+|b|;3+c;\lambda,\mu,1). \end{aligned}$$

Applying Gauss summation theorem

$$F\left(a,b;c;1\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma\left(c\right)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$

for $\Re(c-a-b)>0$, last equation reduces as

$$\begin{aligned}
&S\left(a,b;c;\lambda,\mu,\beta\right)=\frac{\Gamma\left(c\right)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\\
&+\left(\lambda+2\mu+\beta+\beta\lambda+3\mu\beta\right)\frac{|ab|}{c}\frac{\Gamma\left(1+c\right)\Gamma(1+c-1-|a|-1-|b|)}{\Gamma(1+c-1-|a|)\Gamma(1+c-1-|b|)}\\
&+\left(\mu+\beta\lambda+5\mu\beta\right)\frac{|ab|\left(1+|a|\right)\left(1+|b|\right)}{c(1+c)}\frac{\Gamma\left(2+c\right)\Gamma(2+c-2-|a|-2-|b|)}{\Gamma(2+c-2-|a|)\Gamma(2+c-2-|b|)}\\
&+\mu\beta\frac{|ab|\left(1+|a|\right)\left(1+|b|\right)\left(2+|a|\right)\left(2+|b|\right)}{c\left(1+c\right)\left(2+c\right)}\frac{\Gamma\left(3+c\right)\Gamma(3+c-3-|a|-3-|b|)}{\Gamma\left(3+c-3-|a|\right)\Gamma(3+c-3-|b|)}\\
&=\frac{\Gamma\left(c\right)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\\
&+\left(\lambda+2\mu+\beta+\beta\lambda+3\mu\beta\right)\frac{|ab|}{c}\frac{\Gamma\left(1+c\right)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&+\left(\mu+\beta\lambda+5\mu\beta\right)\frac{|ab|\left(1+|a|\right)\left(1+|b|\right)}{c(1+c)}\frac{\Gamma\left(2+c\right)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&+\mu\beta\frac{|ab|\left(1+|a|\right)\left(1+|b|\right)\left(2+|a|\right)\left(2+|b|\right)}{c\left(1+c\right)\left(2+c\right)}\frac{\Gamma\left(3+c\right)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&\Rightarrow\frac{\Gamma\left(c\right)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\\
&+\left(\lambda+2\mu+\beta+\beta\lambda+3\mu\beta\right)\frac{|ab|}{c}\frac{\Gamma\left(1+c\right)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&+\left(\mu+\beta\lambda+5\mu\beta\right)\frac{|ab|\left(1+|a|\right)\left(1+|b|\right)}{c(1+c)}\frac{\Gamma\left(2+c\right)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&+\mu\beta\frac{|ab|\left(1+|a|\right)\left(1+|b|\right)\left(2+|a|\right)\left(2+|b|\right)}{c\left(1+c\right)\left(2+c\right)}\frac{\Gamma\left(3+c\right)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&\leqslant 1
\end{aligned}$$

$$\Rightarrow\frac{\Gamma\left(c\right)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\left(c-|a|-|b|-3\right)\left(c-|a|-|b|-2\right)\left(c-|a|-|b|-1\right)\right.$$
$$+\left(\lambda+\beta+2\mu+\beta\lambda+3\mu\beta\right)|ab|\left(c-|a|-|b|-3\right)\left(c-|a|-|b|-2\right)+\left(\mu+\beta\lambda+5\mu\beta\right)$$

$$|ab|\,(1+|a|)\,(1+|b|)\,(c-|a|-|b|-3)+\mu\beta\,|ab|\,(1+|a|)\,(1+|b|)\,(2+|a|)\,(2+|b|)] \leqslant 2.$$

In the special case when $\lambda = 1$ and $\mu = 0$, Theorem 2.1 immediately yields a following result of Sivasubramanian *et al.* [22].

Corollary 2.2. *Let $a, b \in C \setminus 0$. Also, let c be a real number such that $c > |a| + |b| + 2$. Let $f \in A$ and of the form (1.1). If the hypergeometric inequality*

$$\frac{\Gamma(c)\,\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}\,[(c-|a|-|b|-2)\,(c-|a|-|b|-1)$$

$$+(1+2\beta)\,|ab|\,(c-|a|-|b|-2)+\beta\,|ab|\,(1+|a|)\,(1+|b|)+] \leqslant 2$$

is satisfied, then $zF(a,b;c;z) \in UCD(\beta)$.

Theorem 2.3. *Let $a, b \in C\backslash 0$. Also, let c be a real number such that $c > |a|+|b|+1$. If $f \in R^{\tau}(A,B)$, and if the inequality*

$$\frac{\Gamma(c)\,\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\,\Gamma(c-|b|)}\left\{\begin{array}{c}(\lambda+\mu+\beta-\mu\beta-\beta\lambda)\,[(c-|a|-|b|-1)\,(c-|a|-|b|-2)]\\ +(\mu+\beta\lambda+2\mu\beta)\,|ab|\,(c-|a|-|b|-2)+\mu\beta\,|ab|\,(1+|a|)\,(1+|b|)\\ +\frac{(1-\lambda-\mu)(1-\beta)[(c-|a|-|b|)(c-|a|-|b|-1)(c-|a|-|b|-2)]}{(|a|-1)(|b|-1)}\end{array}\right\}$$

$$\leqslant \frac{1}{(A-B)\,|\tau|}+\frac{(1-\lambda-\mu)\,(1-\beta)\,(c-1)}{(|a|-1)\,(|b|-1)}+1. \tag{2.3}$$

is satisfied, then $I\,(a,b;c;\lambda,\mu,z) \in UCD\,(\beta)$.

Proof. Let f be of the form (1.1) belongs to the class $f \in R^{\tau}(A,B)$. By virtue of Lemma 1.1 it is suffices to show that

$$\sum_{n=2}^{\infty} n\,(1+\beta\,(n-1))\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leqslant 1. \tag{2.4}$$

Taking in to account of the inequality (1.7) and the relation $|\,(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$\sum_{n=2}^{\infty} n\,(1+\beta\,(n-1))\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|$$

$$\leqslant (A-B)\,|\tau|\sum_{n=2}^{\infty}(1+\beta\,(n-1))\left(\frac{\lambda n+\mu n^2+1-\lambda-\mu}{n}\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|$$

$$= (A-B)\left|\tau\right|\sum_{n=2}^{\infty}\left[(\lambda+\beta-\mu\beta-2\beta\lambda)+(\beta\lambda+\mu)\,n+\mu\beta n\,(n-1)+(1-\lambda-\mu)\,(1-\beta)\,\frac{1}{n}\right]$$
$$\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$
$$= (A-B)\left|\tau\right|\sum_{n=2}^{\infty}\left[(\lambda+\beta-\mu\beta-2\beta\lambda+\beta\lambda+\mu)+(\beta\lambda+\mu)\,(n-1)+\mu\beta\,(n-1)\,(n-2)\right.$$
$$\left.+2\mu\beta\,(n-1)+(1-\lambda-\mu)\,(1-\beta)\,\frac{1}{n}\right]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$
$$= (A-B)\left|\tau\right|\sum_{n=2}^{\infty}\left[(\lambda+\mu+\beta-\mu\beta-\beta\lambda)+(\mu+\beta\lambda+2\mu\beta)\,(n-1)+\mu\beta\,(n-1)\,(n-2)\right.$$
$$\left.+(1-\lambda-\mu)\,(1-\beta)\,\frac{1}{n}\right]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$
$$= (A-B)\left|\tau\right|(\lambda+\mu+\beta-\mu\beta-\beta\lambda)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$+(A-B)\left|\tau\right|(\mu+\beta\lambda+2\mu\beta)\sum_{n=2}^{\infty}(n-1)\,\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$+(A-B)\left|\tau\right|\mu\beta\sum_{n=2}^{\infty}(n-1)\,(n-2)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$+(A-B)\left|\tau\right|(1-\lambda-\mu)\,(1-\beta)\sum_{n=2}^{\infty}\frac{1}{n}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$= (A-B)\left|\tau\right|(\lambda+\mu+\beta-\mu\beta-\beta\lambda)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$+(A-B)\left|\tau\right|(\mu+\beta\lambda+2\mu\beta)\,\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(1+|a|)_{n-2}(1+|b|)_{n-2}}{(1+c)_{n-2}(1)_{n-2}}$$
$$+(A-B)\left|\tau\right|\mu\beta\frac{|ab|\,(1+|a|)\,(1+|b|)}{c(1+c)}\sum_{n=3}^{\infty}\frac{(2+|a|)_{n-3}(2+|b|)_{n-3}}{(2+c)_{n-3}(1)_{n-3}}$$
$$+(A-B)\left|\tau\right|(1-\lambda-\mu)\,(1-\beta)\,\frac{(c-1)}{(|a|-1)\,(|b|-1)}\sum_{n=2}^{\infty}\frac{(|a|-1)_{n}(|b|-1)_{n}}{(c-1)_{n}(1)_{n}}$$

Using (1.2) last equation reduces as

$$\sum_{n=2}^{\infty}n\,(1+\beta\,(n-1))\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$
$$\leqslant (A-B)\left|\tau\right|(\lambda+\mu+\beta-\mu\beta-\beta\lambda)\left[I\left(|a|\,,|b|\,;c;\lambda,\mu,1\right)-1\right]$$

$$+ (A-B)|\tau|(\mu+\beta\lambda+2\mu\beta)\frac{|ab|}{c}I(1+|a|,1+|b|;1+c;\lambda,\mu,1)$$
$$+ (A-B)|\tau|\mu\beta\frac{|ab|(1+|a|)(1+|b|)}{c(1+c)}I(2+|a|,2+|b|;2+c;\lambda,\mu,1)$$
$$+ (A-B)|\tau|(1-\lambda-\mu)(1-\beta)\frac{(c-1)}{(|a|-1)(|b|-1)}$$
$$\left[I(|a|-1,|b|-1;c-1;\lambda,\mu,1)-1-\frac{(|a|-1)(|b|-1)}{(c-1)}.\right]$$

Now, applying Gauss summation theorem, we get

$$= (A-B)|\tau|\left\{(\lambda+\mu+\beta-\mu\beta-\beta\lambda)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]\right.$$
$$+ (\mu+\beta\lambda+2\mu\beta)\frac{|ab|}{c}\frac{\Gamma(1+c)\Gamma(1+c-1-|a|-1-|b|)}{\Gamma(1+c-1-|a|)\Gamma(1+c-1-|b|)}$$
$$+ \mu\beta\frac{|ab|(1+|a|)(1+|b|)}{c(1+c)}\frac{\Gamma(2+c)\Gamma(2+c-2-|a|-2-|b|)}{\Gamma(2+c-2-|a|)\Gamma(2+c-2-|b|)}$$
$$+ (1-\lambda-\mu)(1-\beta)\frac{(c-1)}{(|a|-1)(|b|-1)}\left[\frac{\Gamma(c-1)\Gamma(c-1-|a|+1-|b|+1)}{\Gamma(c-1-|a|+1)\Gamma(c-1-|b|+1)}\right.$$
$$\left.\left.-1-\frac{(|a|-1)(|b|-1)}{(c-1)}\right]\right\}$$
$$= (A-B)|\tau|\left\{(\lambda+\mu+\beta-\mu\beta-\beta\lambda)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]\right.$$
$$+ (\mu+\beta\lambda+2\mu\beta)\frac{|ab|}{c}\frac{\Gamma(1+c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$+ \mu\beta\frac{|ab|(1+|a|)(1+|b|)}{c(1+c)}\frac{\Gamma(2+c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$+ (1-\lambda-\mu)(1-\beta)\frac{(c-1)}{(|a|-1)(|b|-1)}\left[\frac{\Gamma(c-1)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right.$$
$$\left.\left.-1-\frac{(|a|-1)(|b|-1)}{(c-1)}\right]\right\}$$
$$= (A-B)|\tau|\frac{\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$\left\{\begin{array}{c}(\lambda+\mu+\beta-\mu\beta-\beta\lambda)[(c-|a|-|b|-1)(c-|a|-|b|-2)]\\ +(\mu+\beta\lambda+2\mu\beta)|ab|(c-|a|-|b|-2)+\mu\beta|ab|(1+|a|)(1+|b|)\\ +\frac{(1-\lambda-\mu)(1-\beta)[(c-|a|-|b|)(c-|a|-|b|-1)(c-|a|-|b|-2)]}{(|a|-1)(|b|-1)}\end{array}\right\}$$

$$-(A-B)|\tau|\left\{(\lambda+\mu+\beta-\mu\beta-\beta\lambda)+\frac{(1-\lambda-\mu)(1-\beta)(c-1)}{(|a|-1)(|b|-1)}+(1-\lambda-\mu)(1-\beta)\right\}$$

$$=(A-B)|\tau|\frac{\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$

$$\left\{\begin{array}{c}(\lambda+\mu+\beta-\mu\beta-\beta\lambda)[(c-|a|-|b|-1)(c-|a|-|b|-2)]\\+(\mu+\beta\lambda+2\mu\beta)|ab|(c-|a|-|b|-2)+\mu\beta|ab|(1+|a|)(1+|b|)\\+\frac{(1-\lambda-\mu)(1-\beta)[(c-|a|-|b|)(c-|a|-|b|-1)(c-|a|-|b|-2)]}{(|a|-1)(|b|-1)}\end{array}\right\}$$

$$-(A-B)|\tau|\left\{(\lambda+\mu+\beta-\mu\beta-\beta\lambda)+\frac{(1-\lambda-\mu)(1-\beta)(c-1)}{(|a|-1)(|b|-1)}\right.$$

$$\left.+(1-\lambda-\mu-\beta+\mu\beta+\beta\lambda)\right\}.$$

$$\Rightarrow\frac{\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$

$$\left\{\begin{array}{c}(\lambda+\mu+\beta-\mu\beta-\beta\lambda)[(c-|a|-|b|-1)(c-|a|-|b|-2)]\\+(\mu+\beta\lambda+2\mu\beta)|ab|(c-|a|-|b|-2)+\mu\beta|ab|(1+|a|)(1+|b|)\\+\frac{(1-\lambda-\mu)(1-\beta)[(c-|a|-|b|)(c-|a|-|b|-1)(c-|a|-|b|-2)]}{(|a|-1)(|b|-1)}\end{array}\right\}$$

$$\leqslant\frac{1}{(A-B)|\tau|}+\frac{(1-\lambda-\mu)(1-\beta)(c-1)}{(|a|-1)(|b|-1)}+1.$$

Repeating the above reasoning for $|b|=|a|$, we can improve the assertion of Theorem 2.3 as follows.

Corollary 2.4. *Let $a,b\in C\setminus 0$. Suppose that $|b|=|a|$. Further, let c be a real number such that $c>2|a|+1$. If $f\in R^{\tau}(A,B)$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-2|a|-2)}{[\Gamma(c-|a|)]^2}\left\{\begin{array}{c}(\lambda+\mu+\beta-\mu\beta-\beta\lambda)[(c-2|a|-1)(c-2|a|-2)]\\+(\mu+\beta\lambda+2\mu\beta)|a|^2(c-2|a|-2)+\mu\beta|a|^2(1+|a|)^2\\+\frac{(1-\lambda-\mu)(1-\beta)[(c-2|a|)(c-2|a|-1)(c-2|a|-2)]}{(|a|-1)^2}\end{array}\right\}$$

$$\leqslant\frac{1}{(A-B)|\tau|}+\frac{(1-\lambda-\mu)(1-\beta)(c-1)}{(|a|-1)^2}+1 \qquad (2.5)$$

is satisfied, then $I_(a,a,c)(f)\in UCD(\beta)$.

Particularly when $\lambda=1$ and $\mu=0$, Theorem 2.3 yields a following result of Sivasubramanian et al [22].

Corollary 2.5. *Let $a,b\in C\setminus 0$. Also, let c be a real number such that $c>|a|+|b|+1$. If $f\in R^{\tau}(A,B)$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}[(c-|a|-|b|-1)+\beta|ab|]\leqslant\frac{1}{(A-B)|\tau|}+1. \qquad (2.6)$$

is satisfied, then $I(a,b;c;\lambda,\mu,z)\in UCD(\beta)$.

Proof. Let f be of the form (1.1) belongs to the class $R^{\tau}(A,B)$. By virtue of

Lemma 1.1 it is suffices to show that

$$\sum_{n=2}^{\infty} n\left(1+\beta\left(n-1\right)\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leqslant 1.$$

Taking in to account of the inequality (1.7) and the relation $\left|(a)_{n-1}\right| \leqslant (|a|)_{n-1}$, we deduce that

$$\sum_{n=2}^{\infty} n\left(1+\beta\left(n-1\right)\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leqslant (A-B)\left|\tau\right|\sum_{n=2}^{\infty}\left(1+\beta\left(n-1\right)\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$

$$= (A-B)\left|\tau\right|\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + \beta\left(A-B\right)\left|\tau\right|\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

By applying Gauss summation Theorem we get required result.

In the special case when $b=1, \lambda=1$ and $\mu=0$, Theorem 2.3 immediately yields a result concerning the Carlson-Shaffer operator $\mathcal{L}\left(a,c\right)\left(f\right) = I_{a,1,c}\left(f\right)$ (see [3]).

Corollary 2.6. *Let $a \in C \setminus 0$. Also, let c be a real number such that $c > |a|+2$. If $f \in R^{\tau}(A,B)$, and if the inequality*

$$\frac{\Gamma\left(c\right)\Gamma\left(c-|a|-2\right)}{\Gamma\left(c-|a|\right)\Gamma\left(c-1\right)}\left[\beta\left|a\right| + \left(c-|a|-2\right)\right] \leqslant \frac{1}{(A-B)\left|\tau\right|}+1 \tag{2.7}$$

is satisfied, then $\mathcal{L}\left(a,c\right)\left(f\right) \in UCD\left(\beta\right)$.

Theorem 2.7. *Let $a, b \in C \setminus 0$. Also, let c be a real number and $P_1 = P_1(k)$ be given by (1.8). If $f \in k-ST$, for some $k (0 \leqslant k < \infty)$, and the inequality*

$$\begin{aligned}&{}_3I_2\left(|a|,|b|,P_1;c,1;\lambda,\mu,1\right) + \left(\lambda+2\mu+\beta\right)\frac{|ab|\,P_1}{c}{}_3I_2\left(1+|a|,1+|b|,1+P_1;1+c,2;\lambda,\mu,1\right)\\ &+\left(\mu+\beta\lambda+3\mu\beta\right)\frac{|ab|\,P_1}{c}{}_3I_2\left(1+|a|,1+|b|,1+P_1;1+c,1;\lambda,\mu,1\right)\\ &+\mu\beta\frac{|ab|\left(1+|a|\right)\left(1+|b|\right)\left(1+P_1\right)}{c\left(1+c\right)}{}_3I_2\left(2+|a|,2+|b|,2+P_1;2+c,3;\lambda,\mu,1\right) \leqslant 2\end{aligned} \tag{2.8}$$

is satisfied, then $I_{a,b,c}(f) \in UCD(\beta)$.

Proof. Let f be given by (1.1). By (1.6), to show $I_{a,b,c}(f) \in UCD(\beta)$ it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\left(1+\beta(n-1)\right)\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leqslant 1.$$

We will repeat the method of proving used in the proof of Theorem 2.1. Applying the estimates for the coefficients given by (1.9), and make use of the (2.2) and $\left|(a)_{n-1}\right| \leqslant (|a|)_{n-1}$, we get

$$\sum_{n=2}^{\infty} n\left(1+\beta(n-1)\right)\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|$$

$$\leqslant \sum_{n=2}^{\infty} n\left(1+\beta(n-1)\right)\left(\lambda+\mu n+\frac{1-\lambda-\mu}{n}\right)\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty} n\left(1+\beta(n-1)\right)\left(\frac{\lambda n+\mu n^2+1-\lambda-\mu}{n}\right)\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty} \left(1+\beta(n-1)\right)\left(\lambda n+\mu n^2+1-\lambda-\mu\right)\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty} \left[\mu\beta n^3+(\beta\lambda+\mu-\mu\beta)n^2+(\lambda+\beta-2\beta\lambda-\mu\beta)n+(1-\lambda-\mu-\beta+\beta\lambda+\mu\beta)\right]$$
$$\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty}[1+(\lambda+2\mu+\beta)(n-1)+(\mu+\beta\lambda+3\mu\beta)(n-1)(n-1)+\mu\beta(n-1)(n-1)(n-2)]$$
$$\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} + (\lambda+2\mu+\beta)\sum_{n=2}^{\infty}(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$+ (\mu+\beta\lambda+3\mu\beta)\sum_{n=2}^{\infty}(n-1)(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$+ \mu\beta\sum_{n=2}^{\infty}(n-1)(n-1)(n-2)\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}.$$

Using the fact that $(a)_n = a(a+1)_(n-1)$, It is so easy to see that

$$= \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}$$
$$+ (\lambda + 2\mu + \beta) \frac{|ab| P_1}{c} \sum_{n=2}^{\infty} \frac{(1+|a|)_{n-2}(1+|b|)_{n-2}(1+P_1)_{n-2}}{(1+c)_{n-2}(2)_{n-2}(1)_{n-2}}$$
$$+ (\mu + \beta\lambda + 3\mu\beta) \frac{|ab| P_1}{c} \sum_{n=2}^{\infty} \frac{(1+|a|)_{n-2}(1+|b|)_{n-2}(1+P_1)_{n-2}}{(1+c)_{n-2}(1)_{n-2}(1)_{n-2}}$$
$$+ \mu\beta \frac{|ab| P_1 (1+|a|)(1+|b|)(1+P_1)}{c(1+c)} \sum_{n=3}^{\infty} \frac{(2+|a|)_{n-3}(2+|b|)_{n-3}(2+P_1)_{n-3}}{(2+c)_{n-3}(3)_{n-3}(1)_{n-3}}.$$

From (1.2),

$$= {}_3I_2(|a|, |b|, P_1; c, 1; \lambda, \mu, 1) - 1$$
$$+ (\lambda + 2\mu + \beta) \frac{|ab| P_1}{c} {}_3I_2(1+|a|, 1+|b|, 1+P_1; 1+c, 2; \lambda, \mu, 1)$$
$$+ (\mu + \beta\lambda + 3\mu\beta) \frac{|ab| P_1}{c} {}_3I_2(1+|a|, 1+|b|, 1+P_1; 1+c, 1; \lambda, \mu, 1)$$
$$+ \mu\beta \frac{|ab| (1+|a|)(1+|b|)(1+P_1)}{c(1+c)} {}_3I_2(2+|a|, 2+|b|, 2+P_1; 2+c, 3; \lambda, \mu, 1).$$
$$\leq 1$$
$$\Rightarrow {}_3I_2(|a|, |b|, P_1; c, 1; \lambda, \mu, 1) + (\lambda + 2\mu + \beta) \frac{|ab| P_1}{c} {}_3I_2(1+|a|, 1+|b|, 1+P_1; 1+c, 2; \lambda, \mu, 1)$$
$$+ (\mu + \beta\lambda + 3\mu\beta) \frac{|ab| P_1}{c} {}_3I_2(1+|a|, 1+|b|, 1+P_1; 1+c, 1; \lambda, \mu, 1)$$
$$+ \mu\beta \frac{|ab| (1+|a|)(1+|b|)(1+P_1)}{c(1+c)} {}_3I_2(2+|a|, 2+|b|, 2+P_1; 2+c, 3; \lambda, \mu, 1) \leqslant 2.$$

If $|b| = |a|$ we can rewrite the Theorem 2.7 as follows.

Corollary 2.8. *Let $a, b \in C \setminus 0$. Suppose that $|b| = |a|$. Also, let c be a real number and $P_1 = P_1(k)$ be given by (1.8). If $f \in k - ST$, for some $k (0 \leq k < \infty)$, and the inequality*

$${}_3I_2(|a|, |a|, P_1; c, 1; \lambda, \mu, 1) + (\lambda + 2\mu + \beta) \frac{|a|^2 P_1}{c} {}_3I_2(1+|a|, 1+|a|, 1+P_1; 1+c, 2; \lambda, \mu, 1)$$

$$+ (\mu + \beta\lambda + 3\mu\beta) \frac{|a|^2 P_1}{c} {}_3I_2(1+|a|, 1+|a|, 1+P_1; 1+c, 1; \lambda, \mu, 1)$$

$$+\mu\beta\frac{|a|^2\,(1+|a|)\,(1+|a|)\,(1+P_1)}{c\,(1+c)}\,{}_3I_2\,(2+|a|\,,2+|a|\,,2+P_1;2+c,3;\lambda,\mu,1)\leqslant 2$$

is satisfied, then $I_{(a,b,c)}(f)\in UCD(\beta)$.

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