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COUPLED FIXED POINT THEOREMS OF WEAKLY C-CONTRACTION WITH MIXED MONOTONE PROPERTY IN ORDERED MODULAR METRIC SPACES

Shishir Jain and Yogita Sharma*

Department of Mathematics,
Shri Vaishnav Vidyapeeth Vishwavidyalaya,
Gram Baroli, Sanwer Road, Indore - 453111, (M. P.), INDIA

E-mail : jainshishir11@rediffmail.com

*Department of Computer Science,
Shri Vaishnav Institute of Management,
Gumashta Nagar, Indore - 452009, (M. P.), INDIA

E-mail : yogitasharma2006@gmail.com

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Abstract: In this paper, we introduce the notion of weakly C-contraction using altering distance function in the setting of modular metric space equipped with partially ordered relation and proved some coupled fixed point results. The results are supported by examples.

Keywords and Phrases: Coupled fixed point, G-monotone mapping, weakly C-contraction, modular metric space, partially ordered set.

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1. Introduction

The Banach contraction principle [4] was introduced by Banach in his thesis in 1922. It is a very popular tool for solving the existing problems in many branches of mathematical analysis. Due to it's applications in mathematics, the Banach contraction principle has been generalized in various settings. In particular, Chatterjea [6] introduced the concept of C-contraction. In 1997, Alber and

Guerre-Delabriere [1] has given a remarkable generalization of Banach contraction by introducing the notion of weakly ϕ-contraction in the context of Hilbert spaces. Choudhury [12] generalized both the mentioned concepts as weakly C-contractive mapping and proved some fixed point results, these results were extended by Harjani et al. [15] to partially ordered metric spaces. Bhaskar and Lakshmikantham [5] introduced the notion of the mixed monotone property and the coupled fixed point of a mapping $F : X \times X \to X$ in partially ordered metric spaces. Since then many authors established coupled fixed point results in various abstract spaces (see in [3, 13, 14, 23, 27]). Lakshmikantham and Ćirić [18] introduced the concept of a mixed g-monotone mapping and proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered metric space, which generalize the main results of Bhaskar and Lakshmikantham [5]. After that many authors generalized the above concept in partially ordered metric spaces (see in [2, 25, 30]). Shatanawi [28] generalized the results of Harjani et al. [15] and Bhaskar and Lakshmikantham [5]. In 2012, Eshaghi Gordji et al. [13] introduced the concept of a mixed weakly monotone pair of mappings and proved some coupled common fixed point theorems, which generalize the results of Bhaskar and Lashmikantham [5] and Kadelburg et al. [16]. One of the useful generalization of metric spaces is modular metric spaces, which was primarily initiated by Nakano [24]. In 2008, Chistyakov [8] introduced the notion of modular metric spaces generated by F-modular (also see in [9, 10]) and using the properties of modular spaces, developed the theory of this space for arbitrary non-empty set in [11]. Author explained that metric modular is a generalized form of metric function. A metric on a set represents non-negative finite distances between any two points of a set; a modular on a set attributes a nonnegative (possible, infinite valued) "fields of velocities": to each 'time $\lambda > 0$', here the distance function $d(x, y)$ has been replaced by the average velocity $\omega_\lambda(x, y)$ for each $\lambda > 0$. In modular metric space, modular convergence, modular limit and modular completeness are "weaker" than the corresponding metric spaces, these characteristics of this space enhance the applicabilities of abstract spaces in many more research areas. In recent years, in many cases, specially in applications to operators, approximation fixed point results and modular type conditions are much more natural because modular type assumptions can be more easily verified than their metric or norm counterparts. In this sequel many mathematicians have done remarkable work on modular metric spaces (see in [20, 21, 22, 24, 26]).

In this paper we introduce the notion of weakly C-contraction using altering distance function in the setting of modular metric space equipped with partially ordered relation and proved some coupled fixed point results. Our results generalize the results of Bhaskar and Lakshmikantham [5], Hajani et al. [15] and Shatanawi [29].

2. Preliminaries

Here, some definitions and results related to our work are discussed.

Chatterjea [6] introduced the concept of C-contraction.

Definition 2.1. [6] *A mapping $F : X \to X$, where (X, d) is a metric space is said to be a C-contraction if there exists $k \in [0, \frac{1}{2})$ such that, the following inequality holds:*

$$d(Fx, Fy) \leq k(d(x, Fy) + d(y, Fx)),$$

for all $x, y \in X$.

In 1997, Alber and Guerre-Delabriere [1] introduced the notion of weakly ϕ-contraction in the context of Hilbert spaces.

Definition 2.2. [1] *A mapping $F : X \to X$, on a metric space X is called weakly ϕ-contractive, if there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) = 0$ if and only if $t = 0$, such that*

$$d(Fx, Fy) \leq d(x, y) - \phi(d(x, y))$$

for all $x, y \in X$.

Choudhury [12] defined a new concept of contractive mapping as a generalization of both the mentioned concepts called weakly C-contractive mapping.

Definition 2.3. [12] *A mapping $F : X \to X$, where (X, d) is a metric space, is said to be weakly C-contractive if the following inequality holds:*

$$d(Fx, Fy) \leq \frac{1}{2}(d(x, Fy) + d(y, Fx)) - \phi(d(x, Fy), d(y, Fx)),$$

where $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(x, y) = 0$ if and only if, $x = y = 0$.

Following result has been proved in [12].

Theorem 2.4. [12] *Let (X, d) be a complete metric space and F be a weakly C-contractive mapping then F has a unique fixed point x^* in X.*

Bhaskar and Lakshmikantham [5] introduced the notion of mixed monotone property and coupled fixed points and proved the results for continuous and non-continuous mappings in partially ordered metric spaces.

Definition 2.5. [5] *Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping, then we say that F has the mixed monotone property if $F(x, y)$ is* *monotone non-decreasing in x and is monotone non-increasing in y that is, for any $x, y \in X$*

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$$

Definition 2.6. [5] *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

The following results were also established in [5].

Theorem 2.7. [5] *Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume there exists $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \tfrac{k}{2}(d(x, u) + d(y, v)) \text{ for all } x \succeq u,\ y \preceq v.$$

If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

The above result is still valid for a mapping F not necessarily continuous if the continuity condition is replaced with an alternative condition discussed in the following theorem.

Theorem 2.8. [5] *Let $(X, \preceq, d)$ be a partially ordered complete metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that X has the following properties.*

(i) *If $\{x_n\}$ is a non-decreasing sequence in X which converges to x, then $x_n \preceq x$ for all $n \in \mathbb{N}$ and*

(ii) *If $\{y_n\}$ is a non-increasing sequence in X which converges to y, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

Suppose that there exists $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \tfrac{k}{2}(d(x, u) + d(y, v)) \text{ for all } x \succeq u,\ y \preceq v.$$

If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Nakano [24] introduced the following concept.

Definition 2.9. [24] *Let X be a linear space on $\mathbb{R}$. A functional $\rho : X \to [0, \infty]$ is called a modular on X if the following conditions hold:*

(A_1) $\rho(0) = 0$;

(A_2) *If* $x \in X$ *and* $\rho(\alpha x) = 0$ *for all numbers* $\alpha > 0$, *then* $x = 0$;

(A_3) $\rho(-x) = \rho(x)$ *for all* $x \in X$;

(A_4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for all* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$ *and* $x, y \in X$.

Chistyakov in [11], introduced the notion of modular metric spaces such as:

Definition 2.10. [11] *Let* X *be a non empty set. A function* $\omega\colon (0, \infty) \times X \times X \to [0, \infty)$ *is said to be a metric modular on* X, *if for all* $x, y, z \in X$ *the following conditions hold:*

(i) $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ *for all* $\lambda, \mu > 0$.

If we replace (i) by $\omega_\lambda(x, x) = 0$ *for all* $\lambda > 0, x \in X$, *then* ω *is said to be a pseudomodular (metric) on* X.
A modular ω on X is said to be regular if the following weaker version of (i) is satisfied:
$x = y$ if and only if $\omega_\lambda(x, y) = 0$ for some $\lambda > 0$. Finally, ω is said to be convex if for $\lambda, \mu > 0$ such that $0 < \mu < \lambda$, and for all $x, y, z \in X$, following inequality holds:

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

An important property of the (metric) psedomodular on set X is that the mapping $\lambda \mapsto \omega_\lambda(x, y)$ is non-increasing for all $x, y \in X$.

Definition 2.11. [11] *Let* ω *be a psedomodular on* X, *then for a fixed* $x_0 \in X$, *the two sets*

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\},$$

$$X^*_\omega = X^*_\omega(x_0) = \{(x \in X : \exists\, \lambda = \lambda(x) > 0) \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

are said to be modular metric spaces (around x_0*). Also, if* ω *is a modular on* X, *then the modular space* X_ω *can be equipped with a (nontrivial) metric* d_ω, *generated by* ω *and given by*

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\}, \quad x, y \in X_\omega.$$

If ω is a convex modular on X, then the two modular spaces coincide, $X_\omega = X^*_\omega$, and this common set can be endowed with a metric d_ω given by

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\}, \quad x,y \in X^*_\omega.$$

Even if ω is a nonconvex modular on X, then $d^*_\omega(x,x) = 0$ and $d_\omega(x,y) = d_\omega(y,x)$.

Definition 2.12. [11] *Let X_ω be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω.*

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is called modular convergent to an element $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$, and any such element x will be called a modular limit of the sequence $\{x_n\}$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X_\omega$ is called modular Cauchy sequence (ω-Cauchy) if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $\omega_\lambda(x_n, x_m) \to 0$ as $n, m \to \infty$, i.e., for all $\epsilon > 0$, there exists $n_0(\epsilon) \in \mathbb{N}$ such that, for all $n, m \ge n_0(\epsilon), \omega_\lambda(x_n, x_m) \le \epsilon$.

(iii) A modular space X_ω is called modular complete if every modular Cauchy sequence $\{x_n\}$ in X_ω is modular convergent in the following sence: if $\{x_n\} \subset X_\omega$ and there exists a $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{n,m\to\infty} \omega_\lambda(x_n, x_m) = 0$, then there exists an $x \in X_\omega$ such that $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$.

Mongkolkeha et al. [19] introduced the contractive condition in modular metric spaces.

Definition 2.13. [19] *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω and $F\colon X_\omega \to X_\omega$ be an arbitrary mapping. A mapping F is called contractive if for each $x, y \in X_\omega$ and for all $\lambda > 0$ there exists $0 \le k < 1$ such that*

$$\omega_\lambda(Fx, Fy) \le k\omega_\lambda(x,y). \tag{1}$$

The following result was also established in [19].

Theorem 2.14. [19] *Let ω be a metric modular on X and X_ω be a complete modular metric space induced by ω and $F : X_\omega \to X_\omega$ if*

$$\omega_\lambda(Fx, Fy) \le k(\omega_{2\lambda}(x, Fx) + \omega_{2\lambda}(y, Fy)) \tag{2}$$

for all $x, y \in X_\omega$ and for all $\lambda > 0$, where $k \in [0, \frac{1}{2})$, then F has a unique fixed point in X_ω. Moreover, for any $x \in X_\omega$, iterative sequence $\{T^n x\}$ converges to a fixed point.

Zhao et al. [31] proved the result of Mongkolkeha et al. [19] for C-contractive mapping.

Theorem 2.15. [31] *Let ω be a metric modular on X, X_ω be a ω-complete modular metric space induced by ω and $F\colon X_\omega \to X_\omega$. If*

$$\omega_\lambda(Fx, Fy) \leq k\,(\omega_{2\lambda}(x, Fy) + \omega_{2\lambda}(y, Fx)),$$

for all $x, y \in X_\omega$ and for all $\lambda > 0$, where $k \in [0, \frac{1}{2})$, then F has a unique fixed point in X_ω.

Zhao et al. [31] also introduced the notion of weakly C-contraction in modular metric space.

Definition 2.16. [31] *Let ω be a metric modular on X, X_ω be a modular metric space induced by ω and $F\colon X_\omega \to X_\omega$ is said to be a weakly C-contraction in X_ω, if for all $x, y \in X_\omega$ and for all $\lambda > 0$, the following inequality holds :*

$$\omega_\lambda(Fx, Fy) \leq \frac{1}{2}\,(\omega_{2\lambda}(x, Fy) + \omega_{2\lambda}(y, Fx)) - \phi(\omega_\lambda(x, Fy), \omega_\lambda(y, Fx)), \qquad (3)$$

where $\phi\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous mapping such that $\phi(x, y) = 0$ if and only if $x = y = 0$.

Authors also proved the following theorem.

Theorem 2.17. [31] *Let ω be a metric modular on X, X_ω be a ω-complete modular metric space induced by ω. Let $F\colon X_\omega \to X_\omega$ be a weakly C-contraction in X_ω such that F is continuous and non-decreasing, then F has a unique fixed point.*

Khan et al. [17] introduced the concept of altering distance function as follows:

Definition 2.18. [17] *The function $\psi\colon [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied.*

(i) ψ is continuous and monotonically non-decreasing.

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Shatanawi [29] proved some fixed point and coupled fixed point theorems in partially ordered metric space for C-contractive mapping F using altering distance function and examined the validity of results without the continuity of function F.

The aim of this paper is to establish some coupled fixed point theorems for non-linear weakly C-contractive type mapping in partially ordered modular metric spaces.

3. Main Results

Here we define the weakly $\mathcal{C}$-contractive mapping using altering distance function in modular metric space.

Definition 3.1. *Let ω be a metric modular on X, X_ω be a modular metric space induced by ω and $F\colon X_\omega \times X_\omega \to X_\omega$ is called weakly $\mathcal{C}$-contractive in X_ω, if for all $x, y \in X_\omega$ and for all $\lambda > 0$, the following inequality holds :*

$$\psi(\omega_\lambda(F(x,y), F(u,v))) \leq \psi(\frac{1}{2}(\omega_{2\lambda}(x,u) + \omega_{2\lambda}(y,v))) - \phi(\omega_\lambda(x,u), \omega_\lambda(y,v)). \quad (4)$$

where $\phi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(t,s) = 0$ if and only if $t = s = 0$.

Remark 3.2. *Every weakly C-contraction with altering distance function is a C-contraction but inverse is not true. The following example justify that if a function is not weakly C-contraction, can be converted into weakly C-contraction by using altering distance function.*

Example 3.3. Let $X_\omega = \mathbb{R}$ where " $\preceq$ " is a usual ordered relation. Then $(X_\omega, \preceq)$ is a partially ordered set with the natural ordering of real numbers. Let $\omega_\lambda\colon (0,\infty) \times X_\omega \times X_\omega \to [0,\infty)$ be defined by $\omega_\lambda(x,y) = \frac{|x-y|}{\lambda}$, for $x, y \in X$ and $\lambda > 0$. Then $(X_\omega, \omega_\lambda, \preceq)$ is a complete partially ordered modular metric space. We define $F\colon X_\omega \times X_\omega \to X_\omega$ such that

$$F(x,y) = \begin{cases} \frac{x-y}{4} & ;\text{if } x \geq y \\ 0 & ;\text{if } x < y \end{cases}$$

Then F is continuous and has mixed monotone property. Define $\phi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\phi(x,y) = \frac{(x+y)^2}{16}$ and $k \in (0, \frac{1}{2})$. Now, we observe that

$$\begin{aligned} \omega_\lambda(F(x,y), F(u,v)) &= \omega_\lambda(\frac{x-y}{4}, \frac{u-v}{4}) \\ &= \frac{1}{4\lambda}(\mid \frac{x-y}{4} - \frac{u-v}{4} \mid) \\ &\leq \frac{1}{4\lambda}(\mid x-u \mid + \mid y-v \mid) \\ &\neq \frac{1}{2}(\omega_{2\lambda}(x,u) + \omega_{2\lambda}(y,v)) - \phi(\omega_\lambda(x,u), \omega_\lambda(y,v)). \end{aligned}$$

So T is not a weakly C-contractive mapping. Further, we defined $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) = t^2$ and $\phi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\phi(x,y) = \frac{(x+y)^2}{16}$

and prove that F is weakly $\mathcal{C}$-contractive. Without loss of generality, assume that $x \succeq u$ and $y \preceq v$. Then, we have

$$\begin{aligned}
\psi(\omega_\lambda(F(x,y),F(u,v))) &= \psi(\omega_\lambda(\frac{x-y}{4},\frac{u-v}{4})) \\
&= (| \frac{x-y-(u-v)}{4\lambda} |)^2 \\
&= (| \frac{x-u-(y-v)}{4\lambda} |)^2 \\
&\leq \frac{1}{4}(\frac{[| x-u | + | y-v |]}{2\lambda})^2 \\
&= \frac{1}{8}(\frac{[| x-u | + | y-v |]}{2\lambda})^2 - \frac{1}{16}(\frac{[| x-u | + | y-v |]}{\lambda})^2 \\
&= \psi(\frac{1}{2}(\omega_{2\lambda}(x,u)+\omega_{2\lambda}(y,v))) - \phi(\omega_\lambda(x,u),\omega_\lambda(y,v)).
\end{aligned}$$

Therefore, it is clear that T is a weakly C-contractive mapping with altering distance function.

Theorem 3.4. *Let $(X_\omega, \preceq)$ be a partially ordered set and ω be a metric modular on X_ω such that $(X_\omega, \omega_\lambda)$ is a complete modular metric space. Let $F\colon X_\omega \times X_\omega \to X_\omega$ be a continuous mapping having the mixed monotone property on X_ω. Assume that for $x, y, u, v \in X_\omega$ with $x \succeq u$ and $y \preceq v$, we have*

$$\psi(\omega_\lambda(F(x,y),F(u,v))) \leq \psi(\frac{1}{2}(\omega_{2\lambda}(x,u)+\omega_{2\lambda}(y,v))) - \phi(\omega_\lambda(x,u),\omega_\lambda(y,v)), \quad (5)$$

where

(i) $\psi\colon [0,\infty) \to [0,\infty)$ is an altering distance function,

(ii) $\phi\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t,s) = 0$ if and only if $t = s = 0$.
If there exists $(x_0, y_0) \in X_\omega \times X_\omega$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. Let $x_0, y_0 \in X_\omega$ such that $F(x_0, y_0) = x_0$ and $F(y_0, x_0) = y_0$, then there is nothing to prove. Suppose that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and let $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, then $x_0 \preceq x_1$ and $y_0 \succeq y_1$.
Again let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$.
We denote

$$F^2(x_0,y_0) = F(F(x_0,y_0),F(y_0,x_0)) = F(x_1,y_1) = x_2$$

and

$$F^2(y_0, x_0) = F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2,$$

with these notation due to mixed monotone property of F, we have

$$x_2 = F^2(x_0, y_0) = F(x_1, y_1) \succeq F(x_0, y_0) = x_1$$

and

$$y_2 = F^2(y_0, x_0) = F(y_1, x_1) \preceq F(y_0, x_0) = y_1.$$

Further for $n = 1, 2, \cdots$ we get

$$x_{n+1} = F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) = F(x_n, y_n)$$

and

$$y_{n+1} = F^{n+1}(y_0, x_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)) = F(y_n, x_n).$$

On continuing this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X_ω such that $x_{n+1} = F(x_n, y_n)$, $y_{n+1} = F(x_n, y_n)$ for all $n \in \mathbb{N}$.
Also, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \text{ and } y_0 \succeq y_1 \succeq y_2 \succeq \cdots,$$

for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} \psi(\omega_\lambda(x_{n+1}, x_{n+2})) &= \psi(\omega_\lambda(F(x_n, y_n)), F(x_{n+1}, y_{n+1})) \\ &\le \psi(\frac{1}{2}(\omega_{2\lambda}(x_n, x_{n+1}) + \omega_{2\lambda}(y_n, y_{n+1}))) \\ &\quad -\phi(\omega_\lambda(x_n, x_{n+1}), \omega_\lambda(y_n, y_{n+1})) \\ &\le \psi(\max(\omega_{2\lambda}(x_n, x_{n+1}), \omega_{2\lambda}(y_n, y_{n+1}))) \\ &\quad -\phi(\omega_\lambda(x_n, x_{n+1}), \omega_\lambda(y_n, y_{n+1}). \end{aligned} \tag{6}$$

Similarly, we have

$$\begin{aligned} \psi(\omega_\lambda(y_{n+1}, y_{n+2})) &\le \psi(\max(\omega_{2\lambda}(x_n, x_{n+1}), \omega_{2\lambda}(y_n, y_{n+1}))) \\ &\quad -\phi(\omega_\lambda(y_n, y_{n+1}), \omega_\lambda(x_n, x_{n+1})). \end{aligned} \tag{7}$$

Since ψ ia a non-decreasing function, by (6) and (7), we have

$$\begin{aligned}\psi(\max(\omega_{2\lambda}(x_n,x_{n+1}),\omega_{2\lambda}(y_n,y_{n+1}))) &= \max(\psi(\omega_{2\lambda}(x_n,x_{n+1}),\psi\omega_{2\lambda}(y_n,y_{n+1})))\\ &\leq \psi(\max(\omega_{2\lambda}(x_n,x_{n+1})),\omega_{2\lambda}(y_n,y_{n+1}))\\ &-\min(\phi(\omega_\lambda(x_n,x_{n+1}),\omega_\lambda(y_n,y_{n+1}),\phi(\omega_\lambda(y_n,y_{n+1}),\omega_\lambda(x_n,x_{n+1})))\\ &\leq \psi(\max(\omega_\lambda(x_n,x_n),\omega_\lambda(x_n,x_{n+1})),(\omega_\lambda(y_n,y_n),\omega_\lambda(y_n,y_{n+1}))\\ &-\min(\phi(\omega_\lambda(x_n,x_{n+1}),\omega_\lambda(y_n,y_{n+1}),\phi(\omega_\lambda(y_n,y_{n+1}),\omega_\lambda(x_n,x_{n+1}))),\\ &\phi(\omega_\lambda(y_n,y_{n+1}),\omega_\lambda(x_n,x_{n+1}))).\\ &= \psi(\max(\omega_\lambda(x_n,x_{n+1})),\omega_\lambda(y_n,y_{n+1}))\\ &-\min(\phi(\omega_\lambda(x_n,x_{n+1}),\omega_\lambda(y_n,y_{n+1}),\phi(\omega_\lambda(y_n,y_{n+1}),\omega_\lambda(x_n,x_{n+1}))).\end{aligned} \tag{8}$$

Since $\phi(x,y)\geq 0$ for all $x,y\in X_\omega$ and ψ is a non-decreasing function, we conclude that
$\psi(\max(\omega_\lambda(x_{n+1},x_{n+2}),\omega_\lambda(y_{n+1},y_{n+2})$ is a non-decreasing sequence. Thus there is $r\geq 0$ such that

$$\lim_{n\to\infty}\max(\omega_\lambda(x_{n+1},x_{n+2}),\omega_\lambda(y_{n+1},y_{n+2}))=r.$$

Letting $n\to\infty$ in (8), we get that

$$\psi(r)=\psi(r)-\lim_{n\to\infty}\min(\phi(\omega_\lambda(x_n,x_{n+1}),\omega_\lambda(y_n,y_{n+1}))). \tag{9}$$

Thus, we have

$$\lim_{n\to\infty}\min(\phi(\omega_\lambda(x_n,x_{n+1}),\omega_\lambda(y_n,y_{n+1})))=0$$

or

$$\lim_{n\to\infty}\min(\phi(\omega_\lambda(y_n,y_{n+1}),\omega_\lambda(x_n,x_{n+1})))=0.$$

In both the cases, we get

$$\lim_{n\to\infty}\phi(\omega_\lambda(x_n,x_{n+1}))=0 \text{ and } \lim_{n\to\infty}\phi(\omega_\lambda(y_n,y_{n+1}))=0.$$

Hence $r=0$ for each $\lambda>0$ and for all $n\in\mathbb{N}$. Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X_ω. Since, $\lim_{n\to\infty}\phi(\omega_\lambda(x_n,x_{n+1}))=0$. So, for each $\lambda>0$ and for each $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that $\omega_\lambda(x_n,x_{n+1})<\epsilon$, for each $n\in\mathbb{N}$ with $n\geq n_0$.
Without loss of generality, suppose $m,n\in\mathbb{N}$ and $m\geq n$ then $\frac{\lambda}{m-n}>0$ and for

above mentioned ϵ there exists $n_{\frac{\lambda}{m-n}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) < \frac{\epsilon}{m-n}$ for all $n \geq n_{\frac{\lambda}{m-n}}$. Now we have

$$\begin{aligned} \omega_\lambda(x_n, x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &< \frac{\epsilon}{m-n} + \frac{\epsilon}{m-n} + \cdots + \frac{\epsilon}{m-n} = \epsilon, \end{aligned} \tag{10}$$

for all $m, n \geq n_{\frac{\lambda}{m-n}} \in \mathbb{N}$. This implies $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
Since X_ω is complete, therefore there exist $x, y \in X_\omega$ such that $x_n \to x$ and $y_n \to y$. Since F is continuous, therefore $x_{n+1} = F(x_n, y_n) \to F(x, y)$ and $y_{n+1} = F(y_n, x_n) \to F(y, x)$.
By the uniqueness of limit, we conclude that $x = F(x, y)$ and $y = F(y, x)$. Thus (x, y) is a coupled fixed point of F.

Example 3.5. Let F, ψ and ϕ be given as in Example 3.3. Here, we observe that for $x_0 = -1$ and $y_0 = 1$ in X_ω, $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ are satisfied. These conditions are also satisfied for the values, $x_0 = 0$ and $y_0 = 0$.
So, by Theorem 3.4 we obtain that F has two coupled fixed points (-1,1) and (0,0).

Example 3.6. Let $X_\omega = [0, 1] \subset \mathbb{R}$ and $(X_\omega, \preceq)$ be a partially ordered set. Let $\omega_\lambda\colon (0, \infty) \times X_\omega \times X_\omega \to [0, \infty]$ be defined by $\omega_\lambda(x, y) = \frac{|x-y|}{\lambda}$ for $x, y \in X$ and $\lambda > 0$. Then $(X_\omega, \omega_\lambda, \preceq)$ is a complete partially ordered modular metric space. We define $F\colon X_\omega \times X_\omega \to X_\omega$ such that

$$F(x, y) = \begin{cases} \frac{x^2}{9} & ; \text{if } x \geq y \\ 0 & ; \text{if } x < y. \end{cases}$$

Obviously, F is continuous and has mixed monotone property. Define $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = t^2$ and $\phi\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ such that $\phi(t, s) = \min(t, s)$. Now, we consider the following cases:
Case I. If $x = y = u = v = 0$, then clearly F satisfies all the conditions of Theorem (3.4), and $x_0 = 0 \preceq F(0, 0)$, $y_0 = 0 \succeq F(0, 0)$.
Case II. If $x = 1, y = 0, u = 0, v = 0$, then

$$\begin{aligned} \psi(\omega_\lambda(F(1, 0), F(0, 0))) &= \psi(\omega_\lambda(\frac{1}{9}, 0)) = \psi(\frac{1}{9\lambda}) = \frac{1}{81\lambda^2}. \\ \psi(\frac{1}{2}(\omega_{2\lambda}(1, 0) + \omega_{2\lambda}(0, 0))) - \phi(\omega_\lambda(1, 0), \omega_\lambda(0, 0)) &= \psi(\frac{1}{2}(\frac{1}{2\lambda})) - \phi(\frac{1}{\lambda}, 0) \\ &= \psi(\frac{1}{4\lambda}) = \frac{1}{16\lambda^2}, \end{aligned}$$

In this case also all the conditions of Theorem (3.4) satisfy such that $x_0 = 0 \preceq F(0, 1)$, $y_0 = 1 \succeq F(1, 0)$.

Case III. If $x = 1, y = 0, u = 0, v = 1$, then

$$\begin{aligned} \psi(\omega_\lambda(F(1,0), F(0,1))) &= \psi(\omega_\lambda(\frac{1}{9}, 0)) = \psi(\frac{1}{9\lambda}) = \frac{1}{81\lambda^2} \\ \psi(\frac{1}{2}(\omega_{2\lambda}(1,0) + \omega_{2\lambda}(0,1))) - \phi(\omega_\lambda(1,0), \omega_\lambda(0,1)) &= \psi(\frac{1}{2}(\frac{1}{2\lambda} + \frac{1}{2\lambda})) - \phi(\frac{1}{\lambda}, 0) \\ &= \psi(\frac{1}{2\lambda}) = \frac{1}{4\lambda^2}. \end{aligned}$$

So, it is clear that in each case all conditions of Theorem 3.4 satisfy, but, $x_0 = 1 \not\preceq F(1,0)$, $y_0 = 0 \succeq F(0,1)$. Thus, in this case coupled fixed point is not possible.

Hence, $(0,0)$ and $(0,1)$ are two coupled fixed points of weakly C-contractive mapping F.

Theorem 3.4 is still valid, if we drop the continuity of F by replacing it with the alternative conditions as discussed in following theorem.

Theorem 3.7. *Suppose that X_ω, F, ψ, ϕ are as in Theorem 3.4 except the continuity of F. Suppose that for a non-decreasing sequence $\{x_n\}$ in X_ω with $x_n \to x$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$ and for a non-increasing $\{y_n\}$ in X_ω with $y_n \to y$, we have $y_n \succeq y$ for all $n \in \mathbb{N}$. If there exists $(x_0, y_0) \in X_\omega \times X_\omega$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.*

Proof. As similar to the proof of Theorem 3.4, we have $\{x_n\}$, a non-decreasing sequence in X_ω which converges to $x \in X_\omega$, and $\{y_n\}$ a non-increasing sequence in X_ω which converges to $y \in X_\omega$. By hypotheses, we have $x_n \preceq x$ for all $n \in \mathbb{N}$ and $y_n \succeq y$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \psi(\omega_\lambda(x_{n+1}, F(x,y))) &= \psi(\omega_\lambda(F(x_n, y_n), F(x,y)) \\ &\leq \psi(\frac{1}{2}(\omega_{2\lambda}(x_n, x) + \omega_{2\lambda}(y_n, y))) \\ &\quad -\phi(\omega_\lambda(x_n, x), \omega_\lambda(y_n, y)). \end{aligned} \tag{11}$$

Letting $n \to \infty$, we get $\psi(\omega_\lambda(x, F(x,y))) = 0$ and hence $x = F(x,y)$. Similarly we can show that $y = F(x,y)$. Thus (x,y) is a coupled fixed point of F.

Now we shall prove the existence and uniqueness theorem of a coupled fixed point. If $(X_\omega, \preceq)$ is a partially ordered set, we endow the product $X_\omega \times X_\omega$ with the following partial order:

$$\text{for } (x,y), (u,v) \in X_\omega \times X_\omega, (x,y) \preceq (u,v) \Leftrightarrow x \succeq u, y \preceq v.$$

Theorem 3.8. *In addition to the hypothesis of Theorem 3.4, suppose that for every $(x,y), (z,t)$ in $X_\omega \times X_\omega$ there exists $(u,v) \in X_\omega \times X_\omega$ that is comparable to*

(x,y) and (z,t), then F has a unique coupled fixed point.

Proof. From Theorem 3.4, the set of coupled fixed points of F is non-empty. Suppose that (x,y) and (z,t) are two coupled fixed points of F, that is , $F(x,y) = x, F(y,x) = y, F(z,t) = z$ and $F(t,z) = t$. We will prove that

$$x = z \text{ and } y = t.$$

By assumption, there exists $(u,v) \in X_\omega \times X_\omega$ such that $(F(u,v), F(v,u))$ is comparable with $(F(x,y), F(y,x))$ and $(F(z,t), F(t,z))$. Put $u_0 = u$ and $v_0 = v$ and choose $u_1, v_1 \in X_\omega$ so that $u_1 = F(u_0, v_0)$ and $v_1 = F(v_0, u_0)$. As done in Theorem 3.4, define sequences $\{u_n\}$, $\{v_n\}$ with

$$u_{n+1} = F(u_n, v_n) \text{ and } v_{n+1} = F(v_n, u_n) \text{ for all n }.$$

Now, set $x_0 = x, y_0 = y, z_0 = z$ and $t_0 = t$, in a similar way, define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}, \{t_n\}$. Then it is easy to show that

$$x_n \to F(x,y), y_n \to F(y,x) \text{ and } z_n \to F(z,t), t_n \to F(t,z) \ \text{ as } \ n \to \infty.$$

Since

$$(F(x,y), F(y,x)) = (x_1, y_1) = (x,y) \text{ and } (F(u,v), F(v,u)) = (u_1, v_1)$$

are comparable, then $x \succeq u$ and $y \preceq v$, or vice versa. it is easy to show that, (x,y) and (u_n, v_n) are comparable for all $n \geq 1$, that is $x \succeq u_n$ and $y \preceq v_n$, or vice versa. Thus from

$$\begin{aligned} \psi(\omega_\lambda(x, u_{n+1})) &= \psi(\omega_\lambda(F(x,y)), F(u_n, v_n)) \\ &\leq \psi(\frac{1}{2}(\omega_{2\lambda}((x, u_n) + \omega_{2\lambda}(y, v_n))) - \phi(\omega_\lambda(x, u_n), \omega_\lambda(y, v_n)) \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} \psi(\omega_\lambda(y, v_{n+1})) &= \psi(\omega_\lambda((F(y,x), F(v_n, u_n)) \\ &\leq \psi(\frac{1}{2}(\omega_{2\lambda}((y, v_n) + \omega_{2\lambda}(x, u_n))) - \phi(\omega_\lambda((y, v_n), \omega_\lambda(x, u_n)) \end{aligned} \tag{13}$$

Since ψ ia a non-decreasing function. By (12) and (13), we have

$$\begin{aligned} \psi(\max(\omega_\lambda(x, u_{n+1}), \omega_\lambda(y, v_{n+1}))) &= \max(\psi(\omega_\lambda(x, u_{n+1}), \psi\omega_\lambda(y, v_{n+1})),) \\ &\leq \psi(\max(\omega_{2\lambda}(x, u_n)), \omega_{2\lambda}(y, v_n)) \\ &- \min(\phi(\omega_\lambda(x, u_n), \omega_\lambda(y, v_n), \phi(\omega_\lambda(y, v_n), \omega_\lambda(x, u_n))) \\ &\leq \psi(\max(\omega_\lambda(x, x) + \omega_\lambda(x, u_n)), (\omega_\lambda(y, y) + \omega_\lambda(y, v_n)) \\ &- \min(\phi(\omega_\lambda(x, u_n), \omega_\lambda(y, v_n))), \phi(\omega_\lambda(y, v_n), \omega_\lambda(x, u_n))) \\ &= \psi(\max(\omega_\lambda(x, u_n)), \omega_\lambda(y, v_n)) \\ &- \min(\phi(\omega_\lambda(x, u_n), \omega_\lambda(y, v_n))), \phi(\omega_\lambda(y, v_n), \omega_\lambda(x, u_n))). \end{aligned}$$

Since $\phi(x,y) \geq 0$ for all $x, y \in X_\omega$ and ψ is a non-decreasing function, we conclude that $\psi(\max(\omega_\lambda(x,u_{n+1}), \omega_\lambda(y,v_{n+1})$ is a non-decreasing sequence. Thus there is $\alpha \geq 0$ such that

$$\lim_{n\to\infty} \max(\omega_\lambda(x,u_{n+1}), \omega_\lambda(y,v_{n+1})) = \alpha.$$

Letting $n \to \infty$ in (13), we get

$$\psi(r) = \psi(r) - \lim_{n\to\infty} \min(\phi(\omega_\lambda(x,u_n), \omega_\lambda(y,y_n)), \phi(\omega_\lambda(y,y_n), \omega_\lambda(x,u_n))). \tag{14}$$

Thus,

$$\lim_{n\to\infty} \min(\phi(\omega_\lambda(x,u_n), \omega_\lambda(y,v_n))), = 0$$

or

$$\lim_{n\to\infty} \phi(\omega_\lambda(y,v_n), \omega_\lambda(x,u_n))) = 0.$$

In both the cases, we get

$$\lim_{n\to\infty} \phi(\omega_\lambda(x,u_n)) = \lim_{n\to\infty} \phi(\omega_\lambda(y,v_n)) = 0. \tag{15}$$

Hence $\alpha = 0$ for each $\lambda > 0$ and for all $n \in \mathbb{N}$. Similarly, we show that

$$\lim_{n\to\infty} \phi(\omega_\lambda(z,u_n)) = \lim_{n\to\infty} \phi(\omega_\lambda(t,v_n)) = 0. \tag{16}$$

From (16), (17) and by the uniqueness of the limit, we have $x = z$ and $y = t$. Hence (x, y) is the unique coupled fixed point of F.

Remark 3.9. *Taking $\psi = I_{[0,\infty]}$ [the identity function] in Theorem 3.4 and Theorem 3.7, we get the following.*

Corollary 3.10. *Let $(X_\omega, \preceq)$ be a partially ordered set and suppose that there exists a metric modular ω_λ on X_ω such that $(X_\omega, \omega_\lambda)$ is a complete modular metric space. Let $F\colon X_\omega \times X_\omega \to X_\omega$ be a weakly $\mathcal{C}$- contractive mapping having mixed monotone property on X_ω. Assume that for $x, y, u, v \in X_\omega$, $x \succeq u$ and $y \preceq v$ for all $\lambda > 0$ such that*

$$\omega_\lambda(F(x,y), F(u,v)) \leq \frac{1}{2}(\omega_{2\lambda}(x,u) + \omega_{2\lambda}(y,v)) - \phi(\omega_\lambda(x,u), \omega_\lambda(y,v)), \tag{17}$$

where $\phi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous mapping such that $\phi(x,y) = 0$ if and only if $x = y$. Suppose that there exists $(x_0, y_0) \in X_\omega \times X_\omega$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and either

(1) F is continuous or

(2) X_ω has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$,then $x_n \preceq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$,then $y_n \preceq y$ for all $n \in \mathbb{N}$.

Then F has a coupled fixed point.

Remark 3.11. *Taking $\phi(a,b) = (\frac{1-k}{2})(a+b)$ and using the properties of modular metric space in corollary 3.10, we get the following result.*

Corollary 3.12. *Let ω be a metric modular on X. Let X_ω be a ω- complete partially ordered modular metric space induced by ω. Let $F\colon X_\omega \times X_\omega \to X_\omega$ be a continuous mapping having the mixed monotone property on X_ω. Assume that for all $x, y, u, v \in X_\omega$, $x \succeq u$ and $y \preceq v$ such that*

$$\omega_\lambda(F(x,y), F(u,v)) \leq \frac{k}{2}\,(\omega_\lambda(x,u) + \omega_\lambda(y,v)). \tag{18}$$

Suppose that there exists $(x_0, y_0) \in X_\omega \times X_\omega$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and either

(1) F is continuous or

(2) X_ω has the following property:

(a) if a non-decreasing sequence $\{x_n\} \to x$,then $x_n \preceq x$ for all $n \in \mathbb{N}$,

(b) if a non-increasing sequence $\{y_n\} \to y$,then $y_n \preceq y$ for all $n \in \mathbb{N}$.

Then F has a coupled fixed point.

4. Conclusion

It is concluded that the obtained results improve, generalize and enrich various recent coupled fixed point theorems in the framework of C-contraction in partially ordered modular metric spaces. Particularly, Our results generalize the results of Bhaskar and Lakshmikantham [5], Hajani et al. [15] and Shatanawi [29]. The theoretical result is accompanied by applied examples.

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