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APPLICATIONS OF $\hat{g}^{**}$s-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: Topology is the branch of Mathematics which was introduced by Johann Benedict Listing in 19^{th} century and its purpose is to investigate the ideas of continuity, within the frame work of Mathematics. The authors introduces a new class of sets namely, $\hat{g}^{**}$s-closed sets [1]. We define $\hat{g}^{**}$s-closed sets by "A subset of a topological space (X, τ) is called a $\hat{g}^{**}$s-closed sets if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is $\hat{g}^{**}$- open" [1]. In this paper using the concept of $\hat{g}^{**}$s-closure, $\hat{g}^{**}$s-interior,$\hat{g}^{**}$s-border, $\hat{g}^{**}$s-frontier and $\hat{g}^{**}$s-exterior and studied some of its properties.

Keywords and Phrases: $\hat{g}^{**}$s-closure, $\hat{g}^{**}$s-interior, $\hat{g}^{**}$s-border, $\hat{g}^{**}$s-frontier, $\hat{g}^{**}$s-exterior.

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1. Introduction

Topology was introduced by Listing in 19^{th} century. In 1963 and 1970 Norman Levine introduced the classes of semi-open and g-closed sets respectively [4]. After these many researches on generalized closed sets have been going on. Crossley and Hildebrand defined semi-closure of sets and irresolute functions [2]. In 1973, Das defined semi-interior point of a subset [3]. In 2019, authors introduced the class of $\hat{g}^{**}s-$closed sets by generalizing the semi-closed sets using $\hat{g}^{*}$-open [1].

2. Preliminaries

In this paper (X,τ) represents the non-empty topological spaces on which no separation axioms are assured unless otherwise mentioned. For a subset A of X, $Cl(A), Int(A), Bd(A), Fr(A), Ext(A)$ denotes the closure, interior, border, frontier and exterior of A respectively.

Definition 2.1. *A subset A of a topological spaces (X,τ) is called*

1. *A semi-open set* [5] *if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$.*
2. *A $\hat{g}$ closed or (w-closed)* [6] *set if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi open in (X,τ).*
3. *A $\hat{g}^*$-closed set* [4] *if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is $\hat{g}$-open in (X,τ).*
4. *A $\hat{g}^{**}s$-closed set* [1] *if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is $\hat{g}^*$-open in (X,τ).*

Definition 2.2. *Let (X,τ) be a topological space.*

1. *For every set $A \subset X$, we define the semi-closure* [3] *of A is the intersection of all semi-closed sets containing A. ie) $sCl(A) = \cap\{U : A \subset U, U \in sc(X,\tau)\}$.*
2. *For every set $A \subset X$, we define the semi-interior* [2] *of A is the union of all semi-open sets containing A.ie) $sInt(A) = \cup\{U : A \subset U, U \in so(X,\tau)\}$.*
3. *If A is a subset of X, border of A is defined by $Bd(A) = A \setminus Int(A)$.*
4. *A subset A of a topological space (X,τ) is called as a frontier of A if $Fr(A) = Cl(A) \setminus Int(A)$.*
5. *A subset A of a topological space (X,τ) is called as a exterior of A if $Ext(A) = Int(X - A)$.*

3. $\hat{g}^{**}s$- Closure

Definition 3.1. *For every set $A \subset X$, we define the $\hat{g}^{**}s$-closure of A is the intersection of all $\hat{g}^{**}s$-closed sets containing A.ie)*
$\hat{g}^{**}sCl(A) = \cap\{U : A \subset U, U \in \hat{g}^{**}s(X,\tau)\}$.

Theorem 3.2. *If A is $\hat{g}^{**}s$-closed in X and B is closed in X, then $A \cup B$ is $\hat{g}^{**}s$-closed in X.*

Proof. By hypothesis $X \setminus A$ is $\hat{g}^{**}s$-open and $X \setminus B$ is open in X $\Rightarrow (X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B)$ is $\hat{g}^{**}s$-open in X. Hence $A \cup B$ is $\hat{g}^{**}s$-closed.

Theorem 3.3. *If A is any subset of a space X, then $\hat{g}^{**}sCl(A)$ is $\hat{g}^{**}s$-closed.*
Proof. By definition: 3.1, the result is obvious.

Theorem 3.4. *A is $\hat{g}^{**}s$-closed iff $\hat{g}^{**}sCl(A) = A$.*
Proof. Necessity condition is obvious.
To prove sufficiency, assume $\hat{g}^{**}sCl(A) = A$, by theorem: 3.3 $\hat{g}^{**}sCl(A)$ is $\hat{g}^{**}s$-closed. Therefore, A is $\hat{g}^{**}s$-closed.

Corollary 3.5. *If A is $\hat{g}^{**}s$-closed and U is open in X, then $A \setminus U$ is $\hat{g}^{**}s$-closed in X.*
Proof. $A \setminus U = A \cap (X \setminus U)$. By theorem: 3.2, the proof follows.

Theorem 3.6. *Let $A \subseteq X$ and let $x \in X$. Then, $x \in \hat{g}^{**}sCl(A)$ iff every $\hat{g}^{**}s$-open set in X containing x intersects A.*
Proof. Suppose $x \notin \hat{g}^{**}sCl(A) \Rightarrow X \setminus \hat{g}^{**}sCl(A)$ is a $\hat{g}^{**}s$-open set in X containing x that does not intersects A.
Conversely, Suppose U is A $\hat{g}^{**}s$-open set containing x that does not intersects A. Then $X \setminus U$ is a $\hat{g}^{**}s$-closed set containing A. Therefore, $\hat{g}^{**}sCl(A) \subseteq X \setminus U$. Hence $x \notin \hat{g}^{**}sCl(A)$. Thus $x \notin \hat{g}^{**}sCl(A)$ iff there is a $\hat{g}^{**}s$-open set containing x that does not intersects A.

Theorem 3.7. *If A is a subset of X, then*
i) $\hat{g}^{**}sCl(X \setminus A) = X \setminus \hat{g}^{**}sInt(A)$
ii) $\hat{g}^{**}sInt(X \setminus A) = X \setminus \hat{g}^{**}sCl(A)$
Proof. i) Let $x \in X \setminus \hat{g}^{**}sInt(A) \Rightarrow x \notin \hat{g}^{**}sInt(A) \Rightarrow x$ does not belong to $\hat{g}^{**}s$-open subset of A. Then, F is a $\hat{g}^{**}s$-closed set containing $X \setminus A$. Hence $X \setminus F$ is a $\hat{g}^{**}s$-open set contained in A. Therefore, $x \notin X \setminus F$ and so, $x \in F$. Hence $x \in \hat{g}^{**}sCl(X \setminus A)$. Then x belongs to every $\hat{g}^{**}s$-closed set containing $X \setminus A$. Hence x does not belong to any $\hat{g}^{**}s$-open subset of A. That is, $x \notin \hat{g}^{**}sInt(A)$. Then $x \notin X \setminus \hat{g}^{**}sInt(A)$.
ii) can be proved by replacing A by $X \setminus A$.

Theorem 3.8. *If a subset A of X is nowhere dense, then $Int\,(\hat{g}^{**}sCl(A)) = \emptyset$.*
Proof. $Int\,(Cl(A)) = \emptyset \Rightarrow Int\,(\hat{g}^{**}sCl(A)) \subseteq Int\,(Cl(A))$.

Theorem 3.9. *In a topological space (X, τ), the following hold:*
i) $\hat{g}^{**}sCl(\emptyset) = \emptyset$
ii) $\hat{g}^{**}sCl(X) = X$
If A and B are subsets of X
iii) $A \subseteq \hat{g}^{**}sCl(A)$
iv) $A \subseteq B \Rightarrow \hat{g}^{**}sCl(A) \subseteq \hat{g}^{**}sCl(B)$
v) $A \subseteq \hat{g}^{**}sCl(A) \subseteq sCl(A) \subseteq Cl(A)$

vi) $\hat{g}^{**}sCl\left(\hat{g}^{**}sCl(A)\right) = \hat{g}^{**}sCl(B)$
vii) $\hat{g}^{**}sCl(A \cup B) \supseteq \hat{g}^{**}sCl(A) \cup \hat{g}^{**}sCl(B)$
viii) $\hat{g}^{**}s(A \cap B) \subseteq \hat{g}^{**}sCl(A) \cap \hat{g}^{**}sCl(B)$
ix) $Cl\left(\hat{g}^{**}sCl(A)\right) = Cl(A)$
x) $\hat{g}^{**}sCl\left(Cl(A)\right) = Cl(A)$

Theorem 3.10. *If A and B are subsets of X such that $A \cap B = \emptyset$ and A is $\hat{g}^{**}s$-open in X, then $A \cap \hat{g}^{**}sCl(B) = \emptyset$.*
Proof. Suppose $x \in A \cap \hat{g}^{**}sCl(B) \Rightarrow x \in A$ and $x \in \hat{g}^{**}sCl(B) \Rightarrow x \in A$ and $x \in B \Rightarrow A \cap B \neq \emptyset$ which is a contradiction to hypothesis. Therefore, $A \cap \hat{g}^{**}sCl(B) = \emptyset$.

4. $\hat{g}^{**}s$ - Interior

Definition 4.1. *For any $A \subset X$, $\hat{g}^{**}s$-interior of A is defined as the union of all $\hat{g}^{**}s$-open sets containing A. ie) $\hat{g}^{**}sInt(A) = \cup\{U : A \subset U, U \in \hat{g}^{**}so(X,\tau)\}$.*

Theorem 4.2. *If the subsets A and B of a topological space (X,τ) are separated $\hat{g}^{**}s$-open, then $A \cup B$ is $\hat{g}^{**}s$-open.*
Proof. Assume that A and B are separated $\hat{g}^{**}s$-open sets. Let G be a $\hat{g}^{*}$-closed set in X such that $G \subseteq A \cup B$. By assumption, we have $Cl(A) \cap B = A \cap Cl(B) = \emptyset$. Then, $G \cap Cl(A) \subseteq (A \cup B) \cap Cl(A) = (A \cap Cl(A)) \cup (B \cap Cl(A)) = (A \cap Cl(A)) \cup \emptyset = A \cup \emptyset = A$. Therefore, $G \cap Cl(A) \subseteq A$. Similarly, $G \cap Cl(A) \subseteq B$. Since G is $\hat{g}^{*}$-closed in X, $G \cap Cl(A)$ and $G \cap Cl(B)$ are $\hat{g}^{*}$-closed. Since A and B are $\hat{g}^{**}s$-open, $G \cap Cl(A) \subseteq sInt(A)$ and $G \cap Cl(B) \subseteq sInt(B)$. Now, $G = G \cap (A \cup B) = (G \cap A) \cup (G \cap B) \Rightarrow G \subseteq (G \cap Cl(A)) \cup (G \cap Cl(B)) \Rightarrow G \subseteq sInt(A) \cup sInt(B) \Rightarrow G \subseteq sInt\left(A \cup B\right)$. Therefore, $A \cup B$ is $\hat{g}^{**}s$-open.

Theorem 4.3. *If a set A is $\hat{g}^{**}s$-open in a topological space (X,τ), then $G = X$, whenever G is $\hat{g}^{*}$-open and $sInt(A) \cup A^c \subseteq G$.*
Proof. Suppose A is $\hat{g}^{**}s$-open and G is $\hat{g}^{*}$-open and $sInt(A) \cup A^c \subseteq G$. Then, $G^c \subseteq (sInt(A) \cup A)^c \Rightarrow G^c \subseteq (sIntA)^c \cap A \Rightarrow G^c \subseteq sCl(A^c) - A^c$. Since, A^c is $\hat{g}^{**}s$-closed, $sCl(A^c) - A^c$ contains no non-empty $\hat{g}^{*}$-closed set in X. Therefore, $G^c = \emptyset \Rightarrow G = X$.

Theorem 4.4. *In a topological space (X,τ), if $Int(A) \cup (X - A)$ is $\hat{g}^{**}s$-closed, then $G \cup Int(A) = A$, for some $\hat{g}^{**}s$-open set in G.*
Proof. Assume that $Int(A) \cup (X - A)$ is $\hat{g}^{**}s$-closed in (X,τ).

Let $U = Int(A) \cup (X - A)$, then U^c is $\hat{g}^{**}s$-open. Now,
$U^c \cup (Int(A)) = [(X - A) \cup (Int(A))]^c \cup (Int(A)) = [(X - A)^c \cap (Int(A))^c] \cup (Int(A)) = [A \cap (Int(A))^c] \cup (Int(A)) = [A \cup (Int(A))] \cap [(Int(A))^c \cup (Int(A))] = A \cap X = A$. Take, $G = U^c$, we have $A = G \cup (Int(A))$ for some $\hat{g}^{**}s$-open set in G.

Theorem 4.5. *If A is any subset of X, $\hat{g}^{**}sInt(A)$ is $\hat{g}^{**}s$-open. Then, $\hat{g}^{**}sInt(A)$ is the largest $\hat{g}^{**}s$-open set contained in A.*
Proof. It is obvious from definition.

Theorem 4.6. *A subset A of X is $\hat{g}^{**}s$-open, then $\hat{g}^{**}sInt(A) = A$.*
Proof. The proof is obvious.

Remark 4.7. *The converse of above theorem is not true.*

Example 4.8. Let $X = \{a, b, c, d\}$. $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and
$\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}$
Here, $\{b, c, d\} = \hat{g}^{**}sInt(\{b, c, d\})$ but $\{b, c, d\}$ is not a $\hat{g}^{**}s$-open set.

Theorem 4.9. *If A is a subset of X, then $\hat{g}^{**}sInt(A)$ is the set of all $\hat{g}^{**}s$-interior points of A.*
Proof. $x \in \hat{g}^{**}sInt(A)$ iff $x \in \hat{g}^{**}s$-open subset U of A. $x \in \hat{g}^{**}sInt(A)$ iff x is a $\hat{g}^{**}s$-interior points of A.

Corollary 4.10. *A subset A of X is $\hat{g}^{**}s$-open iff every point of A is a $\hat{g}^{**}s$-interior points of A.*
Proof. By using theorem: 4.6 and theorem: 4.9.

Theorem 4.11. *Every open set is $\hat{g}^{**}s$-open.*
Proof. Let A be an open set in X. Then, A^c is closed in X. Since every closed set is $\hat{g}^{**}s$-closed, A^c is $\hat{g}^{**}s$-closed in X. Then, A is $\hat{g}^{**}s$-open in X.

Theorem 4.12. *If a subset A of X is $\hat{g}^{**}s$-open and U is open, then $A \cup U$ is $\hat{g}^{**}s$-open.*

Theorem 4.13. *In a topological space (X, τ), the following hold:*
i) $\hat{g}^{**}sInt(\emptyset) = \emptyset$
ii) $\hat{g}^{**}s(X) = X$
If A and B are subsets of X.
iii) $\hat{g}^{**}sInt(A) \subseteq A$
iv) $A \subseteq B \Rightarrow \hat{g}^{**}sInt(A) \subseteq \hat{g}^{**}sInt(B)$
v) $Int(A) \subseteq sInt(A) \subseteq \hat{g}^{**}sInt(A) \subseteq A$
vi) $\hat{g}^{**}sInt(\hat{g}^{**}sInt(A)) \subseteq \hat{g}^{**}sInt(A)$
vii) $\hat{g}^{**}sInt(A \cup B) \supseteq \hat{g}^{**}sInt(A) \cup \hat{g}^{**}sInt(B)$
viii) $\hat{g}^{**}sInt(A \cap B) \subseteq \hat{g}^{**}sInt(A) \cap \hat{g}^{**}sInt(B)$
ix) $Int(\hat{g}^{**}sInt(A)) = Int(A)$
x) $\hat{g}^{**}sInt(Int(A)) = Int(A)$

Theorem 4.14. *Let A be a subset of a space X, then the following are true.*

i) $(\hat{g}^{**}sInt(A))^c = \hat{g}^{**}sCl(A^c)$
ii) $\hat{g}^{**}sInt(A) = (\hat{g}^{**}sCl(A))^c$
iii) $\hat{g}^{**}sCl(A) = (\hat{g}^{**}sInt(A)^c)^c$

Proof. i) Let $x \in (\hat{g}^{**}sInt(A))^c \Rightarrow x \notin \hat{g}^{**}sInt(A)$. That is, every $\hat{g}^{**}s$-open set U containing x is such that $U \notin A \Rightarrow$ every $\hat{g}^{**}s$-open set containing x such that $U \cap A^c \neq \emptyset \Rightarrow x \in \hat{g}^{**}sCl(A^c)$. Therefore, $(\hat{g}^{**}sInt(A))^c \subseteq \hat{g}^{**}sCl(A^c$.
Conversely, $x \in \hat{g}^{**}sCl(A^c)$. By theorem 4.7, every $\hat{g}^{**}s$-open set containing x such that $U \cap A^c \neq \emptyset \Rightarrow x \notin \hat{g}^{**}sInt(A) \Rightarrow x \in (\hat{g}^{**}sInt(A))^c$. Therefore, $\hat{g}^{**}s(A^c) \subseteq (\hat{g}^{**}sInt(A))^c$. Hence, $\hat{g}^{**}s(A^c) = (\hat{g}^{**}sInt(A))^c$.
ii) By taking complement in (i) we get the result. iii) Replace A by A^c in (i).

5. $\hat{g}^{**}s-$ Border

Definition 5.1. *If A is a subset of X, $\hat{g}^{**}s$-border of A is defined by* $\hat{g}^{**}sBd(A) = A \setminus \hat{g}^{**}sInt(A)$.

Theorem 5.2. *In a topological space (X, τ), the following hold:*
i) $\hat{g}^{**}sBd(\emptyset) = \emptyset$
ii) $\hat{g}^{**}sBd(X) = \emptyset$
If A and B are subsets of X.
iii) $\hat{g}^{**}sBd(A) \subseteq A$
iv) $\hat{g}^{**}sInt(A) \cup \hat{g}^{**}sBd(A) = A$
v) $\hat{g}^{**}sInt(A) \cap \hat{g}^{**}sBd(A) = \emptyset$
vi) $\hat{g}^{**}sBd(A) \subseteq sBd(A) \subseteq Bd(A)$
vii) $\hat{g}^{**}sInt(\hat{g}^{**}sBd(A)) = \emptyset$
viii) A is $\hat{g}^{**}sInt(A)$ *iff* $\hat{g}^{**}sBd(A) = \emptyset$
ix) $\hat{g}^{**}sBd(\hat{g}^{**}sInt(A)) = \emptyset$
x) $\hat{g}^{**}sBd(\hat{g}^{**}sBd(A)) = \hat{g}^{**}sBd(A)$
xi) $\hat{g}^{**}sBd(A) = A \cap \hat{g}^{**}sCl(X \setminus A)$
xii) $\hat{g}^{**}sBd(A) = A \cap D_{\hat{g}^{**}s}(X \setminus A)$

6. $\hat{g}^{**}s$ - Frontier

Definition 6.1. *A subset A of a topological space (X, τ) is called as a $\hat{g}^{**}s$-frontier of A if* $\hat{g}^{**}sFr(A) = \hat{g}^{**}sCl(A) \setminus \hat{g}^{**}sInt(A)$.

Theorem 6.2. *In a topological space (X, τ), the following hold:*
i) $\hat{g}^{**}sFr(\emptyset) = \emptyset$
ii) $\hat{g}^{**}sFr(X) = \emptyset$
If A is a subset of X.
iii) $\hat{g}^{**}sCl(A) = \hat{g}^{**}sInt(A) \cup \hat{g}^{**}sFr(A)$
iv) $\hat{g}^{**}sInt(A) \cap \hat{g}^{**}sFr(A) = \emptyset$
v) $\hat{g}^{**}sBd(A) \subseteq \hat{g}^{**}sFr(A) \subseteq \hat{g}^{**}sCl(A)$

vi) A *is* $\hat{g}^{**}s$*-closed iff* $A = \hat{g}^{**}sInt(A) \cup \hat{g}^{**}sFr(A)$

vii) $\hat{g}^{**}sFr(A) \subseteq sFr(A) \subseteq Fr(A)$

viii) $\hat{g}^{**}sFr(A) = \hat{g}^{**}sCl(A) \cap \hat{g}^{**}sCl(X \setminus A)$

ix) $\hat{g}^{**}sFr(A)$ *is* $\hat{g}^{**}s$*-closed and hence* $\hat{g}^{**}sCl(\hat{g}^{**}sFr(A)) = \hat{g}^{**}sFr(A)$

x) $\hat{g}^{**}sFr(A) = \hat{g}^{**}sFr(X \setminus A)$

xi) $\hat{g}^{**}sFr(A)$ *is* $\hat{g}^{**}s$*-closed iff* $\hat{g}^{**}sFr(A) = \hat{g}^{**}sBd(A)$*. Hence* A *is* $\hat{g}^{**}s$ *-closed iff* A *contains its* $\hat{g}^{**}s$*-frontier.*

xii) $\hat{g}^{**}sFr(\hat{g}^{**}sInt(A)) \supseteq \hat{g}^{**}sFr(A)$

xiii) $\hat{g}^{**}sFr(\hat{g}^{**}sCl(A)) \subseteq \hat{g}^{**}sFr(A)$

xiv) $\hat{g}^{**}sFr(\hat{g}^{**}sFr(A)) \subseteq \hat{g}^{**}sFr(A)$

xv) $X - \hat{g}^{**}sInt(A) \cup \hat{g}^{**}sInt(X - A) \cup \hat{g}^{**}sFr(A)$

Theorem 6.3. *If A is a subset of X, then* $\hat{g}^{**}sFr(\hat{g}^{**}sFr(\hat{g}^{**}sFr(A))) = \hat{g}^{**}s(\hat{g}^{**}s(A))$.

Proof. $\hat{g}^{**}s(\hat{g}^{**}s(\hat{g}^{**}s(A)))$

$= \hat{g}^{**}sCl(\hat{g}^{**}s(\hat{g}^{**}s(A))) \cap \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(\hat{g}^{**}sFr(A)))$ —(1)

Now, $X \setminus \hat{g}^{**}sFr(\hat{g}^{**}sFr(A)) = X \setminus [\hat{g}^{**}sCl(\hat{g}^{**}sFr(A)) \cap \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(A))]$

$=[X \setminus \hat{g}^{**}sFr(A)] \cup [X \setminus \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(A))]$.

Consider, $\hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(\hat{g}^{**}sFr(A)))$

$= \hat{g}^{**}sCl[X \setminus \hat{g}^{**}sFr(A)] \cup [X \setminus \hat{g}^{**}sCl(\hat{g}^{**}sFr(A))]$

$= D \cup \hat{g}^{**}s(X \setminus D)$, where $D = \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(A))$

$\supseteq D \cup (X \setminus D) = X$ —(2)

Substitute (2) in (1),

Therefore, (1)$\Rightarrow \hat{g}^{**}sFr(\hat{g}^{**}sFr(\hat{g}^{**}s(A)))$

$= \hat{g}^{**}sCl(\hat{g}^{**}sFr(\hat{g}^{**}sFr(A))) \cap X$

$= \hat{g}^{**}sFr(\hat{g}^{**}sFr(A)) \cap X$

$= \hat{g}^{**}sFr(\hat{g}^{**}sFr(A))$

Theorem 6.4. *If a subset A is* $\hat{g}^{**}s$*-open or* $\hat{g}^{**}s$ *- closed in X, then* $\hat{g}^{**}sFr(\hat{g}^{**}sFr(A)) = \hat{g}^{**}sFr(A)$.

Proof. $\hat{g}^{**}sFr(\hat{g}^{**}sFr(A)) = \hat{g}^{**}sCl(\hat{g}^{**}sFr(A)) \cap \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(A))$

$= \hat{g}^{**}sCl(A) \cap \hat{g}^{**}sCl(X \setminus A) \cap \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(A))$ —(1)

If A is $\hat{g}^{**}s$-open in X and $\hat{g}^{**}sFr(A) \cap A = \emptyset \Rightarrow A \subseteq X \setminus \hat{g}^{**}sFr(A) \Rightarrow \hat{g}^{**}sCl(A) \subseteq \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(A)) \Rightarrow \hat{g}^{**}sCl(A) \cap \hat{g}^{**}sCl(X\ \hat{g}^{**}sFr(A)) = \hat{g}^{**}sCl(A)$.

If A is $\hat{g}^{**}s$-closed in X, $\hat{g}^{**}sFr(A) \subseteq A \Rightarrow X \setminus A \subseteq X \setminus \hat{g}^{**}sFr(A) \Rightarrow \hat{g}^{**}sCl(X \setminus A) \cap \hat{g}^{**}sCl(X \setminus \hat{g}^{**}sFr(A)) = \hat{g}^{**}sCl(X \setminus A)$.

Therefore, $\hat{g}^{**}sFr(\hat{g}^{**}sFr(A)) = \hat{g}^{**}sCl(A) \cap \hat{g}^{**}sCl(X \subseteq A) = \hat{g}^{**}sFr(A)$.

Theorem 6.5. *If A and B are subsets of X such that $A \cap B = \emptyset$ and A is* $\hat{g}^{**}s$*-open in X, then* $A \cap \hat{g}^{**}sFr(B) = \emptyset$.

Proof. Since $\hat{g}^{**}sFr(B) \subseteq \hat{g}^{**}sCl(B)$ and by theorem 3.10, $A \cap \hat{g}^{**}sFr(B) = \emptyset$.

Theorem 6.6. *If A and B are subsets of X, then*
i) $\hat{g}^{**}sFr(A \cup B) \supseteq \hat{g}^{**}sFr(A) \cap \hat{g}^{**}sFr(B)$
ii) $\hat{g}^{**}sFr(A \cap B) \subseteq \hat{g}^{**}sFr(A) \cup \hat{g}^{**}sFr(B)$

7. $\hat{g}^{**}s-$ Exterior

Definition 7.1. *If A is a subset of X,* $\hat{g}^{**}$*s-exterior of A is defined by* $\hat{g}^{**}sExt(A) = \hat{g}^{**}sInt(X - A)$.

Theorem 7.2. *In a topological space* (X, τ)*, the following hold:*
i) $\hat{g}^{**}sExt(\emptyset) = X$
ii) $\hat{g}^{**}sExt(X) = \emptyset$
If A and B are subsets of X *iii)* $A \subseteq B \Rightarrow \hat{g}^{**}sExt(A) \subseteq \hat{g}^{**}sExt(B)$
iv) $\hat{g}^{**}sExt(A)$ *is a* $\hat{g}^{**}$*s-open set.*
v) $Ext(A) \subseteq sExt(A) \subseteq \hat{g}^{**}sExt(A) \subseteq X \setminus A$
vi) A is $\hat{g}^{**}$*s-closed iff* $\hat{g}^{**}sExt(A) = X \setminus A$
vii) $\hat{g}^{**}sExt(A) = X \setminus \hat{g}^{**}sCl(A)$
viii) $\hat{g}^{**}sExt(\hat{g}^{**}sExt(A)) = \hat{g}^{**}sInt(\hat{g}^{**}sCl(A))$
ix) $\hat{g}^{**}sExt(A) = \hat{g}^{**}sExt(X \setminus \hat{g}^{**}sExt(A))$
x) $\hat{g}^{**}sInt(A) \subseteq \hat{g}^{**}sExt(X \setminus \hat{g}^{**}sExt(A))$
xi) $X = \hat{g}^{**}sInt(A) \cup \hat{g}^{**}sExt(A) \cup \hat{g}^{**}sFr(A)$
xii) $\hat{g}^{**}sExt(A \cup B) \subseteq \hat{g}^{**}sExt(A) \cap \hat{g}^{**}sExt(B)$
xiii) $\hat{g}^{**}sExt(A \cap B) \supseteq \hat{g}^{**}sExt(A) \cup \hat{g}^{**}sExt(B)$.

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