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OUTER CONNECTED EQUITABLE DOMINATING SETS IN A GRAPH

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Abstract: Let G be a simple graph with vertex set V and edge set E. An equitable dominating set D of $V(G)$ is called an outer connected equitable dominating set of G if $< V - D >$ is connected. A study of outer connected equitable dominating sets in a graph is initiated.

Keywords and Phrases: Equitable domination, outer connected domination.

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1. Introduction

Graph theoretical terminologies not given here can be founded in [2, 3, 9]. Let $G = (V, E)$ be a simple graph. The neighbourhood of a vertex v, denoted by $N(v)$, is the set of all vertices adjacent to v in G. If v is a vertex of G then the integer $deg(v) = |N(v)|$ is said to be the degree of v in G. The minimum and maximum degree among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree one in a graph is called a pendent vertex or an end vertex. A support is the unique neighbour of an end-vertex.

A set $D \subseteq V(G)$ is a dominating set in G if for every vertex $v \in V(G) - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$.The domination number of a graph G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G.

The concept of equitability was originally conceived in proper colouring of vertices where the cardinalities of any two colour classes differ by at most one [10]. E. Sampathkumar initiated the concept of degree equitability in the vertex set of a graph. Two vertices are said to be degree equitable if their degrees differ by at most one. A subset D of $V(G)$ is called an equitable dominating set of G if for any v in $V - D$, there exists u in D such that u and v are adjacent and degree equitable [1, 4, 5].

Two nice papers by Joanna Cyman on Outer connected domination and doubly connected domination [6], [7] motivated us to define outer-connected equitable domination in graphs. An equitable dominating set D is said to be outer connected if $< V - D >$ is connected. $V(G)$ is obviously an outer connected equitable dominating set. The minimum(maximum) cardinality of an outer connected equitable dominating set of G is called the outer connected equitable domination number (upper outer connected equitable domination number) of G and is denoted by $\widetilde{\gamma}^e_c(G)$ $(\widetilde{\Gamma}^e_c(G))$. A study of this new parameter and its relation with other parameters is initiated in this paper.

2. Basic Results

Definition 2.1. *Let G be a simple graph with vertex set V and edge set E. A subset D of V(G) is called an outer connected equitable dominating set of G if D is an equitable dominating set with the subgraph induced by its complement is connected. The minimum (maximum) cardinality of a minimal outer connected equitable dominating set of G is called the outer connected equitable domination number of G and is denoted by* $\widetilde{\gamma}^e_c(G)$ $(\widetilde{\Gamma}^e_c(G))$.

Observation 2.1. *Let* K_n, P_n, C_n *and* W_n *denote the complete graph, path, cycle and wheel of order n. Let* $K_{1,n}, K_{m,n}, K_{n_1,n_2,\cdots,n_r}$ *denote the star, bi-partite graph and multipartite graphs.*

i) $\widetilde{\gamma}^e_c(K_n) = 1$ for all $n \geq 1$.

ii) $\widetilde{\gamma}^e_c(K_{1,n}) = n + 1$.

iii) $\widetilde{\gamma}^e_c(K_{m,n}) = \begin{cases} 2 & \text{if } |m-n| \leq 1 \\ m+n & \text{otherwise} \end{cases}$

iv) $\widetilde{\gamma}^e_c(P_n) = \widetilde{\gamma}_c(P_n) = \begin{cases} n-1 & \text{if n = 2, 3} \\ n-2 & \text{if } n \geq 4 \end{cases}$.

v) $\widetilde{\gamma}^e_c(C_n) = \widetilde{\gamma}_c(C_n) = n - 2$ for $n \geq 3$.

vi) $\widetilde{\gamma}_c^e(W_n) = n - 2$ for $n \geq 6$

vii) $\widetilde{\gamma}_c^e(K_{n_1,n_2,\cdots,n_r}) = 2t + \sum_{j=1}^{k} |S_j|$, where t is the minimum number of equitable partitions of $n_1, n_2, \cdots, n_r$ and S_j are singleton partitions containing $n_{i_1}, n_{i_2}, \cdots, n_{i_k}$ vertices.
Example 2.1. Consider $G = K_{2,3,5,6,6,8,10,12}$. $\widetilde{\gamma}_c^e(K_{2,3,5,6,6,8,10,12}) = 2 * 2 + 8 + 10 + 12 = 34$, since there are two equitable partitions namely $\{2,3\}$ and $\{5,6,6\}$ and three singleton sets $\{8\}, \{10\}, \{12\}$.

viii) $K_m(a_1, a_2, \cdots, a_m)$ is the graph obtained from K_m by adding a_1 pendents, a_2 pendents, $\cdots, a_m$ pendents at vertices $u_1, u_2, \cdots, u_m$ of K_m respectively. Then,
$\widetilde{\gamma}_c^e((K_m(a_1, a_2, \cdots, a_m)) = a_1 + a_2 + \cdots + a_m + t + 1$ where t is the number of equitable isolates among the vertices of K_m in $K_m(a_1, a_2, \cdots, a_m)$

ix) $\widetilde{\gamma}_c^e(P) = \widetilde{\gamma}_c(P) = 4$, where P is the Petersen graph.

Theorem 2.1. *If $G_1, G_2, \cdots, G_k$ are the components of G and D_i is a minimum outer connected equitable dominating set of $G_i, 1 \leq i \leq k$, then $\widetilde{\gamma}_c^e(G) = |V(G)| - \max\{|V(G_i)| - \widetilde{\gamma}_c^e(G_i) : 1 \leq i \leq k\}$*
Proof follows on the same lines as in Theorem 1 of [6].

Remark 2.1. *If G is an equitable graph, then $\widetilde{\gamma}_c^e(G) = \widetilde{\gamma}_c(G)$*

3. Bounds

Remark 3.1. *For any graph G of order n, $1 \leq \widetilde{\gamma}_c^e(G) \leq n$*

Observation 3.1. *Let the order of G be n. Then, $\widetilde{\gamma}_c^e(G) = 1$ if and only if either $G = K_n$ or $G = H + K_1$ where H is a connected bi-regular graph of order $n - 1$ with bi regularity $(n - 3, n - 2)$ or a regular graph with regularity $n - 3$.*
For, suppose $\widetilde{\gamma}_c^e(G) = 1$. Then, there exists a vertex u in G such that u equitably dominates all the vertices in $V(G) - \{u\}$ and $V(G) - \{u\}$ is connected. That is, any vertex of G other than u is of degree $n - 3$ or $n - 2$ in $< V(G) - \{u\} >$. If every vertex of $< V(G) - \{u\} >$ is of degree $n - 2$, then $G = K_n$. Otherwise, there exists a vertex of degree $n - 3$. If every vertex of $< V(G) - \{u\} >$ is of degree $n - 3$, then $< V(G) - \{u\} >$ is regular of regularity $n - 3$. Otherwise, any vertex of $< (V(G) - \{u\} >$ is of degree $n - 3$ or $n - 2$. That is, $G = (< V(G) - \{u\} >) + \{u\} = H + K_1$, where $H =< V(G) - \{u\} >$ and $K_1 = \{u\}$ with H being connected and bi-regular with bi regularity $(n - 3, n - 2)$. The converse is obvious.

Example 3.1. The fan graph $P_3 + K_1$, $C_4 + K_1$, $(K_n - e) + K_1$ and K_n.

Observation 3.2. *$\widetilde{\gamma}_c^e(G) = n$ if and only if every vertex of G is an equitable isolate of G.*
For, suppose $\widetilde{\gamma}_c^e(G) = n$. If a vertex of u of G is not an equitable isolate of G, then there exists v in V(G) such that u and v are adjacent and degree equitable. Hence $V(G) - \{u\}$ is an outer connected equitable dominating set of G. Therefore, $\widetilde{\gamma}_c^e(G) \leq n - 1$, a contradiction. Hence, every vertex of G is an equitable isolate. The converse is obvious.

Example 3.2. $\overline{K_n}$, $K_{1,n}$, $n \geq 3$, $K_m(a_1, a_2, \cdots, a_m)$ where $a_1, a_2, \cdots, a_m$ form an arithmetic progression with first term $a_1 \geq 1$ and common difference 2.

Observation 3.3. *$\widetilde{\gamma}_c^e(G) \leq n - k$ if and only if there exists a connected subgraph H of G of order k such that every vertex of H is equitably adjacent with some vertex of $V(G) - V(H)$*

Observation 3.4. *$\widetilde{\gamma}_c^e(G) = n - 1$ if and only if G contains an induced subgraph H with $\widetilde{\gamma}_c^e(H) = n - 1$ (that is, every vertex of H is an equitable isolate of H) and a vertex u not in V(H) which is equitably adjacent with a vertex of H or the vertices in H which have equitable neighbour in H are independent and also independent of u except possibly the equitable adjacent vertex of u in H.*
Proof. Suppose $\widetilde{\gamma}_c^e(G) = n - 1$. Let D be a $\widetilde{\gamma}_c^e$ - set of G. Then $|D| = n - 1$. Therefore, there exists u in $V(G) - D$ and u is equitably dominated by D. Further, if $\widetilde{\gamma}_c^e(< D >)$ is less than $n - 1$, then there exists a vertex v in D which is equitably dominated by $D - \{v\}$. If u and v are adjacent, then $D - \{v\}$ is an outer connected equitable dominating set of G and so, $\widetilde{\gamma}_c^e(G) \leq n - 2$, a contradiction. Therefore, those vertices in D which have equitable neighbour in $< D >$ are independent and also not adjacent with u except possibly the vertex in H which is equitably adjacent with u. If $\widetilde{\gamma}_c^e(< D >) = n - 1$, then every vertex of $< D >$ is an equitable isolate. The converse is obvious.

Example 3.3. $tK_2 \cup sK_1 \cup S$ where S is a set of equitable isolates, $t \geq 1$, $s \geq 0$, $|S| \geq 0$, P_3, $D_{r,s}$ where $|r - s| \leq 1$. Also, when $G = 3K_2$, with vertices u_1 to u_6 where u_1 is adjacent with u_2, u_3 is adjacent with u_4, u_5 is adjacent with u_6, $D = V(G) - \{u_1\}$ is an outer connected, equitable dominating set of G and u_3, u_5 are not equitable isolates of $< D >$ but they are independent and also independent of u_1. Note that $\widetilde{\gamma}_c^e(< D >) = 4 < 5 = 6 - 1$.
When $G = 2K_2 \cup K_1 \cup K_{1,3}$, $\widetilde{\gamma}_c^e(G) = 8 = |V(G)| - 1$. $\widetilde{\gamma}_c(G)$ is also 8.
When G is a star $K_{1,n}$, $\widetilde{\gamma}_c^e(G) = n + 1$ and $\widetilde{\gamma}_c(G) = n$.
Let $G = P_3 \circ K_1$ with vertices $u_1, u_2, u_3, v_1, v_2, v_3$ where u_1, u_2, u_3 are the vertices of

P_3 and v_i are the pendents adjacent with u_i, $1 \leq i \leq 3$. The equitable domination number of G is 3, the domination number of G is 3, the outer connected domination number of G is 3 and the outer connected equitable domination number of G is 4. Let $H = P_3 \circ K_1$. Attach one more pendent vertex at the middle vertex of P_3. Join the pendent vertex at one end of P_3 with one of the pendent vertices of the middle vertex of P_3. Let G be the resulting graph. The equitable domination number of G is 4 and the outer connected equitable domination number of G is 5. The domination number of G is 3 and the outer connected domination number of G is 4.

Sampathkumar and Walikar [11] have proved that $\frac{n(G)}{\Delta(G)+1} \leq \gamma_c(G) \leq 2m(G) - n(G)$. In Theorem 2 [6], it has been proved that $\frac{n(G)}{(\Delta(G)+1)} \leq \widetilde{\gamma}_c(G) \leq 2m(G) - n(G) + 1$.

Theorem 3.1. *If G is a connected graph with at least one pair of equitable adjacent vertices, with order of G namely $n(G) \geq 2$, then $\frac{n(G)}{(\Delta(G)+1)} \leq \widetilde{\gamma}_c^e(G) \leq 2m(G) - n(G) + 1$.*

Proof. Since, $\widetilde{\gamma}_c(G) \leq \widetilde{\gamma}_c^e(G)$ and since $\frac{n(G)}{(\Delta(G)+1)} \leq \widetilde{\gamma}_c(G)$, we have $\frac{n(G)}{(\Delta(G)+1)} \leq \widetilde{\gamma}_c^e(G)$. Since G is connected and $n(G) \geq 2$, $m(G) \geq n(G) - 1$. Since, G has a pair of equitable adjacent vertices, $\widetilde{\gamma}_c^e(G) \leq n(G) - 1 \leq 2m(G) - n(G) + 1$.

The bounds are sharp as seen from the following remarks.

Remark 3.2. *Let S be the family of graphs G which contain an outer connected equitable dominating set D such that $pne[v, D] = \Delta_e(G) + 1$.*
For example, in K_n, $n \geq 3$, $D = \{u\}$ for any u in $V(K_n)$ is an outer connected equitable dominating set of K_n. $N_e[u] = V(K_n)$ and $N_e[D - \{u\}] = \varphi$. $|pne[u, D]| = |N_e[u] - N_e[D - \{u\}]| = n - 0 = n$. Also, $\Delta(K_n) = n - 1$. Therefore, $|pne[u, D]| = \Delta_e(G) + 1$. In such graphs, $\frac{n(G)}{(\Delta(G)+1)} = \widetilde{\gamma}_c^e(G)$. Another example is $K_n - e$. For instance, in C_4 with a diagonal, n = 4, $\Delta_e(G) + 1 = 3 + 1 = 4$, $\frac{n(G)}{(\Delta_e(G)+1)} = 1 = \widetilde{\gamma}_c^e(G)$.

Remark 3.3. *Let $D_{r,s}$ be the double star with centres u, v and r pendants attached to u and s pendants attached at v. Let $r = s \geq 3$. Then, m = number of edges $= 2r + 1$. n =number of vertices$= 2r + 2$. $\widetilde{\gamma}_c^e(D_{r,r}) = 2r + 1$. $2m - n + 1 = 4r + 2 - 2r - 2 + 1 = 2r + 1$. Hence, in $D_{r,r}$ the upper bound is realized.*

Theorem 3.2. *If G is a graph, then* $\widetilde{\gamma}_c^e(G) \geq n(G) - \frac{(m(G)+1)}{2}$.

Proof. By Theorem 3 of [6], $\widetilde{\gamma}_c(G) \geq n(G) - \frac{(m(G)+1)}{2}$ and since $\widetilde{\gamma}_c^e(G) \geq \widetilde{\gamma}_c(G)$ the result follows.

Remark 3.4. *Let* $G = P_4$. $\widetilde{\gamma}_c^e(G) = 2, n(G) = 4, m(G) = 3$. $n(G) - \frac{(m(G)+1)}{2} = 2$. *Therefore,* $\widetilde{\gamma}_c^e(G) = n(G) - \frac{(m(G)+1)}{2}$. *Thus the upper bound is realized.*

Remark 3.5. *In a graph G with* $\delta_e(G) \geq 2$ *there is a cycle of length at least* $\delta_e(G) + 1$ *whose vertices are degree equitable in G.*

Proof. Let $(v_0, v_1, \cdots, v_k)$ be a longest equitable path in G. (that is, the degrees of $v_0, v_1, \cdots, v_k$ in G are equitable). Then, $N_e(v_0)$ is a subset of $\{v_1, \cdots, v_k\}$. Therefore, there exists $v_t \in N_e(v_0) \cap \{v_1, \cdots, v_k\}$ for some $t \geq deg_e(v_0) \geq \delta_e(G)$. Consequently, $(v_0, v_1, \cdots, v_t, v_0)$ is a required cycle whose vertices are degree equitable in G.

Theorem 3.3. *If G is a connected graph of order n without degree equitable isolate, then* $\widetilde{\gamma}_c^e(G) \leq n - \delta_e(G)$.

Proof. Suppose $\delta_e(G) \leq 2$. Since G has no degree equitable isolate, $\delta_e(G) \geq 1$. If $\delta_e(G) = 1$, then G has an equitable edge and hence $\widetilde{\gamma}_c^e(G) \leq n - 1 = n - \delta_e(G)$. Suppose $\delta_e(G) = 2$. Then there exists a cycle of length at least three whose vertices are degree equitable in G. Therefore, $\widetilde{\gamma}_c^e(G) \leq n - 2 = n - \delta_e(G)$. Let $\delta_e(G) \geq 3$. Then, by the remark stated above, there exists a cycle of length at least $\delta_e(G)+1$ in G whose vertices are degree equitable in G. Let $C = (v_0, v_1, \cdots, v_t, v_0)$ be a shortest cycle in G of length at least $\delta_e(G)$ whose vertices are degree equitable in G. Let $D = V(G) - V(C)$. Clearly, D is outer connected. Suppose D is not degree equitable dominating. Then there exists u in V(C) such that $N_e(u) \cap D = \varphi$. Let without loss of generality, $u = v_0$. Let $deg_G(v_0) = k$. Then $N_e(v_0) = \{v_1, v_{i_1}, v_{i_2}, \cdots, v_{i_{(k-2)}}, v_t\}$, where $1 < i_1 < i_2 < \cdots < i_{(k-2)} < t$. Hence, $(v_0, v_{i_1}, v_{i_1+1}, \cdots, v_t, v_0)$ is a cycle of length at least $\delta_e(G)$ whose vertices are degree equitable in G which is shorter than C, a contradiction. Therefore, D is an outer connected equitable dominating set of G and $\widetilde{\gamma}_c^e(G) \leq |D| = n - \delta_e(G)$.

Theorem 3.4. *If T is a tree of order* $n \geq 3$, *then* $\widetilde{\gamma}_c^e(T) \geq \Delta_e(T)$. *Equality holds when* $T = P_2$ *or* P_3 *or* P_4 ($\Delta_e(P_4) = 2$ *and* $\widetilde{\gamma}_c^e(P_4) = 2$).

Proof. Let T be a tree of order $n \geq 3$. Since T has at least $\Delta(T)$ end vertices and since any end vertex belongs to every outer connected equitable dominating set of T, $\widetilde{\gamma}_c^e(G) \geq \Delta(T) \geq \Delta_e(T)$.

Remark 3.6. *When T is a star of order $n \geq 4$, $\widetilde{\gamma}_c^e(T) = n + 1 > \Delta(T)$ and $\Delta_e(T) = 0$. When T is a wounded spider of order $n \geq 5$, $\widetilde{\gamma}_c^e(T) > \Delta_e(T)$. When $n = 4$, and exactly one leg is subdivided, then $T = P_4$ and $\widetilde{\gamma}_c^e(T) = \Delta_e(T)$.*

Theorem 3.5. *Let D be an outer connected equitable dominating set of G. Then D is minimal if and only if for any u in D, one of the following holds.*

i) u is an equitable isolate of $< D >$

ii) there exists v in $V - D$ such that v is equitably adjacent with only u in D.

iii) $N(u) \cap (V - D) = \varphi$.

Proof. Let D be a minimal outer connected equitable dominating set of G. Then for any u in D, $D - \{u\}$ is not an outer connected equitable dominating set of G. Either $D - \{u\}$ is not an equitable dominating set of G or an equitable dominating set of G but not outer connected. In the former case, u satisfies (i) or (ii) and in the latter case, u satisfies (iii). The converse is obvious.

Remark 3.7. *Let $G = P_5$ with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ where v_1 and v_5 are end vertices. $D = \{v_2, v_3, v_4\}$ is a minimal outer connected equitable dominating set of G. v_2 and v_4 satisfy (i) as well as (ii) and v_3 satisfies (iii).*

Proposition 3.1. *Let G be a connected graph without equitable isolate. Let $v \in V(G)$ be such that $G - \{v\}$ is also connected. Then $\widetilde{\gamma}_c^e(G) \leq \widetilde{\gamma}_c^e(G - \{v\}) + 1$.*
Proof. Let D be a $\widetilde{\gamma}_c^e$ - set of $G - \{v\}$. Then $D \cup \{v\}$ is an outer connected dominating set of G. Suppose $D \cup \{v\}$ is not an equitable dominating set of G. Then there exists some vertex u in $(V - \{v\}) - D$ which was equitably dominated by some vertex say w of D which was not equitably dominated by $D \cup \{v\}$. That is, $|deg_{G-\{v\}}(u) - deg_{G-\{v\}}(w)| \leq 1$ and $|deg_G(u) - deg_G(w)| \geq 2$. That is v is adjacent with u and not with w. That is, u is an equitable isolate in G but not an equitable isolate in $G - \{v\}$, a contradiction. Therefore, $D \cup \{v\}$ is an equitable dominating set of G. Therefore, $\widetilde{\gamma}_c^e(G) \leq |D \cup \{v\}| = \widetilde{\gamma}_c^e(G - \{v\}) + 1$.

Remark 3.8. *There is no relation between $\widetilde{\gamma}_c^e(G)$ and $\widetilde{\gamma}_c^e(G - \{v\})$.*
For, let $G = P_5$. $\widetilde{\gamma}_c^e(G) = 3$. Let v be a pendent vertex. $\widetilde{\gamma}_c^e(G - \{v\}) = 2$. Hence, $\widetilde{\gamma}_c^e(G) > \widetilde{\gamma}_c^e(G - \{v\})$.
Let G be the subdivision of $K_{1,3}$. Let u be the centre of $K_{1,3}$, v_1, v_2, v_3 be the pendents of $K_{1,3}$. Let u_1, u_2, u_3 be the subdividing vertices of uv_1, uv_2, uv_3. Join v_1 with u_2 and v_2. Let H be the resulting graph. $\widetilde{\gamma}_c^e(H) = 3$ (since $\{v_1, u_3, v_3\}$ is a minimum outer connected equitable dominating set of H). $\widetilde{\gamma}_c^e(H - u_1) = 3$ (since $\{v_2, v_3, u_3\}$ is a minimum outer connected equitable dominating set of $H - u_1$).

Thus, $\widetilde{\gamma}_c^e(H) = \widetilde{\gamma}_c^e(H - u_1) = 3$.
Let $G = P_3 + K_1$. Let $V(K_1) = \{u\}$. $\widetilde{\gamma}_c^e(G) = 1$. $\widetilde{\gamma}_c^e(G - \{u\}) = \widetilde{\gamma}_c^e(P_3) = 2$. G has no equitable isolate, G and $G - \{u\}$ are connected. Thus, $\widetilde{\gamma}_c^e(G - \{v\}) > \widetilde{\gamma}_c^e(G)$.

Remark 3.9. *The condition that G has no equitable isolates can not be dropped from Proposition 3.1.*
For example, let $G = K_{1,3}$. Let u be the central vertex and v_1, v_2, v_3 be the pendents. $\widetilde{\gamma}_c^e(G) = 4$ but $\widetilde{\gamma}_c^e(G - v_1) = 2$ and hence $\widetilde{\gamma}_c^e(G)$ is not less than or equal to $\widetilde{\gamma}_c^e(G - \{v_1\}) + 1$.

Observation 3.5. *Given a positive integer k, there exists a connected graph G such that $\widetilde{\gamma}_c^e(G) - \gamma(G) = k$.*
Proof. Let k be even. Let $t = \frac{k+2}{2}$. Let $n = 3t$. Let $G = P_n$. $\widetilde{\gamma}_c^e(G) = n-2 = 3t-2$. $\gamma(G) = t$. Therefore, $\widetilde{\gamma}_c^e(G) - \gamma(G) = 3t - 2 - t = 2t - 2 = k$. Let k be odd. Let $t = \frac{k+1}{2}$. Let $n = 3t + 2$. $\widetilde{\gamma}_c^e(G) = n - 2 = 3t$ and $\gamma(G) = t + 1$. Therefore, $\widetilde{\gamma}_c^e(G) - \gamma(G) = 3t - (t + 1) = 2t - 1 = k$.

Observation 3.6. *Given a positive integer k, there exists a connected graph G such that $\widetilde{\gamma}_c^e(G) - \gamma_e(G) = k$.*

The example given above serves the purpose since in P_n, $\gamma_e(G) = \gamma(G)$.

Observation 3.7. *Given a positive integer k, there exists a graph G such that $\widetilde{\gamma}_c^e(G) - \widetilde{\gamma}_c(G) = k$.*
For example, when $k \geq 4$, consider $G = K_{2,k}$, $\widetilde{\gamma}_c^e(G) = k+2$, $\widetilde{\gamma}_c(G) = 2$. Therefore, $\widetilde{\gamma}_c^e(G) - \widetilde{\gamma}_c(G) = k$. When $k = 1$, $\widetilde{\gamma}_c^e(K_{m,1}) - \widetilde{\gamma}_c(K_{m,1}) = 1$. Let $k = 2$. Consider, P_3 with vertices u_1, u_2, u_3. Attach three pendents each at each of the vertices of P_3. Let G be the resulting graph. $\widetilde{\gamma}_c^e(G) = 11$ and $\widetilde{\gamma}_c(G) = 9$. Hence, $\widetilde{\gamma}_c^e(G) - \widetilde{\gamma}_c(G) = 2$. Let $k = 3$. Let $G = W_6$. $\widetilde{\gamma}_c^e(G) = 4$, $\widetilde{\gamma}_c(G) = 1$. Hence $\widetilde{\gamma}_c^e(G) - \widetilde{\gamma}_c(G) = 3$.

4. Complexity of $\widetilde{\gamma}_c^e(G)$

Proposition 4.1. *The decision problem of outer connected equitable dominating set is NP - complete even when restricted to bipartite graphs.*
Proof. In Theorem 14 of [6], it is proved that the decision problem of outer connected dominating set is NP - complete. Let G be a bipartite graph with $\widetilde{\gamma}_c(G) = k$. Add suitable number of pendent vertices at each vertex of G so that the resulting graph G_1 is equitable bipartite with the degree of every vertex being $\Delta(G) + 1$. Let all the pendent vertices be joined so that the pendent vertices induces a complete subgraph of G_1. Let G_2 be the resulting graph. Then $\widetilde{\gamma}_c^e(G_2) = \widetilde{\gamma}_c(G) + 1$. Therefore, by Theorem 14 [6], outer connected equitable domination set is NP - complete even when restricted to bipartite graphs.

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