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ON SOME APPLICATION OF BAILEY'S TRANSFORM IN GENERALIZED BASIC HYPERGEOMETRIC TRANSFORMATIONS

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Abstract: The object of this paper is to establish some transformations for generalized basic hypergeometric series by the application of Bailey's transform and some known results.

Keywords and Phrases: Generalized basic hypergeometric function, Bailey's transform,q-shifted factorial.

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1. Introduction

For $q < 1$ and real or complex a, the q-shifted factorial is defined as

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}) & \text{if } n \in N \end{cases} \tag{1.1}$$

For any complex number α,

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \tag{1.2}$$

The general basic hypergeometric series ${}_r\phi_s$ [2, (1.2.22), p.4] is defined as

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ & ; q, z \\ b_1, b_2, \dots, b_s \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \dots (a_r;q)_n}{(q;q)_n (b_1;q)_n \dots (b_s;q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n. \tag{1.3}$$

where $q \neq 0$ when $r > s+1$.The series ${}_r\phi_s$ terminates if one of the numerator parameters is of the form q^{-m} with $m = 0, 1, 2 \dots$, and $q \neq 0$.
Another form of the general basic hypergeometric series (cf. Slater [5]), is defined as

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ & ; q, z \\ b_1, b_2, \dots, b_s \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \dots (b_s;q)_n} \frac{z^n}{(q;q)_n} \tag{1.4}$$

in which there are always r of the a parameters and s of the b parameters.
The q-Gauss summation formula is given by

$${}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty} \quad |c/ab| < 1 \tag{1.5}$$

In 1944, Bailey established a remarkably simple and useful transformation formula which is given in the following form:
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.6}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{1.7}$$

where α_r,δ_r,u_r and v_r are functions of r only such that the series of γ_n exists, then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.8}$$

In this paper, we shall use the following results due to Verma and Jain [6].

$$ {}_4\Phi_3\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; -q^{\frac{1}{2}+n} \\ \sqrt{a}, -\sqrt{a}, aq^{1+n} \end{array}\right] = \frac{(aq;q)_n(-q^{-1/2};q)_n}{2\sqrt{a}} \times $$

$$ \times\left[\frac{1}{(\sqrt{aq};q)_n(-q\sqrt{a};q)_{n-1}} - \frac{1}{(-\sqrt{aq};q)_n(q\sqrt{a};q)_{n-1}}\right] \tag{1.9} $$

$$ {}_3\Phi_2\left[\begin{array}{l} a, q\sqrt{a}, q^{-n}; -q^{1+n} \\ \sqrt{a}, aq^{1+n} \end{array}\right] = \frac{(aq,-1;q)_n}{2\sqrt{2}}\left[\frac{(1+\sqrt{a})}{(aq;q^2)_n} - \frac{(1-\sqrt{a})}{(\sqrt{a};q)_n(-q\sqrt{a};q)_n}\right] \tag{1.10} $$

$$ {}_2\Phi_1\left[\begin{array}{l} a, q^{-n}; -q^{\frac{3}{2}+n} \\ aq^{1+n} \end{array}\right] = \frac{(aq,-\sqrt{q};q)_n}{2} \times $$

$$ \times\left[\frac{(1+\sqrt{a})}{(-\sqrt{aq};q)_n(q\sqrt{a};q)_n} - \frac{(1-\sqrt{a})}{(\sqrt{aq};q)_n(-q\sqrt{a};q)_n}\right] \tag{1.11} $$

We shall also use the following results (cf. Gasper and Rahman [2]).

$$ {}_4\phi_3\left[\begin{array}{l} a, -q\sqrt{a}, b, q^{-n} \\ -\sqrt{a}, aq/b, aq^{1+n} \end{array}; q, \frac{\sqrt{a}q^{1+n}}{b}\right] = \frac{(aq, q\sqrt{a}/b;q)_n}{(q\sqrt{a}, aq/b;q)_n} \tag{1.12} $$

$$ {}_6\phi_5\left[\begin{array}{l} a, -q\sqrt{a}, q\sqrt{a}, b, c, q^{-n} \\ -\sqrt{a}, \sqrt{a}, aq/b, aq/c, aq^{1+n} \end{array}; q, \frac{\sqrt{a}q^{1+n}}{bc}\right] = \frac{(aq, aq/bc;q)_n}{(aq/b, aq/c;q)_n} \tag{1.13} $$

The main object of the present article is to establish transformations formulae by use of Bailey's transform and known results [1, 2, 5, 6]. For more details and further results, the interested reader may be referred to the works presented in [3, 7] (see also the related recent works [4, 8]).

2. Main Results

$$ {}_5\Phi_5\left[\begin{array}{l} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c; q, \frac{-a\sqrt{q}}{bc} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, 0 \end{array}\right] = \frac{1}{2\sqrt{a}}\prod\left[\begin{array}{l} aq, aq/bc; q \\ aq/b, aq/c \end{array}\right] \times $$

$$\times\left\{\sqrt{a}\ {}_3\Phi_2\left[\begin{matrix} b, c, -q^{-\frac{1}{2}}; q, \frac{aq^2}{bc} \\ \sqrt{aq}, -q\sqrt{a} \end{matrix}\right] + {}_3\Phi_2\left[\begin{matrix} b, c, -q^{-\frac{1}{2}}; q, \frac{aq}{bc} \\ \sqrt{aq}, -q\sqrt{a} \end{matrix}\right]\right.$$

$$\left.-\sqrt{a}\ {}_3\Phi_2\left[\begin{matrix} b, c, -q^{-\frac{1}{2}}; q, \frac{aq^2}{bc} \\ -\sqrt{aq}, q\sqrt{a} \end{matrix}\right] + {}_3\Phi_2\left[\begin{matrix} b, c, -q^{-\frac{1}{2}}; q, \frac{aq^2}{bc} \\ -\sqrt{aq}, q\sqrt{a} \end{matrix}\right]\right\} \quad (2.1)$$

$${}_4\phi_4\left[\begin{matrix} a, \ q\sqrt{a}, \ b, \ c \\ & ; q, -\frac{aq^2}{bc} \\ \sqrt{a}, aq/b, aq/c, 0 \end{matrix}\right] = \frac{1}{2\sqrt{2}}\prod\left[\begin{matrix} aq, aq/bc; q \\ aq/b, aq/c \end{matrix}\right]\times$$

$$\left\{(1+\sqrt{a})\ {}_3\Phi_2\left[\begin{matrix} b, c, -1; q, \frac{aq}{bc} \\ \sqrt{aq}, -\sqrt{aq} \end{matrix}\right] - (1-\sqrt{a}){}_3\Phi_2\left[\begin{matrix} b, c, -1; q, \frac{aq}{bc} \\ \sqrt{a}, -q\sqrt{a} \end{matrix}\right]\right\} \quad (2.2)$$

$${}_3\phi_3\left[\begin{matrix} a, \ \ b, \ \ c \\ & ; q, -\frac{aq^{5/2}}{bc} \\ aq/b, aq/c, 0 \end{matrix}\right] = \frac{1}{2\sqrt{2}}\prod\left[\begin{matrix} aq, aq/bc; q \\ aq/b, aq/c \end{matrix}\right]\times$$

$$\left\{(1+\sqrt{a})\ {}_3\Phi_2\left[\begin{matrix} b, c, -\sqrt{q}; q, \frac{aq}{bc} \\ -\sqrt{aq}, q\sqrt{a} \end{matrix}\right] - (1-\sqrt{a}){}_3\Phi_2\left[\begin{matrix} b, c, -\sqrt{q}; q, \frac{aq}{bc} \\ \sqrt{aq}, -q\sqrt{a} \end{matrix}\right]\right\} \quad (2.3)$$

$${}_5\Phi_5\left[\begin{matrix} a, \ -q\sqrt{a}, \ b, \ \alpha, \ \beta \\ & ; q, \frac{a^{3/2}q^2}{b^2c} \\ \sqrt{a}, aq/b, aq/\alpha, aq/\beta, 0 \end{matrix}\right]$$

$$= \prod\left[\begin{matrix} aq, aq/\alpha\beta; q \\ aq/\alpha, aq/\beta \end{matrix}\right]{}_3\phi_2\left[\begin{matrix} \alpha, \ \beta, \ q\sqrt{a}/b \\ & ; q, \frac{aq}{\alpha\beta} \\ aq/b, \ q\sqrt{a} \end{matrix}\right] \quad (2.4)$$

$${}_7\Phi_7\left[\begin{matrix} a, \ q\sqrt{a}, \ -q\sqrt{a}, \ b, \ c, \ \alpha, \ \beta \\ & ; q, \frac{a^2q^2}{\alpha\beta bc} \\ -\sqrt{a}, \sqrt{a}, aq/b, aq/c, aq/\alpha, aq/\beta, 0 \end{matrix}\right]$$

$$= \prod\left[\begin{matrix} aq, aq/\alpha\beta; q \\ aq/\alpha, aq/\beta \end{matrix}\right]{}_3\phi_2\left[\begin{matrix} \alpha, \ \beta, \ q\sqrt{a}/b \\ & ; q, \frac{aq}{\alpha\beta} \\ aq/b, \ q\sqrt{a} \end{matrix}\right] \quad (2.5)$$

3. Proof of the Main Results

To prove the result (2.1).

Take $\alpha_r = \frac{(a, q\sqrt{a}, -q\sqrt{a}; q)_r}{(q, \sqrt{a}, -\sqrt{a}; q)_r} q^{r(r-1)/2} q^{r/2}$, $u_r = \frac{1}{(q;q)_r}$, $v_r = \frac{1}{(aq;q)_r}$ and $\delta_r = (b,c;q)_r \left(\frac{aq}{bc}\right)^r$ in the equations (1.6) and (1.7), we get

$\beta_n = \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}; q)_r q^{r(r-1)/2} q^{r/2}}{(q, \sqrt{a}, -\sqrt{a}; q)_r (q;q)_{n-r} (aq;q)_{n+r}}$ and

$\gamma_n = \sum_{r=0}^{\infty} \frac{(b,c;q)_{n+r}}{(q;q)_r (aq;q)_{2n+r}} \left(\frac{aq}{bc}\right)^r$ these on simplification give

$$\beta_n = \frac{1}{(q;q)_n (aq;q)_n} \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; q)_r \left(-q^{\frac{1}{2}+n}\right)^r}{(q, \sqrt{a}, -\sqrt{a}, aq^{1+n}; q)_r}$$

and

$$\gamma_n = \frac{(b,c;q)_n}{(aq;q)_{2n}} \left(\frac{aq}{bc}\right)^n \sum_{r=0}^{\infty} \frac{(bq^n, cq^n; q)_r}{(q;q)_r (aq^{1+2n};q)_r} \left(\frac{aq}{bc}\right)^r .$$

Now using (1.9) and (1.5) respectively we get the following

$$\beta_n = \frac{(-q^{-1/2};q)_n}{2\sqrt{a}(q;q)_n} \left[\frac{1}{(\sqrt{aq};q)_n (-q\sqrt{a};q)_{n-1}} - \frac{1}{(-\sqrt{aq};q)_n (q\sqrt{a};q)_{n-1}} \right]$$

and

$$\gamma_n = \prod \left[\begin{matrix} aq/b, aq/c; q \\ aq, aq/bc \end{matrix} \right] \frac{(b,c;q)_n}{(aq/b, aq/c; q)_n} \left(\frac{aq}{bc}\right)^n$$

Using these the equation (1.8) can be written as

$$\sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}; q)_n}{(q, \sqrt{a}, -\sqrt{a}; q)_n} q^{n(n-1)/2} q^{n/2} \prod \left[\begin{matrix} aq/b, aq/c; q \\ aq, aq/bc \end{matrix} \right] \frac{(b,c;q)_n}{(aq/b, aq/c; q)_n} \left(\frac{aq}{bc}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-q^{-1/2};q)_n}{2\sqrt{a}(q;q)_n} \left[\frac{1}{(\sqrt{aq};q)_n (-q\sqrt{a};q)_{n-1}} - \frac{1}{(-\sqrt{aq};q)_n (q\sqrt{a};q)_{n-1}} \right] (b,c;q)_n \left(\frac{aq}{bc}\right)^n$$

This on simplification, finally gives the result (2.1).
Proofs of the results (2.2) and (2.3) can be achieved similarly.
To prove (2.4).

Take $\alpha_r = \frac{(a, -q\sqrt{a}, b; q)_r}{(q, -\sqrt{a}, aq/b; q)_r} q^{r(r-1)/2} \left(-\frac{q\sqrt{a}}{b}\right)^r$, $u_r = \frac{1}{(q;q)_r}$, $v_r = \frac{1}{(aq;q)_r}$ and $\delta_r = (\alpha, \beta; q)_r \left(\frac{aq}{\alpha\beta}\right)^r$ in the equations (1.6) and (1.7), we get

$$\beta_n = \sum_{n=0}^{\infty} \frac{(a, -q\sqrt{a}, b; q)_r q^{r(r-1)/2}}{(q, -\sqrt{a}, aq/b; q)_r (q;q)_{n-r} (aq;q)_{n+r}} \left(-\frac{q\sqrt{a}}{b}\right)^r \text{ and}$$

$$\gamma_n = \sum_{r=0}^{\infty} \frac{(\alpha, \beta; q)_{n+r}}{(q;q)_r (aq;q)_{2n+r}} \left(\frac{aq}{\alpha\beta}\right)^r \text{ these on simplification give}$$

$$\beta_n = \frac{1}{(q;q)_n (aq;q)_n} \sum_{n=0}^{\infty} \frac{(a, -q\sqrt{a}, b, q^{-n}; q)_r}{(q, -\sqrt{a}, aq/b, aq^{1+n}; q)_r} \left(-\frac{\sqrt{a}q^{1+n}}{b}\right)^r$$

and

$$\gamma_n = \frac{(\alpha, \beta; q)_n}{(aq;q)_{2n}} \left(\frac{aq}{\alpha\beta}\right)^n \sum_{r=0}^{\infty} \frac{(\alpha q^n, \beta q^n; q)_r}{(q;q)_r (aq^{1+2n}; q)_r} \left(\frac{aq}{\alpha\beta}\right)^r.$$

Now using (1.12) and (1.5) respectively we get the following

$$\beta_n = \frac{(q\sqrt{a}/b; q)_n}{(q, q\sqrt{a}, aq/b; q)_n} \quad \textit{and} \quad \gamma_n = \prod \left[\begin{matrix} aq/\alpha, aq/\beta; q \\ aq, aq/\alpha\beta \end{matrix}\right] \frac{(\alpha, \beta; q)_n}{(aq/\alpha, aq/\beta; q)_n} \left(\frac{aq}{\alpha\beta}\right)^n$$

Using these, the equation (1.8) can be written as

$$\sum_{n=0}^{\infty} \frac{(a, -q\sqrt{a}, b; q)_n}{(q, -\sqrt{a}, aq/b; q)_n} q^{n(n-1)/2} \left(-\frac{q\sqrt{a}}{b}\right)^n \prod \left[\begin{matrix} aq/\alpha, aq/\beta; q \\ aq, aq/\alpha\beta \end{matrix}\right] \times$$

$$\times \frac{(\alpha, \beta; q)_n}{(aq/\alpha, aq/\beta; q)_n} \left(\frac{aq}{\alpha\beta}\right)^n = \sum_{n=0}^{\infty} \frac{(q\sqrt{a}/b; q)_n}{(q, q\sqrt{a}, aq/b; q)_n} (\alpha, \beta; q)_n \left(\frac{aq}{\alpha\beta}\right)^n$$

This on simplification, finally gives the result (2.4).
Proof of the results (2.5) can be achieved similarly.

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