South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 171-182

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

SOME PROPERTIES OF $q-$ANALOGUE OF GENERALIZED MITTAG-LEFFLER FUNCTION ASSOCIATED WITH FRACTIONAL CALCULUS

Krishna Gopal Bhadana and Ashok Kumar Meena

Department of Mathematics,
SPC Government College, Ajmer, Rajasthan,
Maharshi Dayanand Saraswati University, Ajmer,
Rajasthan - 305001, INDIA

E-mail : drbhadanakg@gmail.com, ashokmeena428@gmail.com

(**Received:** *Apr. 30, 2020* **Accepted:** *Feb. 22, 2021* **Published:** *Apr. 30, 2021)*

Abstract: In the present paper we will establish some results and properties of q-generalized Mittag-Leffler function $E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)$. We will get its convergence condition, recurrence relation and many other results associated with fractional calculus such as q-Laplace transform, Riemann-Liouville fractional q-integral operator. We will also discuss some important special cases of main results.

Keywords and Phrases: Generalized q-Mittag Leffler Function, q-Gamma Function, q-Beta Function, q-Laplace transform, q-Derivative, q-Integral.

2020 Mathematics Subject Classification: 33E12, 33D05, 44A20.

1. Introduction

The Mittag-Leffler function has wide applications in many areas of physical sciences, especially in fractional calculus and special functions. The Classical Mittag-Laffler Function [7] (CMLF) is introduced by Swedish Mathematician Gosta Mittag Leffler in 1903. This function was defined as follows for $z \in \mathbb{C}$, $\alpha \in \mathbb{C}$

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}; \quad Re(\alpha) > 0 \tag{1.1}$$

Many researchers have extended the research work of Mittag-Leffler function. In 1930, the solution of the Abel- Voltra type equation in terms of Mittag-Leffler function was given by Hille and Tamarkin [5]. For $\alpha \in (01]$, a statistical distribution [5] was discovered by Pillai (1990) in term of Mittag-Leffler function. This distribution is defined as

$$G_\alpha(z) = \begin{cases} 1 - E_\alpha(-z^\alpha) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{\alpha n}}{\Gamma(\alpha n + 1)} & ; z > 0 \\ 0 & ; z \leq 0 \end{cases} \tag{1.2}$$

The solution of the kinetic equation $N(t) - N_0 = -w^\alpha \ {}_0D_t^{-\alpha} N(t)$ was given by Mathai, Saxena [5] and Haubold (2002) in term of Mittag-Leffler function such that

$$N(t) = N_0 E_\alpha(-w^\alpha t^\alpha) \tag{1.3}$$

Various generalizations of the Classical Mittag-Leffler function $E_\alpha(z)$ with their interesting and useful properties have been given by many researchers, which called Generalized Mittag-Leffler Function (GMLF) or Mittag-Leffler Type Function. A generalization of $E_\alpha(z)$ was studied by Wiman [16], Prabhakar [8], Shukla and Prajapati [10]. A new generalization of Mittag-Leffler function was defined by Salim and Faraj [14] for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $s \leq r + Re(\alpha)$

$$E_{\alpha,\beta,r}^{\gamma,\delta,s}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{sn}}{\gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{rn}}; \ r, s > 0 \min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)\} > 0. \tag{1.4}$$

Some useful definitions, which form the basis of the main results, are given below.

2. Definitions

Definition 2.1. *For $\lambda \in \mathbb{C}$ and $0 < |q| < 1$, the q-shifted factorial is defined as*

$$(\lambda; q)_n = \prod_{k=0}^{n-1} (1 - \lambda q^k) = \frac{(\lambda; q)_\infty}{(\lambda q^n; q)_\infty}; \ n \in \mathbb{N} \ \text{ and } \ (\lambda; q)_0 = 1 \tag{2.1}$$

$$[n]_q! = \frac{(q; q)_n}{(1-q)^n} \tag{2.2}$$

Definition 2.2. *The q-analogue of power function $(a - b)^n$ is*

$$(a - b)^n = \prod_{k=0}^{n-1} (a - bq^k); \quad (a - b)^0 = 1 \tag{2.3}$$

$$(a-b)^n = a^n \left(\frac{b}{a};q\right)_n; \quad a \neq 0 \tag{2.4}$$

Definition 2.3. *The q-Gamma function [2] is defined by*

$$\Gamma_q(\lambda) = (1-q)^{1-\lambda}(1-q)_{\lambda-1} = (1-q)^{1-\lambda}\frac{(q;q)_\infty}{(q^\lambda;q)_\infty} \tag{2.5}$$

and

$$\Gamma_q(\lambda+1) = \frac{1-q^\lambda}{1-q}\Gamma_q(\lambda) \tag{2.6}$$

Definition 2.4. *The q-Beta function [2] is defined by*

$$B_q(a,b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)} = \int_0^1 z^{a-1}(1-qz)_{b-1}d_qz \tag{2.7}$$

Definition 2.5. *The q-derivative of function [2] is defined by*

$$(D_qf)(z) = \frac{f(z)-f(zq)}{z-zq}; \quad z \neq 0, \ q \neq 1 \tag{2.8}$$

and

$$D_q^m z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n-m+1)}z^{n-m} \tag{2.9}$$

Definition 2.6. *The q-integral of function $f(z)$ is defined by*

$$\int_0^a f(z)d_qz = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n) \tag{2.10}$$

Definition 2.7. *The q-Laplace transform [3] is defined as*

$$_qL_s\{f(z)\} = \frac{(q;q)_\infty}{s}\sum_{j=0}^{\infty}\frac{q^j f(s^{-1}q^j)}{(q;q)_j} \tag{2.11}$$

Definition 2.8. *The fractional q-integral operator [6] of order μ is defined as*

$$I_q^\mu f(z) = \frac{1}{\Gamma_q(\mu)}\int_0^z (z-q\xi)^{\mu-1}f(\xi)d_q(\xi); \quad Re(\mu) > 0 \tag{2.12}$$

Definition 2.9. *The fractional q-differential operator [6] of order μ is defined as*

$$D_q^\mu f(z) = D_q^k\{I_q^{k-\mu}f(z)\}; \quad k \text{ is the smallest integer s.t. } k \geq Re(\mu) > 0 \tag{2.13}$$

3. Main Results

Motivated by the applications of q-extension of various special functions in the field of mathematics, we have introduced and defined a new generalized Mittag-Leffler function, called as q-analogue of generalized Mittag-Leffler function $E^{\gamma,\delta,s}_{\alpha,\beta,r}(z,q)$ by the help of equation (1.4) as

$$E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)};\quad r,s>0 \text{ and } s\le r+Re(\alpha), \tag{3.1}$$

$\alpha,\beta,\gamma,\delta\in\mathbb{C};\quad \min\{Re(\alpha),Re(\beta),Re(\gamma),Re(\delta)\}>0$
Where $(\lambda;q)_n$ is q-shifted factorial and $\Gamma_q(n)$ is q-Gamma function.

Special case of $E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)$

(a) For $r=1$, $s=1$, Equation (3.1) reduces to q-GMLF as defined by Sharma and Jain [13]

$$E^{\gamma,\delta,1}_{\alpha,\beta,1}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{n}}{(q^{\delta};q)_{n}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}=E^{\gamma,\delta}_{\alpha,\beta}(z;q) \tag{3.2}$$

(b) Putting $r=1$, $\delta=1$, $s=1$, Equation (3.1) reduces to q-GMLF as defined by Sharma and Jain [12]

$$E^{\gamma,1,1}_{\alpha,\beta,1}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{n}}{(q;q)_{n}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}=E^{\gamma}_{\alpha,\beta}(z;q) \tag{3.3}$$

(c) On taking $r=1$, $\delta=1$, $s=p$ and $z\to z(1-q)$, Equation (3.1) reduces to q-Generalized Mittag-Leffler function as defined by Chanchlani and Garg [3]

$$\begin{aligned}E^{\gamma,1,p}_{\alpha,\beta,1}(z(1-q);q)&=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{pn}}{(q;q)_{n}}\frac{(z(1-q))^n}{\Gamma_q(\alpha n+\beta)}\\&=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{pn}}{\Gamma_q(\alpha n+\beta)}\frac{z^n}{[n]_q!}=E^{\gamma,p}_{\alpha,\beta}(z;q)\end{aligned} \tag{3.4}$$

Theorem 3.1. *Convergence of $E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)$:*
The q-analogue of generalized Mittag-Leffler function introduced by equation (3.1) is convergent for $|z|<(1-q)^{-\alpha}$ when $0<|q|<1$ and if $q\to 1$ then this is absolutely convergent for all value of z, provided that $s<r+Re(\alpha)$.

Proof. From the definition of q-Generalized Mittag-Leffler function given by the Equation (3.1), we have

$$E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}=\sum_{n=0}^{\infty}u_n\ (say)$$

by applying ratio test, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$

$$\Rightarrow \left|\frac{(q^{\gamma};q)_{sn+s}}{(q^{\gamma};q)_{sn}}\frac{(q^{\delta};q)_{rn}}{(q^{\delta};q)_{rn+r}}\frac{\Gamma_q(\alpha n+\beta)}{(\Gamma_q(\alpha n+\alpha+\beta))}\frac{z^{n+1}}{z^n}\right|$$
$$=\left|\frac{(q^{\gamma+sn};q)_{\infty}}{(q^{\gamma+sn+s};q)_{\infty}}\frac{(q^{\delta+rn+r};q)_{\infty}}{(q^{\delta+rn};q)_{\infty}}\frac{(q^{\alpha n+\alpha+\beta};q)_{\infty}}{(q^{\alpha n+\beta};q)_{\infty}}(1-q)^{\alpha}z\right|$$
$$\leq\left|\frac{(1-q)^{\alpha}}{(1-q^{\alpha n+\beta})^{\alpha}}\frac{(1-q^{\gamma+sn})^{s}}{(1-q^{\delta+rn})^{r}}\right|\ |z|$$
$$=\begin{cases}(1-q)^{\alpha}|z| & ;0<|q|<1\\ 0 & ;q=1\end{cases}\quad \text{as } n\to\infty$$

Hence function $E^{\gamma,\delta,s}_{\alpha,\beta,r}(z,q)$ is convergent for $0<|q|<1$ if $|z|<(1-q)^{-\alpha}$. Moreover if $q\to 1$ then equation (3.1) coincide with (1.4), which is generalized Mittag-Leffler function given by Salim and Faraj [14]. which is absolutely convergent for all values of z provided that $s<r+Re(\alpha)$.

Theorem 3.2. *Recurrence Relation:*
If $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\min\{Re(\alpha),Re(\beta),Re(\gamma),Re(\delta)\}>0$ *and* $r,s>0$ *with* $s\leq r+Re(\alpha)$ *then*

$$E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)=E^{\gamma+1,\delta,s}_{\alpha,\beta,r}(z;q)-\frac{q^{r}}{1-q^{\delta}}zE^{\gamma+1,\delta+1,s}_{\alpha,\alpha+\beta,r}(z;q)+\frac{q^{\gamma+1}}{1-q^{\delta}}zE^{\gamma+1,\delta+1,s}_{\alpha,\alpha+\beta,r}(qz;q) \tag{3.5}$$

Proof.

$$E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}=\frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}$$
$$=\frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{(1-q^{\gamma})(q^{\gamma+1};q)_{sn-s}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}$$
$$=\frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{[(1-q^{\gamma+n})-q^{\gamma}(1-q^{n})]\,(q^{\gamma+1};q)_{sn-s}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}$$
$$=\frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{(1-q^{\gamma+n})(q^{\gamma+1};q)_{sn-s}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}$$
$$-q^{\gamma}\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_{sn-s}}{(q^{\delta};q)_{rn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}+q^{\gamma}\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_{sn-s}}{(q^{\delta};q)_{rn}}\frac{q^n z^n}{\Gamma_q(\alpha n+\beta)}$$

Replacing n by $m+1$ in second and third summation

$$= \sum_{n=0}^{\infty} \frac{(q^{\gamma+1};q)_{sn}}{(q^{\delta};q)_{rn}} \frac{z^n}{\Gamma_q(\alpha n+\beta)} - \frac{q^{\gamma}}{1-q^{\delta}} \sum_{m=0}^{\infty} \frac{(q^{\gamma+1};q)_{sm}}{(q^{\delta+1};q)_{rm}} \frac{z^{m+1}}{\Gamma_q(\alpha m+\alpha+\beta)}$$

$$+ \frac{q^{\gamma}}{1-q^{\delta}} \sum_{m=0}^{\infty} \frac{(q^{\gamma+1};q)_{sm}}{(q^{\delta+1};q)_{rm}} \frac{(qz)^{m+1}}{\Gamma_q(\alpha m+\alpha+\beta)}$$

$$= E_{\alpha,\beta,r}^{\gamma+1,\delta,s}(z;q) - \frac{q^{r}}{1-q^{\delta}} z E_{\alpha,\alpha+\beta,r}^{\gamma+1,\delta+1,s}(z;q) + \frac{q^{\gamma+1}}{1-q^{\delta}} z E_{\alpha,\alpha+\beta,r}^{\gamma+1,\delta+1,s}(qz;q)$$

Special case: On taking $r=1$, $s=1$, Equation (3.5) reduces to

$$E_{\alpha,\beta}^{\gamma,\delta}(z;q) = E_{\alpha,\beta}^{\gamma+1,\delta}(z;q) - \frac{q^{r}}{1-q^{\delta}} z E_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(z;q) + \frac{q^{\gamma+1}}{1-q^{\delta}} z E_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(qz;q)$$

which gives the same result as given by Santosh Sharma and Renu Jain [13] in 2016, page 793.

Theorem 3.3. *If the equation (3.1) is satisfied, then*

$$E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = E_{\alpha,\beta,r}^{\gamma,\delta,s}(qz;q) + z(1-q) D_q E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) \tag{3.6}$$

Proof.

$$E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}} \frac{z^n}{\Gamma_q(\alpha n+\beta)} = f(z) \quad \text{(say)}$$

$$\Rightarrow f(qz) = \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}} \frac{q^n z^n}{\Gamma_q(\alpha n+\beta)}$$

$$D_q E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = \frac{f(z)-f(qz)}{z-qz} = \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}} \frac{z^n}{\Gamma_q(\alpha n+\beta)} \frac{(1-q^n)}{(z-zq)}$$

$$z(1-q) D_q E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}} \frac{z^n}{\Gamma_q(\alpha n+\beta)} - \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}} \frac{q^n z^n}{\Gamma_q(\alpha n+\beta)}$$

$$z(1-q) D_q E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) - E_{\alpha,\beta,r}^{\gamma,\delta,s}(qz;q)$$

Theorem 3.4. *Riemann-Liouville fractional q-integral of $E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q)$: If $E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q)$ is q-analogue of generalized Mittag-Leffler function (3.1), then for fractional q-integral operator of order μ*

$$I_q^{\mu} E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = z^{\mu} \frac{B_q(n+1,\mu)}{\Gamma_q(\mu)} E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q); \quad Re(\mu) > 0 \tag{3.7}$$

Proof. On using equation (3.1) and (2.12), we have

$$I_q^{\mu}E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = \frac{1}{\Gamma_q(\mu)}\int_0^z (z-q\xi)^{\mu-1}E_{\alpha,\beta,r}^{\gamma,\delta,s}(\xi;q)d_q\xi$$
$$= \frac{1}{\Gamma_q(\mu)}\int_0^z (z-q\xi)^{\mu-1}\left(\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{\xi^n}{\Gamma_q(\alpha n+\beta)}\right)d_q\xi$$
$$= \frac{1}{\Gamma_q(\mu)}\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{1}{\Gamma_q(\alpha n+\beta)}\int_0^z (z-q\xi)^{\mu-1}\xi^n d_q\xi$$
$$= \frac{1}{\Gamma_q(\mu)}\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{1}{\Gamma_q(\alpha n+\beta)}\int_0^z z^{\mu-1}\frac{\left(\frac{q\xi}{z};q\right)_{\infty}}{\left(\frac{q^{\mu}\xi}{z};q\right)_{\infty}}\xi^n d_q\xi$$

setting $\xi = zt$, then we get

$$= \frac{1}{\Gamma_q(\mu)}\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{sn}}{(q^{\delta};q)_{rn}}\frac{z^{\mu+n}}{\Gamma_q(\alpha n+\beta)}\int_0^1 t^n\frac{(qt;q)_{\infty}}{(q^{\mu}t;q)_{\infty}}d_qt$$
$$\Rightarrow I_q^{\mu}E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q) = z^{\mu}\frac{B_q(n+1,\mu)}{\Gamma_q(\mu)}E_{\alpha,\beta,r}^{\gamma,\delta,s}(z;q)$$

Corollary.

$$I_q^{\mu}E_{1,1,r}^{\gamma,\delta,s}(z;q) = z^{\mu}E_{1,\mu+1,r}^{\gamma,\delta,s}(z;q); \quad Re(\mu) > 0 \tag{3.8}$$

Proof. On setting $\alpha = 1$, $\beta = 1$ and $B_q(n+1,\mu) = \dfrac{\Gamma_q(n+1)\Gamma_q(\mu)}{\Gamma_q(n+\mu+1)}$ in equation (3.7)

Special case. Putting $r = 1$, $s = 1$, $\delta = 1$ in above corollary, we get

$$I_q^{\mu}E_{1,1}^{\gamma}(z;q) = z^{\mu}E_{1,\mu+1}^{\gamma}(z;q) \tag{3.9}$$

which gives the same result as given by Sharma and Jain [12] in 2014, page 617.

Theorem 3.5. *If the condition of (3.1) is satisfied, then for $v \in \mathbb{C}$, $\alpha > 0$ and $v \neq 0, -1, -2, -3, ...$ the following integral formula holds true*

$$\int_0^1 z^{\beta-1}(1-qz)_{v-1}E_{\alpha,\beta,r}^{\gamma,\delta,s}(wz^{\alpha};q)d_qz = \Gamma_q(v)E_{\alpha,\beta+v,r}^{\gamma,\delta,s}(w;q) \tag{3.10}$$

Proof. Let

$$\begin{aligned}
&\int_0^1 z^{\mu-1}(1-qz)_{v-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(wz^p;q)d_qz \\
&= \int_0^1 z^{\mu-1}(1-qz)_{v-1}\left(\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{w^n z^{pn}}{\Gamma_q(\alpha n+\beta)}\right)d_qz \\
&= \sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{w^n}{\Gamma_q(\alpha n+\beta)}\int_0^1 z^{\mu+pn-1}(1-qz)_{v-1}d_qz \\
&= \sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{w^n}{\Gamma_q(\alpha n+\beta)}B_q(\mu+pn,v) \\
&= \sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{w^n}{\Gamma_q(\alpha n+\beta)}\frac{\Gamma_q(\mu+pn)\Gamma_q(v)}{\Gamma_q(\mu+pn+v)}
\end{aligned}$$

In particular, setting $p=\alpha$ and $\mu=\beta$, we get

$$\int_0^1 z^{\beta-1}(1-qz)_{v-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(wz^\alpha;q)d_qz = \sum_{n=0}^{\infty}\frac{(q^r;q)_{sn}}{(q^\delta;q)_{rn}}\frac{w^n\Gamma_q(v)}{\Gamma_q(\alpha n+\beta+v)}$$

$$\int_0^1 z^{\beta-1}(1-qz)_{v-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(wz^\alpha;q)d_qz = \Gamma_q(v)E^{\gamma,\delta,s}_{\alpha,\beta+v,r}(w;q)$$

Theorem 3.6. *Fractional q-Derivative of $E^{\gamma,\delta,s}_{\alpha,\beta,r}(z,q)$: if $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ and $r,s>0$ with $\mu=1,2,3,\ldots$ then*

$$D^\mu_q\left[z^{\beta-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(z^\alpha;q)\right] = z^{\beta-\mu-1}E^{\gamma,\delta,s}_{\alpha,\beta-\mu,r}(z^\alpha;q) \tag{3.11}$$

Proof. Let

$$\begin{aligned}
f(z) &= z^{\beta-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(z^\alpha;q) \\
&= z^{\beta-1}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{z^{\alpha n}}{\Gamma_q(\alpha n+\beta)} \\
f(zq) &= q^{\beta-1}z^{\beta-1}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{(zq)^{\alpha n}}{\Gamma_q(\alpha n+\beta)}
\end{aligned}$$

by the help of equation (2.8)

$$\begin{aligned} D_q\left\{z^{\beta-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(z^\alpha;q)\right\} &= z^{\beta-1}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{z^{\alpha n}}{\Gamma_q(\alpha n+\beta)}\frac{(1-q^{\alpha n+\beta-1})}{(z-qz)} \\ &= z^{\beta-2}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{(z)^{\alpha n}}{\Gamma_q(\alpha n+\beta-1)} \\ &= z^{\beta-2}E^{\gamma,\delta,s}_{\alpha,\beta-1,r}(z^\alpha;q) \end{aligned}$$

Again operating D_q both sides, we get

$$D_q^2\left\{z^{\beta-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(z^\alpha;q)\right\} = z^{\beta-3}E^{\gamma,\delta,s}_{\alpha,\beta-2,r}(z^\alpha;q)$$

If the process is repeated, then we get

$$D_q^\mu\left\{z^{\beta-1}E^{\gamma,\delta,s}_{\alpha,\beta,r}(z^\alpha;q)\right\} = z^{\beta-\mu-1}E^{\gamma,\delta,s}_{\alpha,\beta-\mu,r}(z^\alpha;q)$$

Corollary. *if ω any arbitrary constant and $\gamma,\delta\in\mathbb{C}$, $r,s>0$ then following result hold true*

$$D_q^\mu\left\{E^{\gamma,\delta,s}_{1,1,r}(wz;q)\right\} = (wz)^{-\mu}E^{\gamma,\delta,s}_{1,1-\mu,r}(wz;q); \quad Re(\gamma)>0, Re(\delta)>0 \tag{3.12}$$

Proof. Replacing z by wz in equation (3.8), we have

$$\begin{aligned} &I_q^\mu E^{\gamma,\delta,s}_{1,1,r}(wz;q) = (wz)^\mu E^{\gamma,\delta,s}_{1,\mu+1,r}(wz;q) \\ &\Rightarrow I_q^{k-\mu}E^{\gamma,\delta,s}_{1,1,r}(wz;q) = (wz)^{k-\mu}E^{\gamma,\delta,s}_{1,k-\mu+1,r}(wz;q) \end{aligned}$$

Now if k is smallest integer with $k\geq Re(\mu)$, then

$$\begin{aligned} &D_q^k\left\{I_q^{k-\mu}E^{\gamma,\delta,s}_{1,1,r}(wz;q)\right\} = D_q^k\left\{(wz)^{k-\mu}E^{\gamma,\delta,s}_{1,k-\mu+1,r}(wz;q)\right\} \\ &D_q^\mu E^{\gamma,\delta,s}_{1,1,r}(wz;q) = (wz)^{-\mu}E^{\gamma,\delta,s}_{1,1-\mu,r}(wz;q) \end{aligned}$$

Theorem 3.7. *q-Laplace Transform of $E^{\gamma,\delta,s}_{\alpha,\beta,r}(z;q)$;*

$${}_qL_s\left\{E^{\gamma,\delta,s}_{\alpha,\beta,r}(wz^p;q)\right\} = \frac{1}{s}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}}\frac{\Gamma_q(1+pn)}{\Gamma_q(\alpha n+\beta)}\left(\frac{\omega(1-q)^p}{s^p}\right)^n \tag{3.13}$$

Proof. q-Laplace transform of $f(z)$ is given by the definition 2.7 now according (3.1), we have

$$E^{\gamma,\delta,s}_{\alpha,\beta,r}(wz^p;q) = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}} \frac{\omega^n z^{pn})}{\Gamma_q(\alpha n+\beta)} = f(z) \text{ say} \tag{3.14}$$

$$\Rightarrow f(s^{-1}q^k) = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}} \frac{\omega^n s^{-pn} q^{kpn}}{\Gamma_q(\alpha n+\beta)} \tag{3.15}$$

Now using (3.15) in equation (2.11), we have

$$_qL_s\left\{E^{\gamma,\delta,s}_{\alpha,\beta,r}(\omega z^p;q)\right\} = \frac{(q;q)_\infty}{s} \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \left(\sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}} \frac{\omega^n s^{-pn} q^{kpn}}{\Gamma_q(\alpha n+\beta)}\right)$$

By interchanging the order of summations

$$= \frac{(q;q)_\infty}{s} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}} \frac{\omega^n s^{-pn}}{\Gamma_q(\alpha n+\beta)} \sum_{k=0}^{\infty} \frac{q^{(1+pn)k}}{(q;q)_k}$$

$$= \frac{1}{s} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}} \frac{\omega^n s^{-pn}}{\Gamma_q(\alpha n+\beta)} \sum_{k=0}^{\infty} (q^{1+k};q)_\infty q^{(1+pn)k}$$

Using result $\sum_{j=0}^{\infty}(q^{1+j};q)_\infty q^{(1+\sigma n)j} = \Gamma_q(1+\sigma n)(1-q)^{\sigma n}$

$$= \frac{1}{s} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{sn}}{(q^\delta;q)_{rn}} \frac{\Gamma_q(1+pn)}{\Gamma_q(\alpha n+\beta)} \left(\frac{\omega(1-q)^p}{s^p}\right)^n$$

4. Acknowledgement

The support provided by Council of Scientific and Industrial Research, New Delhi, India, through Junior Research Fellowship to one of the authors Mr. Ashok Kumar Meena is gratefully acknowledged.

References

[1] Agarwal, R.P., Certain Fractional q- Integrals and q-Derivatives, Proceedings of the Cambridge Philosophical Society, 66 (1969), 365-370.

[2] Gasper, G. and Rahman, M., Basic Hypergeometric Series, 2^{nd} ed, Encyclopedia of Mathematics and its Applications 96, Cambridge University Press, Cambridge, (2004).

[3] Garg, M. Chanchlani, L. and Kalla, S. L., A q-Analogue of Generalized Mittag-Leffler Function, Algebras Groups and Geometries, 28 (2011), 205-223.

[4] Garg, M. Chanchlani, L., Kober Fractional q-Derivative Operates, Le Matematiche, vol. LXVI, (2011), 13-26. doi:10.4418/2011.66.1.2

[5] Haubold, H. J., Mathai, A. M. and Saxena, R. K., Mittag-Leffler Functions and Their Applications, Journal of Applied Mathematics, Volume 2011, Article ID: 298628, doi:10.1155/2011/298628.

[6] Miller, K. S. and Ross, B., An Introduction to Fractional Calculus and Fractional Differential Equations, Wiley-New York, (1993).

[7] Mittag-Leffler, G. M., Sur la nouvelle function $E_\alpha(x)$, C. R. Acad. Sci. Paris., 137 (1903), 554-558.

[8] Prabhakar, T. R., A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel, Yokohama Mathematical Journal, 19 (1971), 7-15.

[9] Rainville, E. D., Special Functions, New York: Chelsea Publication Co., (1960).

[10] Shukla, A. K. and Prajapati, J. C., On a Generalization of Mittag-Leffler Function and its Properties, J. Math. Anal. Appl., 336 (2007), 797-811.

[11] Shukla, A. K. and Prajapati, J. C., On a Recurrence Relation of Generalized Mittag-Leffler Function, Survey in Mathematics and its Applications, 4 (2009), 133-138.

[12] Sharma, S. K. and Jain, R., On Some Properties of Generalized q-Mittag Leffler Function, Mathematica Aeterna, 4, no.6 (2014), 613-619.

[13] Sharma Santosh and Jain Renu, On Some Recurrence Relations of Generalized q-Mittag-Leffler Function, Mathematica Aeterna, 6, no. 5 (2016), 791-795.

[14] Salim, T. O. and Faraj, A. W., A Generalization of Mittag-Leffler Function and Integral Operator Associated with Fractional Calculus, Journal of Fractional Calculus and Applications, 3, no.5 (2012), 1-13.

[15] Salim, T. O., Some Properties Relating to the Generalized Mittag-Leffler Function, Advances in Applied Mathematical Analysis, 4, no.1 (2009), 21-30.

[16] Wiman, A., Uber den Fundamental satz in der Theorie de Funktionen $E_\alpha(x)$, Acta Math., 29 (1905), 191-201.