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INTUITIONISTIC LEVEL SUBGROUPS IN CYCLIC GROUPS OF ORDER p^n

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Abstract: In this paper, we investigate whether the level subgroups of intuitionistic fuzzy subgroups of a group form a chain. The fuzzy counterpart of the same has already been proved affirmative. We ratify the result for cyclic groups of prime order and then for cyclic groups of prime power order. Finally, we illustrate our results through some numerical examples.

Keywords and Phrases: Intuitionistic fuzzy set, intuitionistic fuzzy subgroup, level subgroup, prime power, cyclic group.

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1. Introduction

The idea of a *fuzzy subset* of a non-empty set was proposed by L. A. Zadeh [22] in 1965. He defined a fuzzy subset A of a universal set X as a membership function $A : X \to I$ where $I = [0, 1]$. Now, fuzzy mathematics has become an area of meticulous research, having applications in a variety of fields such as engineering [19], computer science [8], medical diagnosis [10], social behavior studies [4], decision making [11] etc. After the introduction of fuzzy sets, several generalizations of abstract mathematical structures to the fuzzy context have come up. In the course of this advancement, many researchers were inspired to inquire into generalizing various concepts of abstract algebra in the fuzzy setting, as algebraic structures have a key role in mathematics and have countless applications in various disciplines such as computer sciences, information sciences, cryptography, coding theory, etc. The keystone of this quest was laid by Rosenfeld [16] - known as the father of fuzzy abstract algebra - who applied the fuzzy approach to the theory of groups and defined the notion of fuzzy subgroups of a group. Thereafter, many works appeared in the literature regarding various fuzzy algebraic structures. Furthermore, as an extension of the fuzzy set theory, the notion of intuitionistic fuzzy sets was presented by K. T. Atanassov [2] in 1983. In 1989 Biswas [3] applied Atanassov's idea of intuitionistic fuzzy sets to the theory of groups and developed the theory of intuitionistic fuzzy subgroups of a group. Later in 2000, Lee [9] introduced the concept of bipolar valued fuzzy sets, as a further extension of the fuzzy set theory and in 2013 Mahmood [12] applied this concept to group theory. [13] and [14] are some other significant works in this area. Still now, many new findings keep coming out in these fields of research.

P S Das [7] has proved that level subgroups of fuzzy subgroups of a group form a chain. Later in 2006, eventhough Ahn et.al. [20] studied some properties of level subgroups of intuitionistic fuzzy subgroups of cyclic groups, no attempt was made to check whether these level subgroups form a chain in the intuitionistic fuzzy context also. The purpose of our work is to investigate whether this result can be carried over to the realm of Intuitionistic Fuzzy Subgroups of a group. This paper is organized into three main sections as follows. In the first section we describe the basic concepts which are required to understand our work. We state our main findings in the second section. In the third section we reinforce our findings with the help of illustrative examples.

It is well known that, cyclic groups of prime power order and their subgroups have widespread applications in cryptography and coding theory, both of which are some of the ever evolving branches of mathematics. Cryptographic models and signature schemes [21] which incorporate fuzzy and intuitionistic fuzzy concepts, and the concept of fuzzy code [1], which can tackle the uncertainties or vagueness that may arise during communication, are the newbies in this realm. Since our work concentrates on the intuitionistic fuzzy subgroups of cyclic groups of prime power order, it can find applications in the realms of fuzzy and intuitionistic fuzzy cryptography.

2. Notations and Definitions

In this section we present some basic definitions and results regarding Fuzzy Subgroups (FSGs) and Intuitionistic Fuzzy Subgroups (IFSGs). Here G denotes a multiplicative group, unless otherwise stated, and $\vee$ and $\wedge$ denote the "min" and "max" operators respectively on I.

Definition 2.1. [16] *A fuzzy subset A of a group G is said to be a* ***Fuzzy Subgroup*** *(FSG) of G if, for $x, y \in G$*
(1) $A(xy) \geq \wedge[A(x), A(y)]$
(2) $A(x^{-1}) = A(x)$.

Proposition 2.2. [16] *If A is FSG of a group G with identity element e, then*
(1) $A(e) \geq A(x), \forall x \in G$, and
(2) $G_A = \{x \in G : A(x) = A(e)\}$ is a subgroup of G.

Definition 2.3. [7] *If A is a fuzzy subset of a set X and $t \in I$, then* ***Level Subset*** *of A at t (or t-cut of A) denoted by A_t is defined as $A_t = \{x \in X : A(x) \geq t\}$.*

Proposition 2.4. [7] *In a group G, a Fuzzy Subset A will be a FSG of G if and only if A_t is a subgroup of G for $0 \leq t \leq A(e)$.*

Definition 2.5. [7] *For a FSG A of a group G, the subgroup A_t is called* ***Level Subgroup*** *of A at t, for $0 \leq t \leq A(e)$.*

Proposition 2.6. [17] *If A is a FSG of a group G then for all $t_1, t_2 \in I$ with $t_1 > t_2$, $A_{t_1} \subseteq A_{t_2}$.*

Proposition 2.7. [7] *Let A be a FSG of a finite group G with $Im(A) = \{t_i : i = 1, 2, 3, ..., n\}$. Then the collection $\{A_{t_i} : i = 1, 2, 3, ..., n\}$ contains all level subgroups of A.*
Moreover, if $t_1 > t_2 > t_3 > ... > t_n$, then all these level subgroups will form a chain $G_A = A_{t_1} \subseteq A_{t_2} \subseteq A_{t_3} \subseteq ... \subseteq A_{t_n} = G$.

Definition 2.8. [2] *An* ***Intuitionistic Fuzzy Subset*** *(IFS) of a set X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A, \nu_A : X \to I$ represent the degree of membership and degree of non membership of any element $x \in X$ and should satisfy the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall\ x \in X$.*

Definition 2.9. [15] *An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ of a group G is said to be an* ***Intuitionistic Fuzzy Subgroup*** *(IFSG) of G if for all $x, y \in G$*
(1) $\mu_A(xy) \geq \wedge[\mu_A(x), \mu_A(y)]$
(2) $\mu_A(x^{-1}) = \mu_A(x)$
(3) $\nu_A(xy) \leq \vee[\nu_A(x), \nu_A(y)]$, and
(4) $\nu_A(x^{-1}) = \nu_A(x)$.

Proposition 2.10. [15] *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of a group G with identity element e. Then,*

(1) $\mu_A(e) \geq \mu_A(x)$ *and* $\nu_A(e) \leq \nu_A(x)$, $\forall\ x \in G$.
(2) $G_A = \{x \in G : \mu_A(x) = \mu_A(e)$ *and* $\nu_A(x) = \nu_A(e)\}$ *is a subgroup of* G.

Definition 2.11. [18] *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ *be an IFS of a set* X *and* $\alpha,\ \beta \in I$. *Then the* ***Intuitionistic Level Subset*** *(ILS) of* A *at* $(\alpha,\ \beta)$ *(or* $(\alpha,\ \beta)$*–cut of IFS A) is the crisp set* $A_{\alpha,\beta} = \{x \in X : \mu_A(x) \geq \alpha$ *and* $\nu_A(x) \leq \beta\}$.

Proposition 2.12. [18] *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ *be an IFS of a group* G. *Then,*
(1) $A_{\alpha,\beta} = \phi, \forall\ \alpha > \mu_A(e)$ *and* $\beta < \nu_A(e)$
(2) A *is an IFSG of* $G \Leftrightarrow A_{\alpha,\beta}$ *is a subgroup of* G *for* $0 \leq \alpha \leq \mu_A(e)$ *and* $\nu_A(e) \leq \beta \leq 1$.

Definition 2.13. [5] *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ *be an IFSG of a group* G. *Then the subgroup* $A_{\alpha,\beta}$, $0 \leq \alpha \leq \mu_A(e)$ *and* $\nu_A(e) \leq \beta \leq 1$, *of* G *is called* ***Intuitionistic Level Subgroup*** *(ILSG) of* A *at* (α, β).

Proposition 2.14. [18] *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ *be an IFSG of a group* G *and* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I$ *be such that* $\alpha_1 \geq \alpha_2, \beta_1 \leq \beta_2$. *Then* $A_{\alpha_1,\beta_1} \subseteq A_{\alpha_2,\beta_2}$.

Proposition 2.15 [5] *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ *be an IFSG of a finite group* G, $Im(\mu_A) = \{t_i : i = 1, 2, 3, ..., n\}$ *and* $Im(\nu_A) = \{s_j : j = 1, 2, 3, ..., n\}$. *Then the collection*

$$\{\ A_{t_i,s_j} : i = 1, 2, 3, ..., n; j = 1, 2, 3, ..., n\}$$

contains all ILSG's of G.

3. Results and Discussion

Proposition 2.15 states the Intuitionistic Fuzzy analogue of the first part of Proposition 2.7. We now check whether the second part of Proposition 2.7 holds true in the intuitionistic fuzzy context. We do this for the case when G is a cyclic group of prime power order. In this section we discuss some results about cyclic subgroups of IFSG's in finite cyclic groups and prove that if A is an IFSG of a cyclic group G of prime power order, then the ILSG's of A in G form a chain.

Proposition 3.1. [6] *In any IFSG of a finite cyclic group, the generators will have the minimum membership value and maximum non-membership value.*

Proposition 3.2. *Let* G *be a cyclic group of prime order and* $A = \{\langle x, \mu_A(x), \nu_A(x)\ \rangle : x \in G\}$ *be an IFSG of* G. *Then the ILSG's of* A *in* G *form a chain.*
Proof. Let G be any finite cyclic group of prime order p and $A = \{\langle x,\ \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be any IFSG of G. Since all non-identity elements are generators of G, there can be atmost two membership levels (say) $1 \geq t_1 > t_2 \geq 0$ and two

non-membership levels (say) $0 \leq s_1 < s_2 \leq 1$ and μ_A, ν_A would have to be defined as in table 1, where e is the identity element in G.

	μ_A	ν_A
e	t_1	s_1
$x \in G - \{e\}$	t_2	s_2

Table 1: μ_A and ν_A for the IFSG A in proposition 3.2.

Hence the ILSG's of A in G are $A_{t_1,s_1} = \{e\}$ and $A_{t_2,s_2} = G$ which form a chain $\{e\} = A_{t_1,s_1} \subseteq A_{t_2,s_2} = G$, consisting of the trivial subgroups of G.

Proposition 3.3. *Let G be any group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of G with $Im(\mu_A) = \{\ t_i : i = 1, 2, 3, ..., n\}$ and $Im(\nu_A) = \{\ s_j : i = 1,\ 2,\ 3,\ ...,\ m\}$. Then $s_i \leq s_j \Leftrightarrow A_{t_k,s_i} \subseteq A_{t_k,s_j}$ and $t_i \leq t_j \Leftrightarrow A_{t_j,s_k} \subseteq A_{t_i,s_k}$ for any $t_k, s_k \in I$.*

Proof. Follows from Proposition 2.14.

Proposition 3.4. *Let G be any group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of G. Let $a, b \in G$ be such that $\langle a\rangle \subseteq \langle b\rangle$. Then $\mu_A(a) \geq \mu_A(b)$ and $\nu_A(a) \leq \nu_A(b)$.*

Proof. $\langle a\rangle \subseteq \langle b\rangle \Rightarrow a = b^n$. Hence $\mu_A(a) \geq \mu_A(b)$ by first axiom of IFSG and $\nu_A(a) \leq \nu_A(b)$ by third axiom of IFSG.

The following example shows that the converse of the above proposition is not true.

Example 3.5. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbf{Z}_{10}\}$ be an IFSG of $(\mathbf{Z}_{10}, +_{10})$ defined as in table 2.

	μ_A	ν_A
0	0.8	0.1
2,4,6,8	0.5	0.3
1,3,5,7,9	0.3	0.5

Table 2: μ_A and ν_A for the IFSG A in example 3.5.

Then $\mu_A(2) > \mu_A(5)$ and $\nu_A(2) < \nu_A(5)$, but $\langle 2\rangle = \{0, 2, 4, 6, 8\}$ is not a subset of $\langle 5\rangle = \{0, 5\}$.

Now we will move on to extend the above results to cyclic groups of prime power order. The following proposition is the analogue of proposition 3.2 in the case of cyclic groups of prime power order.

Proposition 3.6. *Let G be a cyclic group of prime power order and $A = \{\langle x,$*

$\mu_A(x)$, $\nu_A(x)\rangle : x \in G\}$ be an IFSG of G. Then ILSG's of G w.r.t A form a chain.
Proof. Since every finite cyclic group of order n is isomorphic to $\mathbf{Z}_n, G \cong \mathbf{Z}_{p^n}$ for some prime number p and positive integer n. All elements of this group except $0, p, p^2, ..., p^{n-1}$ are generators of G and the only subgroups of G are $\langle 0\rangle \subseteq \langle p^{n-1}\rangle \subseteq \langle p^{n-2}\rangle \subseteq ... \subseteq \langle p\rangle \subseteq G$. If a is any generator of G, then by propositions 2.10, 3.1 and 3.4 we get:

$$\mu_A(0) \geq \mu_A(p^{n-1}) \geq \mu_A(p^{n-2}) \geq ... \geq \mu_A(p) \geq \mu_A(a)$$

$$\nu_A(0) \leq \nu_A(p^{n-1}) \leq \nu_A(p^{n-2}) \leq ... \leq \nu_A(p) \leq \nu_A(a)$$

Hence A can have atmost $n+1$ levels of membership (say) $1 \geq t_1 > t_2 > ... > t_{n+1} \geq 0$ and $n+1$ levels of non-membership (say) $0 \leq s_1 < s_2 ... < s_{n+1} \leq 1$. In order to satisfy all these conditions μ_A and ν_A should be defined as follows: (the same is given in table 3 also)

$$\mu_A(x) = \begin{cases} t_1; & if\ x = 0 \\ t_2; & if x \in \langle p^{n-1}\rangle - \{0\} \\ t_3; & if\ x \in \langle p^{n-2}\rangle - \langle p^{n-1}\rangle \\ ... & \\ ... & \\ t_{n-1}; & if\ x \in \langle p^2\rangle - \langle p^3\rangle \\ t_n; & if\ x \in \langle p\rangle - \langle p^2\rangle \\ t_{n+1}; & if\ x\ is\ a\ generator \end{cases} \qquad \nu_A(x) = \begin{cases} s_1; & if\ x = 0 \\ s_2; & if x \in \langle p^{n-1}\rangle - \{0\} \\ s_3; & if\ x \in \langle p^{n-2}\rangle - \langle p^{n-1}\rangle \\ ... & \\ ... & \\ s_{n-1}; & if\ x \in \langle p^2\rangle - \langle p^3\rangle \\ s_n; & if\ x \in \langle p\rangle - \langle p^2\rangle \\ s_{n+1}; & if\ x\ is\ a\ generator \end{cases}$$

Then the distinct ILSG's of A in G are:

$$\begin{gathered} A_{t_1,s_1} = A_{t_1,s_2} = A_{t_1,s_3} = ... = A_{t_1,s_{n+1}} = A_{t_2,s_1} = A_{t_3,s_1} = ... = A_{t_{n+1},s_1} = \{0\} \\ A_{t_2,s_2} = A_{t_2,s_3} = A_{t_2,s_4} = ... = A_{t_2,s_{n+1}} = A_{t_3,s_2} = A_{t_4,s_2} = ... = A_{t_{n+1},s_2} = \langle p^{n-1}\rangle \\ A_{t_3,s_3} = A_{t_3,s_4} = A_{t_3,s_5} = ... = A_{t_3,s_{n+1}} = A_{t_4,s_3} = A_{t_5,s_3} = ... = A_{t_{n+1},s_3} = \langle p^{n-2}\rangle \\ ... \\ ... \\ A_{t_{n-1},s_{n-1}} = A_{t_{n-1},s_n} = A_{t_{n-1},s_{n+1}} = A_{t_n,s_{n-1}} = A_{t_{n+1},s_{n-1}} = \langle p^2\rangle \\ A_{t_n,s_n} = A_{t_n,s_{n+1}} = A_{t_{n+1},s_n} = \langle p\rangle \\ A_{t_{n+1},s_{n+1}} = G \end{gathered}$$

Hence the set $\{A_{t_1,s_1}, A_{t_2,s_2}, A_{t_3,s_3}, ..., A_{t_{n+1},s_{n+1}}\}$ contains all the distinct ILSG's of G and these ILSG's form a chain given by:

$$A_{t_1,s_1} \subseteq A_{t_2,s_2} \subseteq A_{t_3,s_3} \subseteq ... \subseteq A_{t_{n-1},s_{n-1}} \subseteq A_{t_n,s_n} \subseteq A_{t_{n+1},s_{n+1}}$$

(By Proposition 3.3 the above arguments will hold true even if any of the membership values or non-membership values coincide.)

For x in	μ_A	ν_A
$\{0\}$	t_1	s_1
$\langle p^{n-1}\rangle - \{0\}$	t_2	s_2
$\langle p^{n-2}\rangle - \langle p^{n-1}\rangle$	t_3	s_3
...	...	...
$\langle p^2\rangle - \langle p^3\rangle$	t_{n-1}	s_{n-1}
$\langle p\rangle - \langle p^2\rangle$	t_n	s_n
$G - \langle p\rangle$	t_{n+1}	s_{n+1}

Table 3: μ_A and ν_A as defined in the proof of proposition 3.6

Remark. Let X be any set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an IFS of X. To represent A and its ILS geometrically, we take μ_A along x-axis and ν_A along y-axis. Then an element x in A is represented by the point $(\mu_A(x), \nu_A(x))$ in the coordinate plane. In this representation all elements of A will lie inside the triangle bounded by $x = 0, y = 0$ and the line $x + y = 1$.

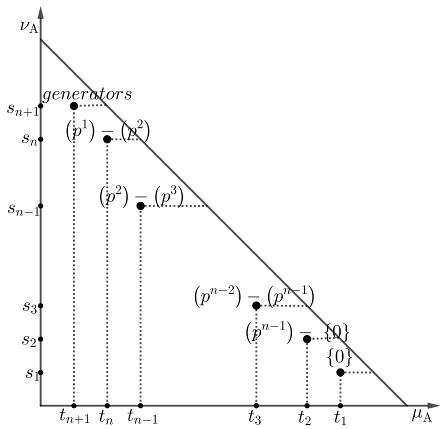

Figure 1: Geometric representation of ILSG's in proof of proposition 3.6

Corollary 3.7. *Let G be a cyclic group of prime power order and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of G. Then for $a, b \in G$, $\langle a\rangle \subseteq \langle b\rangle \Leftrightarrow \mu_A(a) \geq \mu_A(b)$ and $\nu_A(a) \leq \nu_A(b)$.*

Proof. The fact that $\langle a\rangle \subseteq \langle b\rangle \Rightarrow \mu_A(a) \geq \mu_A(b)$ and $\nu_A(a) \leq \nu_A(b)$ follows from Proposition 3.4.

Conversely, suppose $\mu_A(a) \geq \mu_A(b)$ and $\nu_A(a) \leq \nu_A(b)$ for $a, b \in G$, where G is a cyclic group of prime power order (say) p^n. Then $G \cong \mathbf{Z}_{p^n}$ and as stated in the proof of proposition 3.6, the membership function μ_A and non-membership function ν_A in the IFSG A should be defined as in Table 3.
Now, $\mu_A(a) \geq \mu_A(b)$ and $\nu_A(a) \leq \nu_A(b) \Rightarrow a \in \langle p^{k-1} \rangle - \langle p^k \rangle$ and $b \in \langle p^{l-1} \rangle - \langle p^l \rangle$ for some $k \geq l$. Then $\langle a \rangle = \langle p^{k-1} \rangle$ and $\langle b \rangle = \langle p^{l-1} \rangle$, and hence $\langle a \rangle \subseteq \langle b \rangle$ as $k - 1 \geq l - 1$.
(The above arguments can be applied with some necessary modifications even if any of the membership values or non-membership values coincide.)

Remark. Corollary 3.7 is the analogue of Proposition 3.4 in the case of cyclic groups of prime power order and it states that, the converse of Proposition 3.4 holds true in cyclic groups of prime power order.

4. Illustrative Examples

Example 4.1. This example illustrates a case in which all the membership and non-membership values are distinct.
Let $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in \mathbf{Z}_{27}\}$ be an IFSG of $(\mathbf{Z}_{27}, +_{27})$ defined as in table 4.

For x in	μ_A	ν_A
$\{0\}$	$t_1 = 0.8$	$s_1 = 0.1$
$\langle 3^2 \rangle - \{0\}$	$t_2 = 0.6$	$s_2 = 0.2$
$\langle 3 \rangle - \langle 3^2 \rangle$	$t_3 = 0.5$	$s_3 = 0.3$
$\mathbf{Z}_{27} - \langle 3 \rangle$	$t_4 = 0.3$	$s_4 = 0.5$

Table 4: μ_A and ν_A for the IFSG A in example 4.1.

Then the distinct ILSG's of $\mathbf{Z}_{27}$ *w.r.t* A are:

$$A_{t_1,s_1} = A_{t_1,s_2} = A_{t_1,s_3} = A_{t_1,s_4} = A_{t_2,s_1} = A_{t_3,s_1} = A_{t_4,s_1} = \langle 3^3 \rangle = \{0\}$$
$$A_{t_2,s_2} = A_{t_2,s_3} = A_{t_2,s_4} = A_{t_3,s_2} = A_{t_4,s_2} = \langle 3^2 \rangle = \{0, 9, 18\}$$
$$A_{t_3,s_3} = A_{t_3,s_4} = A_{t_4,s_3} = \langle 3^1 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$$
$$A_{t_4,s_4} = \langle 3^0 \rangle = \mathbf{Z}_{27}$$

which forms a chain as $\{0\} \subset \{0, 9, 18\} \subset \{0, 3, 6, 9, 12, 15, 18, 21, 24\} \subset \mathbf{Z}_{27}$.

In proof of proposition 3.6 we have stated that, the arguments given in the proof will hold true even if any of the membership values or non-membership values coincide. The following examples (example 4.2 to example 4.5) illustrate this fact.

Example 4.2. This example depicts a case in which some of the membership values coincide.

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbf{Z}_{27}\}$ be an IFSG of $(\mathbf{Z}_{27}, +_{27})$ defined as in table 5.

For x in	μ_A	ν_A
$\{0\}$	$t_1 = 0.8$	$s_1 = 0.1$
$\langle 3^2\rangle - \{0\}$	$t_2 = 0.6$	$s_2 = 0.2$
$\langle 3\rangle - \langle 3^2\rangle$	$t_3 = 0.6$	$s_3 = 0.3$
$\mathbf{Z}_{27} - \langle 3\rangle$	$t_4 = 0.3$	$s_4 = 0.5$

Table 5: μ_A and ν_A for the IFSG A in example 4.2.

The elements in $\langle 3^2\rangle - \{0\}$ and $\langle 3\rangle - \langle 3^2\rangle$ have the same degree of membership 0.6. Here also the distinct ILSG's of $\mathbf{Z}_{27}$ *w.r.t* A are:

$$A_{t_1,s_1} = A_{t_1,s_2} = A_{t_1,s_3} = A_{t_1,s_4} = A_{t_2,s_1} = A_{t_3,s_1} = A_{t_4,s_1} = \langle 3^3\rangle = \{0\}$$
$$A_{t_2,s_2} = A_{t_3,s_2} = A_{t_4,s_2} = \langle 3^2\rangle = \{0, 9, 18\}$$
$$A_{t_3,s_3} = A_{t_2,s_3} = A_{t_2,s_4} = A_{t_3,s_4} = A_{t_4,s_3} = \langle 3^1\rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$$
$$A_{t_4,s_4} = \langle 3^0\rangle = \mathbf{Z}_{27}$$

which forms a chain as $\{0\} \subset \{0, 9, 18\} \subset \{0, 3, 6, 9, 12, 15, 18, 21, 24\} \subset \mathbf{Z}_{27}$.

Example 4.3. This example depicts a case in which some of the non-membership values coincide.
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbf{Z}_{27}\}$ be an IFSG of $(\mathbf{Z}_{27}, +_{27})$ defined as in table 6.

For x in	μ_A	ν_A
$\{0\}$	$t_1 = 0.8$	$s_1 = 0.1$
$\langle 3^2\rangle - \{0\}$	$t_2 = 0.6$	$s_2 = 0.2$
$\langle 3\rangle - \langle 3^2\rangle$	$t_3 = 0.5$	$s_3 = 0.3$
$\mathbf{Z}_{27} - \langle 3\rangle$	$t_4 = 0.3$	$s_4 = 0.3$

Table 6: μ_A and ν_A for the IFSG A in example 4.3.

Here also the distinct ILSG's of $\mathbf{Z}_{27}$ *w.r.t* A are:

$$A_{t_1,s_1} = A_{t_1,s_2} = A_{t_1,s_3} = A_{t_1,s_4} = A_{t_2,s_1} = A_{t_3,s_1} = A_{t_4,s_1} = \langle 3^3\rangle = \{0\}$$
$$A_{t_2,s_2} = A_{t_2,s_3} = A_{t_2,s_4} = A_{t_3,s_2} = A_{t_4,s_2} = \langle 3^2\rangle = \{0, 9, 18\}$$
$$A_{t_3,s_3} = A_{t_3,s_4} = \langle 3^1\rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$$
$$A_{t_4,s_4} = A_{t_4,s_3} = \langle 3^0\rangle = \mathbf{Z}_{27}$$

which forms a chain as $\{0\} \subset \{0, 9, 18\} \subset \{0, 3, 6, 9, 12, 15, 18, 21, 24\} \subset \mathbf{Z}_{27}$.

Example 4.4. This example deals with a case in which some of both membership

and non-membership values coincide.
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbf{Z}_{27}\}$ be an IFSG of $(\mathbf{Z}_{27}, +_{27})$ defined as in table 7.

For x in	μ_A	ν_A
$\{0\}$	$t_1 = 0.8$	$s_1 = 0.1$
$\langle 3^2\rangle - \{0\}$	$t_2 = 0.6$	$s_2 = 0.2$
$\langle 3\rangle - \langle 3^2\rangle$	$t_3 = 0.6$	$s_3 = 0.3$
$\mathbf{Z}_{27} - \langle 3\rangle$	$t_4 = 0.3$	$s_4 = 0.3$

Table 7: μ_A and ν_A for the IFSG A in example 4.4.

Here also the distinct ILSG's of $\mathbf{Z}_{27}$ $w.r.t$ A are:

$$A_{t_1,s_1} = A_{t_1,s_2} = A_{t_1,s_3} = A_{t_1,s_4} = A_{t_2,s_1} = A_{t_3,s_1} = A_{t_4,s_1} = \langle 3^3\rangle = \{0\}$$
$$A_{t_2,s_2} = A_{t_3,s_2} = A_{t_4,s_2} = \langle 3^2\rangle = \{0, 9, 18\}$$
$$A_{t_3,s_3} = A_{t_2,s_3} = A_{t_2,s_4} = A_{t_3,s_4} = \langle 3^1\rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$$
$$A_{t_4,s_4} = A_{t_4,s_3} = \langle 3^0\rangle = \mathbf{Z}_{27}$$

which forms a chain as $\{0\} \subset \{0, 9, 18\} \subset \{0, 3, 6, 9, 12, 15, 18, 21, 24\} \subset \mathbf{Z}_{27}$.

Example 4.5. This example deals with a case in which some of the membership values and the corresponding non-membership values coincide.
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbf{Z}_{27}\}$ be an IFSG of $(\mathbf{Z}_{27}, +_{27})$ defined as in table 8.

For x in	μ_A	ν_A
$\{0\}$	$t_1 = 0.8$	$s_1 = 0.1$
$\langle 3^2\rangle - \{0\}$	$t_2 = 0.6$	$s_2 = 0.2$
$\langle 3\rangle - \langle 3^2\rangle$	$t_3 = 0.6$	$s_3 = 0.2$
$\mathbf{Z}_{27} - \langle 3\rangle$	$t_4 = 0.3$	$s_4 = 0.3$

Table 8: μ_A and ν_A for the IFSG A in example 4.5.

The elements in $\langle 3^2\rangle - \{0\}$ and $\langle 3\rangle - \langle 3^2\rangle$ are having the same degree of membership 0.6 and same degree of non-membership 0.2.
Here also the distinct ILSG's of $\mathbf{Z}_{27}$ $w.r.t$ A are:

$$A_{t_1,s_1} = A_{t_1,s_2} = A_{t_1,s_3} = A_{t_1,s_4} = A_{t_2,s_1} = A_{t_3,s_1} = A_{t_4,s_1} = \langle 3^3\rangle = \{0\}$$
$$A_{t_2,s_2} = A_{t_2,s_3} = A_{t_2,s_4} = A_{t_3,s_2} = A_{t_4,s_2} = A_{t_3,s_3} = A_{t_3,s_4} = A_{t_4,s_3} = \langle 3^1\rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$$
$$A_{t_4,s_4} = \langle 3^0\rangle = \mathbf{Z}_{27}$$

which forms a chain as $\{0\} \subset \{0, 3, 6, 9, 12, 15, 18, 21, 24\} \subset \mathbf{Z}_{27}$.

Remark. The above examples hold true for all values of t_i and s_j with $1 \geq t_1 \geq t_2 \geq t_3 \geq t_4 \geq 0$, $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq 1$.

Remark. In examples 4.1 to 4.4, the chain of ILSG's of the IFSG A in $\mathbf{Z}_{27}$ consists of all the crisp subgroups of $\mathbf{Z}_{27}$. But this need not be the case always, as shown in example 4.5. That is, if some of the membership values and the corresponding non-membership values coincide, then the chain of ILSG's of an IFSG A in a group G of prime power order need not always contain all crisp subgroups of G.

5. Conclusion

In the process of fuzzifying the abstract algebraic concepts, it was proved that the level subgroups of a fuzzy subgroup of any group form a chain. For the time being, nothing has been done to extend this result to the context of intuitionistic fuzzy groups. The purpose of our work was to investigate whether this result can be carried over affirmatively to the scenario of intuitionistic fuzzy subgroups. We have successfully proved it in positive for cyclic groups of prime and prime power orders.

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