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(ϑ, ϱ)-FUZZY Z_α OPEN SETS IN DOUBLE FUZZY TOPOLOGICAL SPACES

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Abstract: In this paper we introduce (ϑ, ϱ)- fuzzy Z_α-open, (ϑ, ϱ)-fuzzy Z_α-closed sets, (ϑ, ϱ)- fuzzy Z_α-clopen sets and study some of their properties in double fuzzy topological spaces.

Keywords and Phrases: (ϑ, ϱ)- fuzzy Z_α-open, (ϑ, ϱ)-fuzzy Z_α-closed sets, (ϑ, ϱ)-fuzzy Z_α-clopen sets, (ϑ, ϱ)-fuzzy Z_α-interior, (ϑ, ϱ)-fuzzy Z_α-closure.

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1. Introduction

Intuitionistic fuzzy sets were first introduced by Atanassov [1] in 1993, then Coker [2] introduced the notion of Intuitionistic fuzzy topological space in 1997. In 2005, Garcia and Rodabaugh [3] proved that the term intuitionistic is unsuitable in mathematics and applications and they introduced the name double for the term intuitionistic. In the past two decades many researchers [5, 6, 13] doing more applications on double fuzzy topological spaces. From 2017, Mubarki et al., [7] introduced and studied some properties on Z_α-open sets and maps in topological spaces. In this paper we introduce (ϑ, ϱ)- fuzzy Z_α-open, (ϑ, ϱ)-fuzzy Z_α-closed sets and study some of their properties in double fuzzy topological spaces.

2. Preliminaries

Throughout this paper, U will be a non-empty set, I is the closed unit interval [0,1], $I_0 = (0, 1]$ and $I_1 = [0, 1)$. A fuzzy set μ is quasi-coincident with a fuzzy set ν denoted by $\mu q \nu$ iff there exists $u \in U \ni \mu(u) + \nu(u) > 1$ and otherwise they are not quasi-coincident which denoted by $\mu \bar{q} \nu$. The family of all fuzzy sets on U is denoted by I^U. By $\underline{0}$ and $\underline{1}$, we denote the smallest and the largest fuzzy sets on U. For a fuzzy set $\mu(u) \in I^U$, $\underline{1} - \mu(u)$ denotes its complement. For $u \in U, \vartheta \in I_0$, a fuzzy point u_ϑ is defined by $u_\vartheta(v) = \vartheta$ if $u = v$ for all other $v, u_\vartheta(v) = 0$.

Definition 2.1. [10] *A double fuzzy topology $(\mathit{\Gamma}, \mathit{\Gamma}^*)$ on U is a pair of maps $\mathit{\Gamma}, \mathit{\Gamma}^* : I^U \to I$, which satisfies the following properties:*

(O1) $\mathit{\Gamma}(\eta) \leq \underline{1} - \mathit{\Gamma}^*(\eta)$ *for each* $\eta \in I^U$.

(O2) $\mathit{\Gamma}(\eta_1 \wedge \eta_2) \geq \mathit{\Gamma}(\eta_1) \wedge \mathit{\Gamma}(\eta_2)$ *and* $\mathit{\Gamma}^*(\eta_1 \wedge \eta_2) \leq \mathit{\Gamma}^*(\eta_1) \vee \mathit{\Gamma}^*(\eta_2)$ *for each* $\eta_1, \eta_2 \in I^U$.

(O3) $\mathit{\Gamma}(\bigvee_{j \in \Gamma} \eta_j) \leq \bigwedge_{j \in \Gamma} \mathit{\Gamma}(\eta_j))$ *and* $\mathit{\Gamma}^*(\bigvee_{j \in \Gamma} \eta_j) \leq \bigvee_{j \in \Gamma} \mathit{\Gamma}^*(\eta_j)$ *for each* $\eta_j, \in I^U, j \in \Gamma$.

The triplet $(U, \mathit{\Gamma}, \mathit{\Gamma}^*)$ is called a double fuzzy topological space (briefly, dfts). A fuzzy set η is called an (ϑ, ϱ)-fuzzy open (briefly (ϑ, ϱ)-fo) set if $\mathit{\Gamma}(\eta) \geq \vartheta$ and $\mathit{\Gamma}^*(\eta) \leq \varrho$, η is called an (ϑ, ϱ)-fuzzy closed (briefly (ϑ, ϱ)-fc) set iff $\underline{1} - \eta$ is an (ϑ, ϱ)-fo set.

Definition 2.2. [4] *Let $(U, \mathit{\Gamma}, \mathit{\Gamma}^*)$ be a dfts. Then double fuzzy interior (briefly, $I_{\mathit{\Gamma},\mathit{\Gamma}^*}(\rho, \vartheta, \varrho)$) and double fuzzy closure (briefly, $C_{\mathit{\Gamma},\mathit{\Gamma}^*}(\rho, \vartheta, \varrho)$) operators are defined from $I^U \times I_0 \times I_1 \to I^U$ as follows*

$$I_{\mathit{\Gamma},\mathit{\Gamma}^*}(\rho, \vartheta, \varrho) = \bigvee \left\{ \mu \in I^U | \mu \leq \rho, \mathit{\Gamma}(\mu) \geq \vartheta, \mathit{\Gamma}^*(\mu) \leq \varrho \right\},$$

$$C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\bigwedge\left\{\mu\in I^U|\mu\geq\rho,\Gamma(\underline{1}-\mu)\geq\vartheta,\Gamma^*(\underline{1}-\mu)\leq\varrho\right\},$$

where $\vartheta\in I_0$ and $\varrho\in I_1$ such that $\vartheta+\varrho\leq 1$.

Definition 2.3. [9] *Let (U,Γ,Γ^*) be a dfts. Then for each $\vartheta\in I_0$, a fuzzy set $\rho\in I^U$, is said to be (ϑ,ϱ)-fuzzy regular open (resp. closed) (briefly (ϑ,ϱ)-fro (resp. (ϑ,ϱ)-frc)) set if $\rho=I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$ (resp. $\underline{1}-\rho$ is (ϑ,ϱ)-fro set).*

Definition 2.4. [8] *Let (U,Γ,Γ^*) be a dfts. Then for each $\vartheta\in I_0$, and for fuzzy set $\rho\in I^U$, we define the operators $\delta C_{\Gamma,\Gamma^*}$ and $\delta I_{\Gamma,\Gamma^*}$: $I^U\times I_0\times I_1\to I^U$ as follows*

$$\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\bigvee\left\{\mu\in I^U|\mu\leq\rho,\mu\text{ is an }(\vartheta,\varrho)\text{-}fro\right\},$$

$$\delta C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\bigwedge\left\{\mu\in I^U|\mu\geq\rho,\mu\text{ is an }(\vartheta,\varrho)\text{-}frc\right\}.$$

Definition 2.5. [8, 5, 12] *Let (U,Γ,Γ^*) be a dfts. Then for each $\vartheta\in I_0$, a fuzzy set $\rho\in I^U$, is said to be (ϑ,ϱ)-fuzzy*

(i) δ pre (resp. semi & e) open (briefly (ϑ,ϱ)-$f\delta po$ (resp. (ϑ,ϱ)-$f\mathcal{S}o$ & (ϑ,ϱ)-feo)) set if $\rho\leq I_{\Gamma,\Gamma^*}(\delta C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$ (resp. $\rho\leq C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$ & $\rho\leq C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)\vee I_{\Gamma,\Gamma^*}(\delta C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$).

(ii) α (resp. δ semi, b & Z) open (briefly (ϑ,ϱ)-$f\alpha o$ (resp. (ϑ,ϱ)-$f\delta\mathcal{S}o$, (ϑ,ϱ)-fbo & (ϑ,ϱ)-fZo)) set if $\rho\leq I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)$ (resp. $\rho\leq C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$, $\rho\leq C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)\vee I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$& $\rho\leq C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)\vee I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$).

(iii) δ pre (resp. semi, α, e, δ semi, b & Z) closed (briefly (ϑ,ϱ)-$f\delta pc$ (resp. (ϑ,ϱ)-$f\mathcal{S}c$, (ϑ,ϱ)-$f\alpha c$, (ϑ,ϱ)-fec, (ϑ,ϱ)-$f\delta\mathcal{S}c$, (ϑ,ϱ)-fbc & (ϑ,ϱ)-fZc)) set if $\underline{1}-\rho$ is an (ϑ,ϱ)-$f\delta po$ (resp. (ϑ,ϱ)-$f\mathcal{S}o$, (ϑ,ϱ)-$f\alpha o$, (ϑ,ϱ)-feo, (ϑ,ϱ)-$f\delta\mathcal{S}o$, (ϑ,ϱ)-fbo & (ϑ,ϱ)-fZo).

Definition 2.6. [8, 5, 12] *Let (U,Γ,Γ^*) be a dfts. Then for each $\vartheta\in I_0$, and for fuzzy set $\rho\in I^U$, we define the operators $\delta\mathcal{S}C_{\Gamma,\Gamma^*}$ (resp. $\alpha C_{\Gamma,\Gamma^*}$) and $\delta\mathcal{S}I_{\Gamma,\Gamma^*}$ (resp. $\alpha I_{\Gamma,\Gamma^*}$): $I^U\times I_0\times I_1\to I^U$ as follows*

$\delta\mathcal{S}I_{\Gamma,\Gamma^*}$ (resp. $\alpha I_{\Gamma,\Gamma^*}$)$(\rho,\vartheta,\varrho)=\bigvee\{\mu\in I^U|\mu\leq\rho,\mu$ *is an* (ϑ,ϱ)-$f\delta\mathcal{S}o$ (*resp.* (ϑ,ϱ)-$f\alpha o$)$\}$,

$\delta\mathcal{S}C_{\Gamma,\Gamma^*}$ (*resp.* $\alpha C_{\Gamma,\Gamma^*}$)$(\rho,\vartheta,\varrho)=\bigwedge\{\mu\in I^U|\mu\geq\rho,\mu$ *is an* (ϑ,ϱ)-$f\delta\mathcal{S}c$ (*resp.* (ϑ,ϱ)-$f\alpha c$)$\}$.

Definition 2.7. [11, 12] *In a dfts, (U,Γ,Γ^*) A fuzzy set $\nu\in I^U$ is called an (ϑ,ϱ)-*

fuzzy Z clopen (resp. (ϑ,ϱ)- fuzzy δ clopen, (ϑ,ϱ)- fuzzy α clopen, (ϑ,ϱ)- fuzzy semi clopen, (ϑ,ϱ)- fuzzy γ clopen, (ϑ,ϱ)- fuzzy e clopen and (ϑ,ϱ)- fuzzy δ semi clopen) (briefly (ϑ,ϱ)-$fZclo$ (resp. (ϑ,ϱ)-$f\delta clo$, (ϑ,ϱ)-$f\alpha clo$, (ϑ,ϱ)-$f\mathcal{S}clo$, (ϑ,ϱ)-$f\gamma clo$, (ϑ,ϱ)-$feclo$ and (ϑ,ϱ)-$f\delta\mathcal{S}clo$)) if ν is both (ϑ,ϱ)-fZo (resp. (ϑ,ϱ)-$f\delta o$, (ϑ,ϱ)-$f\alpha o$, (ϑ,ϱ)-$f\mathcal{S}o$, (ϑ,ϱ)-$f\gamma o$, (ϑ,ϱ)-feo and (ϑ,ϱ)-$f\delta\mathcal{S}o$) set and (ϑ,ϱ)-fZc (resp. (ϑ,ϱ)-$f\delta c$, (ϑ,ϱ)-$f\alpha c$, (ϑ,ϱ)-$f\mathcal{S}c$, (ϑ,ϱ)-$f\gamma c$, (ϑ,ϱ)-fec and (ϑ,ϱ)-$f\delta\mathcal{S}c$) set.

3. An (ϑ,ϱ)- fuzzy Z_α open sets

Definition 3.1. *Let (U,Γ,Γ^*) be a dfts, $\rho,\mu\in I^U$, $\vartheta\in I_0$ and $\varrho\in I_1$ such that $\vartheta+\varrho\le 1$, then the fuzzy set ρ is called an*

(i) (ϑ,ϱ)-fuzzy Z_α open (briefly (ϑ,ϱ)-$fZ_\alpha o$) set if $\rho\le I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)\vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$.

(ii) (ϑ,ϱ)-fuzzy Z_α closed (briefly (ϑ,ϱ)-$fZ_\alpha c$) set if $\rho\ge C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)\wedge I_{\Gamma,\Gamma^*}(\delta C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$.

Definition 3.2. *Let (U,Γ,Γ^*) be a dfts, then the*

(i) union of all (ϑ,ϱ)-$fZ_\alpha o$ sets contained in ρ is called the (ϑ,ϱ)-fuzzy Z_α interior of ρ (briefly, $Z_\alpha I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)$).

(ii) intersection of all (ϑ,ϱ)-$fZ_\alpha c$ sets containing ρ is called the (ϑ,ϱ)-fuzzy Z_α closure of ρ (briefly, $Z_\alpha C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)$).

Proposition 3.1. *Let (U,Γ,Γ^*) be a dfts, $\rho,\mu\in I^U$, $\vartheta\in I_0$ and $\varrho\in I_1$ then (i) $Z_\alpha I_{\Gamma,\Gamma^*}(\underline{0},\vartheta,\varrho)=\underline{0}$, and $Z_\alpha I_{\Gamma,\Gamma^*}(\underline{1},\vartheta,\varrho)=\underline{1}$, $\forall r\in I_0$, $s\in I_1$. (ii) $Z_\alpha C_{\Gamma,\Gamma^*}(\underline{0},\vartheta,\varrho)=\underline{0}$, and $Z_\alpha C_{\Gamma,\Gamma^*}(\underline{1},\vartheta,\varrho)=\underline{1}$, $\forall r\in I_0$, $s\in I_1$. (iii) $\underline{1}-Z_\alpha I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=Z_\alpha C_{\Gamma,\Gamma^*}(\underline{1}-\rho,\vartheta,\varrho)$. (iv) $\underline{1}-Z_\alpha C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=Z_\alpha I_{\Gamma,\Gamma^*}(\underline{1}-\rho,\vartheta,\varrho)$. (v) If $\rho<\mu$ then $Z_\alpha I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)<Z_\alpha I_{\Gamma,\Gamma^*}(\mu,\vartheta,\varrho)$. (vi) If $\rho\le\mu$ then $Z_\alpha C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)\le Z_\alpha C_{\Gamma,\Gamma^*}(\mu,\vartheta,\varrho)$.*

Definition 3.3. *Let (U,Γ,Γ^*) be a dfts, $\rho\in I^U$, $\vartheta\in I_0$ and $\varrho\in I_1$, ρ is called an (ϑ,ϱ)- fuzzy $Z_\alpha Q$-neighborhood of $u_t\in P_t(U)$ if there exists an (ϑ,ϱ)-$fZ_\alpha o$ set $\eta\in I^U$ such that $u_t\ q\ \eta$ and $\eta\le\rho$.*

The family of all (ϑ,ϱ)-fuzzy $Z_\alpha Q$-neighborhood of u_t is dented by $Z_\alpha Q$-(u_t,ϑ,ϱ).

Theorem 3.1. *Let (U,Γ,Γ^*) be a dfts, for each $\rho,\eta\in I^U$, $\vartheta\in I_0$ and $\varrho\in I_1$ such that $\vartheta+\varrho\le 1$, then the operator (ϑ,ϱ)-$Z_\alpha C_{\Gamma,\Gamma^*}$ satisfies the following statements*

(i) $\rho\le Z_\alpha C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)$.

(ii) $Z_\alpha C_{\Gamma,\Gamma^*}(\rho \vee \eta, \vartheta, \varrho) \geq Z_\alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \vee Z_\alpha C_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$.

(iii) $Z_\alpha C_{\Gamma,\Gamma^*}(\rho \wedge \eta, \vartheta, \varrho) \leq Z_\alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \wedge Z_\alpha C_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$.

(iv) $Z_\alpha C_{\Gamma,\Gamma^*}(Z_\alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho) = Z_\alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

(v) *If* ρ *is* (ϑ, ϱ)-$fZ_\alpha C$ *set then* $Z_\alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) = \rho$.

(vi) *If* η *is* (ϑ, ϱ)-$fZ_\alpha o$ *set then* $\eta \; q \; \rho$ *iff* $\eta \; q \; Z_\alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

Proof. Straight forward.

Theorem 3.2. *Let* (U, Γ, Γ^*) *be a dfts, for each* $\rho, \eta \in I^U$, $\vartheta \in I_0$ *and* $\varrho \in I_1$ *such that* $\vartheta + \varrho \leq 1$, *then the operator* (ϑ, ϱ)-$Z_\alpha I_{\Gamma,\Gamma^*}$ *satisfies the following statements*

(i) $\rho \geq Z_\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

(ii) $Z_\alpha I_{\Gamma,\Gamma^*}(\rho \vee \eta, \vartheta, \varrho) \geq Z_\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \vee Z_\alpha I_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$.

(iii) $Z_\alpha I_{\Gamma,\Gamma^*}(\rho \wedge \eta, \vartheta, \varrho) \leq Z_\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \wedge Z_\alpha I_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$.

(iv) $Z_\alpha I_{\Gamma,\Gamma^*}(Z_\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho) = Z_\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

(v) *If* ρ *is* (ϑ, ϱ)-$fZ_\alpha o$ *set then* $Z_\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) = \rho$.

Proof: Straight forward.

Theorem 3.3. *Let* (U, Γ, Γ^*) *be a dfts, for each* $\rho \in I^U$, $\vartheta \in I_0$ *and* $\varrho \in I_1$ *such that* $\vartheta + \varrho \leq 1$, *then*

(i) *Every* (ϑ, ϱ)-$f\delta o$ *set is* (ϑ, ϱ)-fo *set.*

(ii) *Every* (ϑ, ϱ)-fo *set is* (ϑ, ϱ)-$f\alpha o$ *set.*

(iii) *Every* (ϑ, ϱ)-$f\alpha o$ *set is* (ϑ, ϱ)-$fZ_\alpha o$ *set.*

(iv) *Every* (ϑ, ϱ)-$f\delta o$ *set is* (ϑ, ϱ)- $f\delta\mathcal{S}o$ *set.*

(v) *Every* (ϑ, ϱ)-$f\delta\mathcal{S}o$ *set is* (ϑ, ϱ)-$fZ_\alpha o$ *set.*

(vi) *Every* (ϑ, ϱ)-$f\delta\mathcal{S}o$ *set is* (ϑ, ϱ)- feo *set.*

(vii) *Every* (ϑ, ϱ)-$fZ_\alpha o$ *set is* (ϑ, ϱ)- fZo *set.*

(viii) *Every* (ϑ, ϱ)-$fZ_\alpha o$ *set is* (ϑ, ϱ)- $f\mathcal{S}o$ *set.*

(ix) Every (ϑ, ϱ)-$f\mathcal{S}o$ set is (ϑ, ϱ)- $f\gamma o$ set.

(x) Every (ϑ, ϱ)-fZo set is (ϑ, ϱ)- $f\gamma o$ set.

(xi) Every (ϑ, ϱ)-fZo set is (ϑ, ϱ)- feo set.

Proof: Obvious.

Remark 3.1. *The converse of the above theorem, in general, need not be true by the following examples.*

Example 3.1. Let $U = \{5_a, 5_b, 5_c\}$ and let the fs's α_1, α_2 and α_3 defined as $\alpha_1(5_a) = 0.3,\ \alpha_1(5_b) = 0.4,\ \alpha_1(5_c) = 0.5,\ \alpha_2(5_a) = 0.6,\ \alpha_2(5_b) = 0.9,\ \alpha_2(5_c) = 0.5,\ \alpha_3(5_a) = 0.4,\ \alpha_3(5_b) = 0.7$ and $\alpha_3(5_c) = 0.5$. Consider the double topology (Γ, Γ^*) defined as

$$\Gamma(\rho) = \begin{cases} 1, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \rho \in \{\alpha_1, \alpha_2\}, \\ 0, & \text{Otherwise.} \end{cases} \qquad \Gamma^*(\rho) = \begin{cases} 0, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \rho \in \{\alpha_1, \alpha_2\}, \\ 1, & \text{Otherwise.} \end{cases}$$

Then the fuzzy set

(i) α_2 is an $(\frac{1}{2}, \frac{1}{2})$-$fZ_\alpha o$ set but not an $(\frac{1}{2}, \frac{1}{2})$- $f\delta\mathcal{S}o$ set.

(ii) α_3 is an $(\frac{1}{2}, \frac{1}{2})$-$fZo$ set but not an $(\frac{1}{2}, \frac{1}{2})$- $fZ_\alpha o$ set.

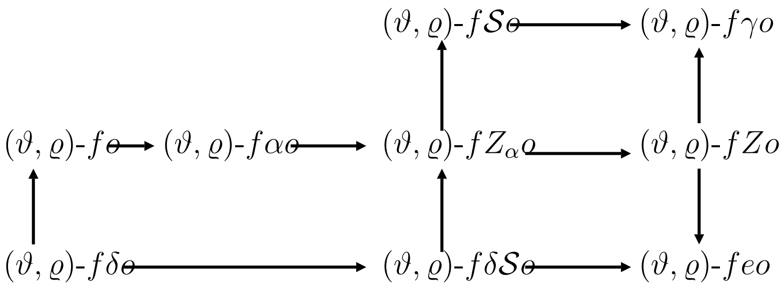
Example 3.2. Let $U = \{5_a, 5_b, 5_c\}$ and let the fs's α_1 to α_7 defined as $\alpha_1(5_a) = 0.2,\ \alpha_1(5_b) = 0.3,\ \alpha_1(5_c) = 0.5;\ \ \alpha_2(5_a) = 0.6,\ \alpha_2(5_b) = 0.5,\ \alpha_2(5_c) = 0.5;$ $\alpha_3(5_a) = 0.6,\ \alpha_3(5_b) = 0.7,\ \alpha_3(5_c) = 0.5;\ \alpha_4(5_a) = 0.7,\ \alpha_4(5_b) = 0.6,\ \alpha_4(5_c) = 0.5;$ $\alpha_5(5_a) = 0.7,\ \alpha_5(5_b) = 0.7,\ \alpha_5(5_c) = 0.5;\ \alpha_6(5_a) = 0.61,\ \alpha_6(5_b) = 0.6,\ \alpha_6(5_c) = 0.5$ $\alpha_7(5_a) = 1.0,\ \alpha_7(5_b) = 0.6,\ \alpha_7(5_c) = 0.5$. Consider the double topology (Γ, Γ^*) defined as

$$\Gamma(\rho) = \begin{cases} 1, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \rho \in \{\underline{0.5}, \alpha_1, \alpha_2, \alpha_3\}, \\ 0, & \text{Otherwise.} \end{cases} \qquad \Gamma^*(\rho) = \begin{cases} 0, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \rho \in \{\underline{0.5}, \alpha_1, \alpha_2, \alpha_3\}, \\ 1, & \text{Otherwise.} \end{cases}$$

Then the fuzzy set (i) α_4 is an $(\frac{1}{2}, \frac{1}{2})$- $f\mathcal{S}o$ set but not an $(\frac{1}{2}, \frac{1}{2})$- $fZ_\alpha o$ set. (ii) α_5 is an $(\frac{1}{2}, \frac{1}{2})$- $fZ_\alpha o$ set but not an $(\frac{1}{2}, \frac{1}{2})$- $f\alpha o$ set. (iii) α_6 is an $(\frac{1}{2}, \frac{1}{2})$- $f\alpha o$ set but not an $(\frac{1}{2}, \frac{1}{2})$- fo set. (iv) α_7 and α_5 are $(\frac{1}{2}, \frac{1}{2})$- $fZ_\alpha o$ set but $\alpha_5 \wedge \alpha_7 = \alpha_4$ is not an $(\frac{1}{2}, \frac{1}{2})$- $fZ_\alpha o$ set.

The others are in [12].

From the above theorem and examples, the following implications are hold.



Theorem 3.4. *Let (U,Γ,Γ^*) be a dfts,*

(i) $\bigvee_{i\in I}\gamma_i$ *is an* (ϑ,ϱ)-$fZ_\alpha o$ *set if* $\forall\ i\in I,\ \gamma_i$ *be an* (ϑ,ϱ)-$fZ_\alpha o$ *set.*

(ii) $\bigwedge_{i\in I}\gamma_i$ *is an* (ϑ,ϱ)-$fZ_\alpha c$ *set if* $\forall\ i\in I,\ \gamma_i$ *be an* (ϑ,ϱ)-$fZ_\alpha c$ *set.*

Proof. (i) Let γ_i be an (ϑ,ϱ)-$fZ_\alpha o$ set, $\forall i\in I$ then

$$\begin{aligned}
\gamma_i &\leq I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\gamma_i,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)\vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\gamma_i,\vartheta,\varrho),\vartheta,\varrho)\ \forall i\in I,\\
\Rightarrow \bigvee_{i\in I}\gamma_i &\leq \bigvee_{i\in I}(I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\gamma_i,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)\vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\gamma_i,\vartheta,\varrho),\vartheta,\varrho))\\
&\leq I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\bigvee_{i\in I}\gamma_i,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)\vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\bigvee_{i\in I}\gamma_i,\vartheta,\varrho),\vartheta,\varrho)
\end{aligned}$$

Thus $\bigvee_{i\in I}\gamma_i$ is an (ϑ,ϱ)-$fZ_\alpha o$ set.

(ii) Similar to (i).

Proposition 3.2. *Let (U,Γ,Γ^*) be a dfts, for $\rho\in I^U$ $\vartheta\in I_0$ and $\varrho\in I_1$. Then the statement*

(i) $\alpha C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\rho\vee C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)$ *and* $\alpha I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\rho\wedge I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)$.

(ii) $\delta\mathcal{S}C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\rho\vee I_{\Gamma,\Gamma^*}(\delta C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$ *and* $\delta\mathcal{S}I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\rho\wedge C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$.

(iii) $\mathcal{P}C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\rho\vee C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$ *and* $\mathcal{P}I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)=\rho\wedge I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)$

are hold.

Theorem 3.5. *Let (U, Γ, Γ^*) be a dfts, for $\rho \in I^U$ $\vartheta \in I_0$ and $\varrho \in I_1$. Then the statements*

(i) ρ *is* (ϑ, ϱ)-$fZ_\alpha o$ *set.*

(ii) $\rho = Z_\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

(iii) $\rho = \alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \vee \delta \mathcal{S} I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

are equivalent.

Proof. (i) $\Leftrightarrow$ (ii): Obvious.

(i) $\Rightarrow$ (iii): Let ρ be a (ϑ, ϱ)-$fZ_\alpha o$ set. Then $\rho \leq I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho), \vartheta, \varrho) \vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho)$. By Proposition **??**,

$\alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \vee \delta \mathcal{S} I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$

$$\begin{aligned} &= (\rho \wedge I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho), \vartheta, \varrho)) \vee (\rho \wedge C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho)) \\ &= \rho \wedge (I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho), \vartheta, \varrho)) \vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho) = \rho. \end{aligned}$$

(iii) $\Rightarrow$ (i): Let $\rho = \alpha I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \vee \delta \mathcal{S} I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$. Then by Proposition 3.2, we have

$$\begin{aligned} \rho &= (\rho \wedge I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho), \vartheta, \varrho)) \vee (\rho \wedge C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho)) \\ &\leq I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho), \vartheta, \varrho) \vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho), \vartheta, \varrho). \end{aligned}$$

Therefore ρ is (ϑ, ϱ)-$fZ_\alpha o$ set.

Theorem 3.6. *Let (U, Γ, Γ^*) be a dfts, for $\rho \in I^U$ $\vartheta \in I_0$ and $\varrho \in I_1$. Then the following statements are equivalent.*

(i) ρ *is* (ϑ, ϱ)-$fZ_\alpha c$ *set.*

(ii) $\rho = Z_\alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

(iii) $\rho = \alpha C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho) \wedge \delta \mathcal{S} C_{\Gamma,\Gamma^*}(\rho, \vartheta, \varrho)$.

Theorem 3.7. *Let (U, Γ, Γ^*) be a dfts, for $\rho, \eta \in I^U$ $\vartheta \in I_0$ and $\varrho \in I_1$, where η is an crisp subset such that*

(i) $\Gamma(\eta) \geq \vartheta,\ \Gamma^*(\eta) \leq \varrho$ *if ρ is an (ϑ, ϱ)-$fZ_\alpha o$ set, then $\rho \wedge \eta$ is an (ϑ, ϱ)-$fZ_\alpha o$ set.*

(ii) $\Gamma(\underline{1}-\eta)\geq\vartheta,\ \Gamma^*(\underline{1}-\eta)\leq\varrho$ *if* ρ *is an* (ϑ,ϱ)-$fZ_\alpha c$ *set, then* $\rho\vee\eta$ *is an* (ϑ,ϱ)-$fZ_\alpha c$ *set.*

Proof. (i) Let ρ is an (ϑ,ϱ)-$fZ_\alpha o$ set, and a crisp set $\eta\in I^U$ with $\Gamma(\eta)\geq\vartheta,\ \Gamma^*(\eta)\leq\varrho$, then

$$\begin{aligned}
&\rho\wedge\eta\\
\leq\ & (I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)\vee C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho))\wedge\eta\\
=\ & (I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho)\wedge\eta)\vee(C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho),\vartheta,\varrho)\wedge\eta)\\
\leq\ & (I_{\Gamma,\Gamma^*}(C_{\Gamma,\Gamma^*}(I_{\Gamma,\Gamma^*}(\rho\wedge\eta,\vartheta,\varrho),\vartheta,\varrho),\vartheta,\varrho))\vee(C_{\Gamma,\Gamma^*}(\delta I_{\Gamma,\Gamma^*}(\rho\wedge\eta,\vartheta,\varrho),\vartheta,\varrho))
\end{aligned}$$

Hence $\rho\wedge\eta$ is an (ϑ,ϱ)-$fZ_\alpha o$ set.
(ii) Similar to (i).

Theorem 3.8. *Let* (U,Γ,Γ^*) *be a dfts, for* $\rho\in I^U,\ \vartheta\in I_0$ *and* $\varrho\in I_1$

(i) If $\Gamma(\rho)\geq r$ *and* $\Gamma^*(\rho)\leq\varrho$ *then* ρ *is an* (ϑ,ϱ)-$fZ_\alpha o$ *set.*

(ii) $I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)$ *is an* (ϑ,ϱ)-$fZ_\alpha o$ *set.*

(iii) $C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)$ *is an* (ϑ,ϱ)-$fZ_\alpha c$ *set.*

Proof. Form the definition of dfts, $I_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)$ and $C_{\Gamma,\Gamma^*}(\rho,\vartheta,\varrho)$, it can be easily verified.

4. (ϑ,ϱ)- fuzzy Z_α clopen sets

Definition 4.1. *In a dfts,* (U,Γ,Γ^*) *a fuzzy set* $\nu\in I^U$ *is called an* (ϑ,ϱ)*- fuzzy* Z_α *clopen (briefly* (ϑ,ϱ)-$fZ_\alpha clo$*) if* ν *is both* (ϑ,ϱ)-$fZ_\alpha o$ *set and* (ϑ,ϱ)-$fZ_\alpha c$ *set.*

Proposition 4.1. *In a dfts* (U,Γ,Γ^*),

(i) $\underline{0}$ *and* $\underline{1}$ *are* (ϑ,ϱ)-$fZ_\alpha clo$ *sets.*

(ii) If $\nu\in I^U$ *is* (ϑ,ϱ)-$fZ_\alpha clo$ *set then so is* $(\underline{1}-\nu)$.

(iii) If $\nu,\mu\in I^U$ *are* (ϑ,ϱ)-$fZ_\alpha clo$ *sets then* $\nu\vee\mu$ *and* $\nu\wedge\mu$ *are* (ϑ,ϱ)-$fZ_\alpha clo$ *set.*

(iv) The set of all (ϑ,ϱ)-$fZ_\alpha clo$ *sets may be used as a basis for a double fuzzy topology. Whereas the set of all* (ϑ,ϱ)-$fZ_\alpha o$ *sets do not form a basis for a double fuzzy topology.*

Definition 4.2. *Let* (U,Γ,Γ^*) *be a dfts, then the* *(i) union of all (ϑ, ϱ)-$fZ_\alpha clo$ sets contained in η is called the (ϑ, ϱ)- fuzzy Z_α clopen interior of η (briefly, $Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$).*

(ii) intersection of all (ϑ, ϱ)-$fZ_\alpha clo$ sets containing η is called the (ϑ, ϱ)- fuzzy Z_α clopen closure of η (briefly, $Z_\alpha C^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$).

Proposition 4.2. *In a dfts (U, Γ, Γ^*), $\forall \gamma, \nu \in I^U$,*

(a) $Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\underline{0}, \vartheta, \varrho) = \underline{0}$, *and* $Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\underline{1}, \vartheta, \varrho) = \underline{1}$.

(b) If $\eta \leq \nu$ *then* $Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho) \leq Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\nu, \vartheta, \varrho)$.

(c) $Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho) \leq Z_\alpha I_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho) \leq \eta \leq Z_\alpha C_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho) \leq Z_\alpha C^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$.

(d) $Z_\alpha I^{co}_{\Gamma,\Gamma^*}(Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho), \vartheta, \varrho) = Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$.

(e) $\underline{1} - Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho) = Z_\alpha C^{co}_{\Gamma,\Gamma^*}(\underline{1} - \eta, \vartheta, \varrho)$.

(f) If η *is* (ϑ, ϱ)-$fZ_\alpha clo$ *set then* $Z_\alpha C^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho) = \eta = Z_\alpha I^{co}_{\Gamma,\Gamma^*}(\eta, \vartheta, \varrho)$.

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