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ON ALAN DAY'S DOUBLING CONSTRUCTION IN BOOLEAN ALGEBRA

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Abstract: In this paper, we prove that in a Boolean Algebra, doubling of an interval makes it distributive but not Boolean.

Keywords and Phrases: Lattices, boolean algebra, doubling construction in lattices.

2020 Mathematics Subject Classification: 44A99.

1. Introduction

G. Gratzer in his paper [4] introduced a new lattice L^U from a given lattice L by adding an element a^U called the double of $a \neq 0, 1$ in L where $L^U = L \cup \{a^U\}$ with a new order denoted by $\leq^U$. Following that construction, A. Day [1] introduced a similar construction $L[I]$ by doubling an interval I of a given lattice L. After that it witnessed many developments, e.g. see [2], [3], [6]. In the paper [3] entitled 'Doubling Constructions in Lattice Theory', Alan Day mentioned the following result which appeared in [2]: Let L be a distributive lattice and take $I = [u, v]$ in L, $L[I]$ is again distributive if and only if $L = [u, 1] \cup [0, v]$. The proof there is implicit. For Boolean algebras, we give in this paper an explicit proof.

In this section, we give some preliminary definitions needed for the development of the paper. In section 2, we give the proof of the main result and in section 3, we give a counter-example to show that $B_n(I)$ is not distributive if I is an intermediate interval. In section 4, we give the conclusion of this paper.

Definition 1.1. [5] *A lattice L satisfying the following identities*

- $(x \wedge y) \vee z = (x \wedge y) \bigvee (x \wedge z)$
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

for all $x, y, z \in L$ is called a distributive lattice.

If not, it is a non-distributive lattice.

Definition 1.2. [5] *A Boolean lattice is a complemented and bounded distributive lattice.*

Definition 1.3. [4] *Let L be a lattice and let $a \in L$ such that $a \neq 0, 1$. Now, we construct a lattice $L^U = L \cup \{a^U\}$ by adding the double of a: the element a^U, using the order relation stated as follows:*

For $x, y \in L$, let $x \leq^U y$ if $x \leq y$;

for $x \leq a$, let $x <^U a^U$;

for $a < x$, let $a^U \leq^U x$.

Definition 1.4. [4] *Let $I = [a, b]$ be an interval of a lattice L. The set $I \times C_2$ is formed using the two-element chain $C_2 = \{0, 1\}$. The set $L[I] = (L \backslash I) \cup (I \times C_2)$ is a lattice given by the ordering for $x, y \in L$ and $i, j \in C_2$;*

$x \leq y$ if $x \leq y$ in L;

$(x, i) \leq y$ if $x \leq y$ in L;

$x \leq (y, j)$ if $x \leq y$ in L;

$(x, i) \leq (y, j)$ if $x \leq y$ in L and $i \leq j$ in C_2

$L(I)$ is a lattice got by doubling of the interval I in L. This is Day's definition of doubling of intervals.

2. Main Results - Doubling in Boolean Algebras

Theorem 2.1. *Doubling construction of a Boolean algebra by an interval containing 0 is always distributive.*

Proof. Let B_n denote the Boolean algebra of rank n. Let $a_1, a_2, ..., a_n$ be the atoms of B_n. Let I be an interval of B_n containing 0 of B_n, denoted by 0_L. Then,

$$I \simeq B_k, \text{ for some } k \leq n.$$

Without loss of generality, let us assume that

$$B_k = [0, a_1a_2...a_k]$$

which has $a_1, a_2, ..., a_k$ as its atoms and where we write $a_1a_2...a_k$ for $a_1 \vee a_2 \vee ... \vee a_k$.
When we double the interval B_k, we have $B_k \times C_2 \simeq B_{k+1}$.
Now, $B_n(I) = (B_n \setminus I) \cup (I \times C_2)$ is the new lattice formed by doubling the interval I.
The elements of B_{k+1} are of the form $(a_{m1}...a_{ms}, 0)$ or $(a_{p1}...a_{pq}, 1)$,
where $a_{m1}, a_{m2}, ..., a_{ms} \in \{a_1, a_2, ..., a_k\}$ and $a_{p1}, ..., a_{pq} \in \{a_1, a_2, ..., a_k\}$.
We claim that $B_n(I)$ is distributive.
Let $x, y, z \in B_n(I)$.
Let $x, y \in (B_k \times C_2) = B_{k+1}$ and $z \in B_n \setminus B_k$.

Case 1.

Let $x = (a_{m1}...a_{ms}, 0), y = (a_{p1}...a_{pq}, 1)$ and $z = (a_p...a_r), p < r \leq n$, where $a_{m1}, ..., a_{ms}$ and $a_{p1}, ..., a_{pq}$ are distinct.

Subcase 1a.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\}, a_{p1}, ..., a_{pq} \notin \{a_p, ..., a_r\}$
Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$
$= (a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}a_p...a_r)$
$= (a_{m1}...a_{ms}, 0)$
$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$
$= (0_L, 0) \vee (a_{m1}...a_{ms}, 0)$, where 0_L denotes the lowest element of B_n to distinguish it from 0 of C_2.
$= (a_{m1}...a_{ms}, 0)$
Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Subcase 1a$_1$.

Let $a_{m1} = a_{p1}, a_{m2} = a_{p2}, ..., a_{mt} = a_{pt}, t < s$,
$a_{m1}, ..., a_{ms}, a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$
Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$
$= (a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)$
$= (a_{m1}...a_{mt}, 0)$
$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$
$= (0_L, 0) \vee (a_{m1}...a_{mt}, 0)$
$= (a_{m1}...a_{mt}, 0)$
Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 1b.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)$

$= (0_L, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (0_L, 0)$

$= (0_L, 0)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 1c.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)$

$= (a_{m1}...a_{ms}, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (a_{m1}...a_{ms}, 0)$

$= (a_{m1}...a_{ms}, 0)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 1d.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \notin \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}a_p...a_r)$

$= (0_L, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (0_L, 0)$

$= (0_L, 0)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Case 2.

Let $x = (a_{m1}...a_{ms}, 1), y = (a_{p1}...a_{pq}, 0)$ and $z = (a_p...a_r), p < r \leq n$

Subcase 2a.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \notin \{a_p...a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}a_p...a_r)$

$= (a_{m1}...a_{ms}, 1)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (a_{m1}...a_{ms}, 1)$

$= (a_{m1}...a_{ms}, 1)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 2a$_1$.

Let $a_{m1} = a_{p1}, a_{m2} = a_{p2}, ..., a_{mt} = a_{pt}, t < s,$

$a_{m1}, ..., a_{ms}, a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)$

$= (a_{m1}...a_{ms}, 1)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (a_{m1}...a_{ms}, 1)$

$= (a_{m1}...a_{ms}, 1)$

Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 2b.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\},$

$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)$

$= (0_L, 1)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (0_L, 1)$

$= (0_L, 1)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 2c.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\},$

$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)$

$= (a_{m1}...a_{ms}, 1)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (a_{m1}...a_{ms}, 1)$

$= (a_{m1}...a_{ms}, 1)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 2d.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\},$

$a_{p1}, ..., a_{pq} \notin \{a_p...a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}a_p...a_r)$

$= (0_L, 1)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (0_L, 1)$

$= (0_L, 1)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Case 3.

Let $x = (a_{m1}...a_{ms}, 0), y = (a_{p1}...a_{pq}, 0)$ and $z = (a_p...a_r), p < r \leq n$

Subcase 3a.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \notin \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}a_p...a_r)$

$= (a_{m1}...a_{ms}, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (a_{m1}...a_{ms}, 0)$

$= (a_{m1}...a_{ms}, 0)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 3a$_1$.

Let $a_{m1} = a_{p1}, a_{m2} = a_{p2}, ..., a_{mt} = a_{pt}, t < s$,

$a_{m1}, ..., a_{ms}, a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)$

$= (a_{m1}...a_{ms}, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (a_{m1}...a_{ms}, 0)$

$= (a_{m1}...a_{ms}, 0)$

Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 3b.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}a_p...a_r)$

$= (0_L, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (0_L, 0)$

$= (0_L, 0)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 3c.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)$

$= (a_{m1}...a_{ms}, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (a_{m1}...a_{ms}, 0)$

$= (a_{m1}...a_{ms}, 0)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 3d.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \notin \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 0) \wedge [(a_{p1}...a_{pq}, 0) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}a_p...a_r)$

$= (0_L, 0)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 0) \wedge (a_{p1}...a_{pq}, 0)] \vee [(a_{m1}...a_{ms}, 0) \wedge (a_p...a_r)]$

$= (0_L, 0) \vee (0_L, 0)$

$= (0_L, 0)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Case 4.

Let $x = (a_{m1}...a_{ms}, 1), y = (a_{p1}...a_{pq}, 1)$ and $z = (a_p...a_r), p < r \leq n$

Subcase 4a.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\}$,

$a_{p1}, ..., a_{pq} \notin \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}a_p...a_r)$

$= (a_{m1}...a_{ms}, 1)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$

$= (0_L, 1) \vee (a_{m1}...a_{ms}, 1)$

$= (a_{m1}...a_{ms}, 1)$

Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 4a$_1$.

Let $a_{m1} = a_{p1}, a_{m2} = a_{p2}, ..., a_{mt} = a_{pt}, t < s$,

$a_{m1}, ..., a_{ms}, a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$

Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$

$= (a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)$

$= (a_{m1}...a_{ms}, 1)$

$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$

$= (0_L, 1) \vee (a_{m1}...a_{ms}, 1)$
$= (a_{m1}...a_{ms}, 1)$
Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 4b.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\}$,
$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$
Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$
$= (a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)$
$= (0_L, 1)$
$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$
$= (0_L, 1) \vee (0_L, 1)$
$= (0_L, 1)$
Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 4c.

Let $a_{m1}, ..., a_{ms} \in \{a_p, ..., a_r\}$,
$a_{p1}, ..., a_{pq} \in \{a_p, ..., a_r\}$
Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$
$= (a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)$
$= (a_{m1}...a_{ms}, 1)$
$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$
$= (0_L, 1) \vee (a_{m1}...a_{ms}, 1)$
$= (a_{m1}...a_{ms}, 1)$
Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Subcase 4d.

Let $a_{m1}, ..., a_{ms} \notin \{a_p, ..., a_r\}$,
$a_{p1}, ..., a_{pq} \notin \{a_p, ..., a_r\}$
Now, $x \wedge (y \vee z) = (a_{m1}...a_{ms}, 1) \wedge [(a_{p1}...a_{pq}, 1) \vee (a_p...a_r)]$
$= (a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}a_p...a_r)$
$= (0_L, 1)$
$(x \wedge y) \vee (x \wedge z) = [(a_{m1}...a_{ms}, 1) \wedge (a_{p1}...a_{pq}, 1)] \vee [(a_{m1}...a_{ms}, 1) \wedge (a_p...a_r)]$
$= (0_L, 1) \vee (0_L, 0)$
$= (0_L, 1)$
Hence, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Hence, in all the above cases, we see that when $x, y \in B_k \times C_2$ and $z \in B_n \setminus B_k$, x, y, z satisfy the distributive law.

In a similar way, in the cases when $x \in (B_k \times C_2) = B_{k+1}$ and $y, z \in B_n \setminus B_k$, it can be proved to satisfy the distributive law. In the cases when $x, y, z \in B_{k+1}$ and when $x, y, z \in B_n \setminus B_k$ the result follows, as B_n is distributive.

Thus, we conclude that $B_n(I)$ is distributive.

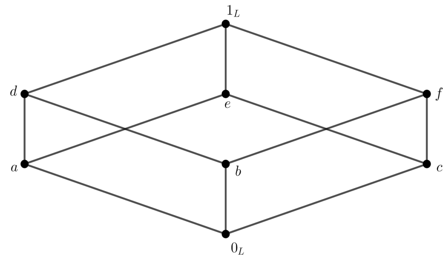

Figure 1: B_3

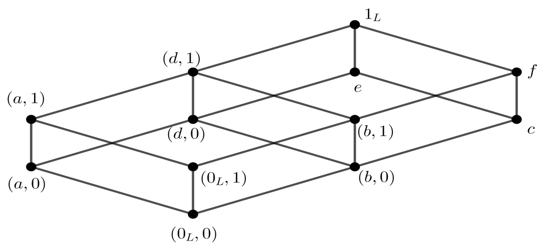

Figure 2: $B_3(I)$ where $I = [0_L, d]$

This $B_3(I)$ is distributive but not Boolean, as it is not complemented.

Corollary 2.2. *Doubling construction of a Boolean algebra by an interval containing 1 is always distributive.*

3. Special Cases - A Counter example

In this section, we give a counter example in which doubling of an intermediate interval of B_3 is not distributive. The following figure $B_3(I)$ contains the sublattice

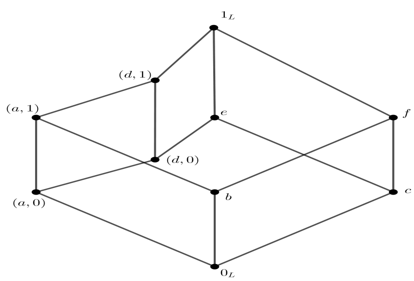

Figure 3: $B_3(I)$ where $I = [a, d]$

$\{0_L, (d, 0), (d, 1), c, 1_L\}$ in the form of N_5, a non-modular lattice which shows that $B_3(I)$ is not distributive.

4. Conclusion

There is a scope of examining the effect of doubling construction in other types of lattices.

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