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DERIVED STRUCTURES AND THEIR GRAPH INVARIANT

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Abstract: The symmetric division deg invariant is one of the 148 discrete Adriatic invariants introduced several years ago. In this paper, we obtained the expressions for symmetric division deg invariant of some derived graph and its complements.

Keywords and Phrases: Degree, Zagreb invariant, symmetric division deg invariant.

2020 Mathematics Subject Classification: 05C07, 05C09.

1. Introduction and Preliminaries

Molecular descriptors, results of functions mapping molecule's chemical information [8] into a number have found applications in modeling many physicochemical properties in QSAR and QSPR studies [1]. Damir Vukicević et al. [10] observed that many of these descritors are defined simply as the sum of individual bond contributions. Among the 148 discrete Adriatic invariants studied in [10], whose predictive properties were evaluated against the benchmark datasets of the International Academy of Mathematical Chemistry, 20 invariants were selected as significant predictors of physicochemical properties. One of these useful discrete Adriatic invariants is the symmetric division deg invariant which is defined as $SDD(\Gamma) = \sum\limits_{xy\in E(\Gamma)} \left(\frac{d_\Gamma(x)}{d_\Gamma(y)} + \frac{d_\Gamma(y)}{d_\Gamma(x)}\right)$, where $d_\Gamma(x)$ is the degree of vertex x in Γ. Among all the existing molecular descriptors, SDD invariant has the best correlating ability for predicting the total surface area of polychlorobiphenys [10], see more details in [2], [3], [4], [5], [6], [9].

Let Γ be a connected graph. The derived graph Γ^{ab} is obtained from Γ, whose vertex set $V(\Gamma) \cup E(\Gamma)$. The two vertices t_1 and t_2 are adjacent in Γ^{ab} if and only if the following conditions are holds:
(i) t_1, $t_2 \in V(\Gamma)$, t_1 and t_2 are adjacent in Γ if $a = +$ and t_1 and t_2 are not adjacent in Γ if $a = -$.
(ii) $t_1 \in V(\Gamma)$, $t_2 \in E(\Gamma)$, t_1 and t_2 are incident in Γ if $b = +$ and t_1 and t_2 are not incident in Γ if $b = -$.

2. SDD Invariant of Γ^{ab}

In this section, we present the results for symmetric division deg invariant of the derived graph Γ^{ab} of Γ. The first and second Zagreb invariants of Γ are defined as $M_1(\Gamma) = \sum\limits_{xy\in E(\Gamma)} (\lambda_\Gamma(x) + \lambda_\Gamma(y))$ and $M_2(\Gamma) = \sum\limits_{xy\in E(\Gamma)} (\lambda_\Gamma(x)\lambda_\Gamma(y))$.

Theorem 2.1. *Let Γ be a (n, m)-graph. Then $SDD(\Gamma^{++}) = SDD(\Gamma) + M_1(\Gamma) + n$.*
Proof. Partition the edge set of the graph Γ^{++} into two subsets, namely, $A_1 = \{xy|xy \in E(\Gamma)\}$ and $A_2 = \{xe|$the vertex x is incident to the edge e in $\Gamma\}$. One can observe that the cardinality of both the subsets A_1 and A_2 are m and $2m$, respectively. By the structure of the graph Γ^{++}, if $x \in V(\Gamma)$ then $\lambda_{\Gamma^{++}}(x) = 2\lambda_\Gamma(x)$ and if $e \in E(\Gamma)$ then $\lambda_{\Gamma^{++}}(e) = 2$. Hence

$$\begin{aligned}
SDD(\Gamma^{++}) &= \sum_{xy\in E(\Gamma^{++})} \frac{\lambda^2_{\Gamma^{++}}(x) + \lambda^2_{\Gamma^{++}}(y)}{\lambda_{\Gamma^{++}}(x)\lambda_{\Gamma^{++}}(y)} \\
&= \sum_{xy\in A_1} \frac{\lambda^2_{\Gamma^{++}}(x) + \lambda^2_{\Gamma^{++}}(y)}{\lambda_{\Gamma^{++}}(x)\lambda_{\Gamma^{++}}(y)} + \sum_{xe\in A_2} \frac{\lambda^2_{\Gamma^{++}}(x) + \lambda^2_{\Gamma^{++}}(e)}{\lambda_{\Gamma^{++}}(x)\lambda_{\Gamma^{++}}(e)} \\
&= \sum_{xy\in E(\Gamma)} \frac{4\lambda^2_\Gamma(x) + 4\lambda^2_\Gamma(y)}{4\lambda_\Gamma(x)\lambda_\Gamma(y)} + \sum_{xe\in A_2} \frac{4\lambda^2_\Gamma(x) + 4}{4\lambda_\Gamma(x)} \\
&= SDD(\Gamma) + \sum_{xe\in A_2} \frac{\lambda^2_\Gamma(x) + 1}{\lambda_\Gamma(x)}, \text{ by the definition of } SDD \text{ index of } \Gamma.
\end{aligned}$$

In the second sum, the quantity $\frac{\lambda^2_\Gamma(x)+1}{\lambda_\Gamma(x)}$ appears $\lambda_\Gamma(x)$ times. Hence the sum is equivalent to

$$\begin{aligned} SDD(\Gamma^{++}) &= SDD(\Gamma) + \sum_{x \in V(\Gamma)} \frac{\lambda_\Gamma(x)(\lambda_\Gamma^2(x)+1)}{\lambda_\Gamma(x)} \\ &= SDD(\Gamma) + M_1(\Gamma) + n. \end{aligned}$$

Theorem 2.2. *Let Γ be a (n,m)-graph. Then $SDD(\Gamma^{+-}) = m(m+2) + (n-2)^2$.*
Proof. Partition the edge set of the graph Γ^{+-} into two subsets, namely, $A_1 = \{xy | xy \in E(\Gamma)\}$ and $A_2 = \{xe|\text{the vertex } x \text{ is not incident to the edge } e \text{ in } \Gamma\}$. One can observe that the cardinality of both sets A_1 and A_2 are m and $m(n-2)$, respectively. By the structure of the graph Γ^{++}, if $x \in V(\Gamma)$ then $\lambda_{\Gamma^{+-}}(x) = m$ and if $e \in E(\Gamma)$ then $\lambda_{\Gamma^{+-}}(e) = n-2$. Therefore

$$\begin{aligned} SDD(\Gamma^{+-}) &= \sum_{xy \in E(\Gamma^{+-})} \frac{\lambda^2_{\Gamma^{+-}}(x) + \lambda^2_{\Gamma^{+-}}(y)}{\lambda_{\Gamma^{+-}}(x)\lambda_{\Gamma^{+-}}(y)} \\ &= \sum_{xy \in A_1} \frac{\lambda^2_{\Gamma^{+-}}(x) + \lambda^2_{\Gamma^{+-}}(y)}{\lambda_{\Gamma^{+-}}(x)\lambda_{\Gamma^{+-}}(y)} + \sum_{xe \in A_2} \frac{\lambda^2_{\Gamma^{+-}}(x) + \lambda^2_{\Gamma^{+-}}(e)}{\lambda_{\Gamma^{+-}}(x)\lambda_{\Gamma^{+-}}(e)} \\ &= \sum_{xy \in E(\Gamma)} \frac{m^2+m^2}{m^2} + \sum_{xe \in A_2} \frac{m^2+(n-2)^2}{m(n-2)}. \end{aligned}$$

In the first sum m times and second sum $m(n-2)$ times of the above equation. Hence the total sum is equivalent to

$$SDD(\Gamma^{+-}) = m(m+2) + (n-2)^2.$$

Theorem 2.3. *Let Γ be a (n,m)-graph. Then $SDD(\Gamma^{-+}) = \frac{n(n-1)^2 + m(n^2-4n+6)}{n-1}$.*
Proof. Partition the edge set of Γ^{-+} into two subsets, namely, $A_1 = \{xy | xy \notin E(\Gamma)\}$ and $A_2 = \{xe|\text{the vertex } x \text{ is incident to the edge } e \text{ in } \Gamma\}$. One can see that the number of elements in the sets A_1 and A_2 are $\binom{n}{2} - m$ and $2m$, respectively. By the structure of the graph Γ^{-+}, if $x \in V(\Gamma)$ then $\lambda_{\Gamma^{-+}}(x) = n-1$ and if $e \in E(\Gamma)$ then $\lambda_{\Gamma^{-+}}(e) = 2$. Hence

$$\begin{aligned} SDD(\Gamma^{-+}) &= \sum_{xy \in E(\Gamma^{-+})} \frac{\lambda^2_{\Gamma^{-+}}(x) + \lambda^2_{\Gamma^{-+}}(y)}{\lambda_{\Gamma^{-+}}(x)\lambda_{\Gamma^{-+}}(y)} \\ &= \sum_{xy \in A_1} \frac{\lambda^2_{\Gamma^{-+}}(x) + \lambda^2_{\Gamma^{-+}}(y)}{\lambda_{\Gamma^{-+}}(x)\lambda_{\Gamma^{-+}}(y)} + \sum_{xe \in A_2} \frac{\lambda^2_{\Gamma^{-+}}(x) + \lambda^2_{\Gamma^{-+}}(e)}{\lambda_{\Gamma^{-+}}(x)\lambda_{\Gamma^{-+}}(e)} \\ &= \sum_{xy \notin E(\Gamma)} \frac{((n-1)^2 + (n-1)^2}{(n-1)^2} + \sum_{xe \in A_2} \frac{(n-1)^2 + 2^2}{2(n-1)}. \end{aligned}$$

In the first sum $\binom{n}{2} - m$ times and second sum $2m$ times of the above equation. Hence the total sum is equivalent to

$$SDD(\Gamma^{-+}) = 2\Big(\tbinom{n}{2} - m\Big) + 2m\Big(\frac{n^2 - 2n + 5}{2(n-1)}\Big) = n(n-1) + \frac{m(n^2 - 4n + 6)}{n-1}.$$

Theorem 2.4. *Let* Γ *be a* (n, m)*-graph. Then* $\frac{\alpha}{(n+m-1-2\Delta(\Gamma))^2} + \frac{\beta}{(n+m-1-2\Delta(\Gamma))(n-2)} \leq SDD(\Gamma^{--}) \leq \frac{\alpha}{(n+m-1-2\delta(\Gamma))^2} + \frac{\beta}{(n+m-1-2\delta(\Gamma))(n-2)}$, *where* $\alpha = (n + m - 1)^2(n(n - 1) - 2m) + 4\overline{F}(\Gamma) - 4(n + m - 1)\overline{M_1}(\Gamma)$ *and* $\beta = m(n - 2)(n + m - 1)^2 + m(n - 2)^3 - 8m^2(n + m - 1) + 4(2m + n - 1)M_1(\Gamma) - 4F(\Gamma)$.

Proof. Partition the edge set of Γ^{--} into two subsets, namely, $A_1 = \{xy|xy \notin E(\Gamma)\}$ and $A_2 = \{xe|\text{the vertex } x \text{ is not incident to the edge } e \text{ in } \Gamma\}$. One can observe that $|A_1| = \binom{n}{2} - m$ and $|A_1| = m(n-2)$. By the structure of the graph Γ^{--}, if $x \in V(\Gamma)$ then $\lambda_{\Gamma^{--}}(x) = n+m-1-2\lambda_\Gamma(x)$ and if $e \in E(\Gamma)$ then $\lambda_{\Gamma^{--}}(e) = n-2$. Thus

$$\begin{aligned}
SDD(\Gamma^{--}) &= \sum_{xy\in E(\Gamma^{--})} \frac{\lambda^2_{\Gamma^{--}}(x) + \lambda^2_{\Gamma^{--}}(y)}{\lambda_{\Gamma^{--}}(x)\lambda_{\Gamma^{--}}(y)} \\
&= \sum_{xy\in A_1} \frac{\lambda^2_{\Gamma^{--}}(x) + \lambda^2_{\Gamma^{--}}(y)}{\lambda_{\Gamma^{--}}(x)\lambda_{\Gamma^{--}}(y)} + \sum_{xe\in A_2} \frac{\lambda^2_{\Gamma^{--}}(x) + \lambda^2_{\Gamma^{--}}(e)}{\lambda_{\Gamma^{--}}(x)\lambda_{\Gamma^{--}}(e)} \\
&= \sum_{xy\notin E(\Gamma)} \frac{(n + m - 1 - 2\lambda_\Gamma(x))^2 + (n + m - 1 - 2\lambda_\Gamma(y))^2}{(n + m - 1 - 2\lambda_\Gamma(x))(n + m - 1 - 2\lambda_\Gamma(y))} \\
&\quad + \sum_{xe\in A_2} \frac{(n + m - 1 - 2\lambda_\Gamma(x))^2 + (n-2)^2}{(n + m - 1 - 2\lambda_\Gamma(x))(n-2)} \\
&= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \sum_{xy\notin E(\Gamma)} \frac{(n + m - 1 - 2\lambda_\Gamma(x))^2 + (n + m - 1 - 2\lambda_\Gamma(y))^2}{(n + m - 1 - 2\lambda_\Gamma(x))(n + m - 1 - 2\lambda_\Gamma(y))} \\
&= \sum_{xy\notin E(\Gamma)} \frac{2(n + m - 1)^2 + 4(\lambda^2_\Gamma(x) + \lambda^2_\Gamma(y)) - 4(n + m - 1)(\lambda_\Gamma(x) + \lambda_\Gamma(y))}{(n + m - 1 - 2\lambda_\Gamma(x))(n + m - 1 - 2\lambda_\Gamma(y))}.
\end{aligned}$$

Since for any vertex x in Γ, $\delta(\Gamma) \leq \lambda_\Gamma(x) \leq \Delta(\Gamma)$. Therefore

$$I_1 \leq \sum_{xy\notin E(\Gamma)} \frac{2(n + m - 1)^2 + 4(\lambda^2_\Gamma(x) + \lambda^2_\Gamma(y)) - 4(n + m - 1)(\lambda_\Gamma(x) + \lambda_\Gamma(y))}{(n + m - 1 - 2\delta(\Gamma))^2}$$

$$= \frac{2(n+m-1)^2(\binom{n}{2}-m)+4\overline{F}(\Gamma)-4(n+m-1)\overline{M_1}(\Gamma)}{(n+m-1-2\delta(\Gamma))^2}.$$

$$I_2 = \sum_{xe\in A_2} \frac{(n+m-1-2\lambda_\Gamma(x))^2+(n-2)^2}{(n+m-1-2\lambda_\Gamma(x))(n-2)}$$

$$= \sum_{xe\in A_2} \frac{(n+m-1)^2+4\lambda_\Gamma^2(x)-4(n+m-1)\lambda_\Gamma(x)+(n-2)^2}{(n+m-1-2\lambda_\Gamma(x))(n-2)}$$

$$\leq \sum_{x\in V(\Gamma)} (m-\lambda_\Gamma(x))\Big(\frac{(n+m-1)^2+4\lambda_\Gamma^2(x)-4(n+m-1)\lambda_\Gamma(x)+(n-2)^2}{(n+m-1-2\delta(\Gamma))(n-2)}\Big)$$

$$= \frac{m\Big(n(n+m-1)^2+4M_1(\Gamma)-8m(n+m-1)+n(n-2)^2\Big)}{(n+m-1-2\delta(\Gamma))(n-2)}$$

$$- \frac{\Big(2m(n+m-1)^2+4F(\Gamma)-4(n+m-1)M_1(\Gamma)+2m(n-2)^2\Big)}{(n+m-1-2\delta(\Gamma))(n-2)}$$

$$= \frac{m(n-2)(n+m-1)^2+m(n-2)^3-8m^2(n+m-1)}{(n+m-1-2\delta(\Gamma))(n-2)}$$

$$+ \frac{4(2m+n-1)M_1(\Gamma)-4F(\Gamma)}{(n+m-1-2\delta(\Gamma))(n-2)}.$$

Adding I_1 and I_2 we get the result.

3. SDD Invariant of $\overline{\Gamma^{ab}}$

In this section, we provide the formulae for symmetric division deg invariant of the complement of the derived graph Γ^{ab} of Γ.

Theorem 3.1. *Let Γ be a (s,m)-graph. Then* $\frac{\alpha'}{(s+m-1-2\Delta(\Gamma))^2}+\frac{\beta'}{(s+m-1-2\Delta(\Gamma))(s+m-3)}$ $+m(m-1) \leq SDD(\overline{\Gamma^{++}}) \leq \frac{\alpha'}{(s+m-1-2\delta(\Gamma))^2}+\frac{\beta'}{(s+m-1-2\delta(\Gamma))(s+m-3)}+m(m-1)$, *where* $\alpha' = (s+m-1)^2(s(s-1)-2m)+4\overline{F}(\Gamma)-4(s+m-1)\overline{M_1}(\Gamma)$ *and* $\beta' = m(s-2)(s+m-1)^2+m(s-2)(s+m-3)^2-8m^2(s+m-1)+4(2m+s-1)M_1(\Gamma)-4F(\Gamma)$.

Proof. Partition the edge set of $\overline{\Gamma^{++}}$ into three subsets, namely, $A_1 = \{xy|xy \notin E(\Gamma)\}$, $A_2 = \{xe|$the vertex x is not incident to the edge e in $\Gamma\}$ and $A_3 = \{e_1e_2|e_1, e_2 \in E(\Gamma)\}$. One can check that $|A_1| = \binom{n}{2}-m$, $|A_2| = m(s-2)$ and $|A_3| = \binom{m}{2}$. By the structure of the graph $\overline{\Gamma^{++}}$, if $x \in V(\Gamma)$ then $\lambda_{\overline{\Gamma^{++}}}(x) = s+m-1-2\lambda_\Gamma(x)$ and if $e \in E(\Gamma)$ then $\lambda_{\overline{\Gamma^{++}}}(e) = s+m-3$. Therefore

$$SDD(\overline{\Gamma^{++}}) = \sum_{xy\in E(\overline{\Gamma^{++}})} \frac{\lambda^2_{\overline{\Gamma^{++}}}(x)+\lambda^2_{\overline{\Gamma^{++}}}(y)}{\lambda_{\overline{\Gamma^{++}}}(x)\lambda_{\overline{\Gamma^{++}}}(y)}$$

$$\begin{aligned}
&= \sum_{xy\in A_1} \frac{\lambda^2_{\overline{\Gamma^{++}}}(x)+\lambda^2_{\overline{\Gamma^{++}}}(y)}{\lambda_{\overline{\Gamma^{++}}}(x)\lambda_{\overline{\Gamma^{++}}}(y)} + \sum_{xe\in A_2} \frac{\lambda^2_{\overline{\Gamma^{++}}}(x)+\lambda^2_{\overline{\Gamma^{++}}}(e)}{\lambda_{\overline{\Gamma^{++}}}(x)\lambda_{\overline{\Gamma^{++}}}(e)} + \sum_{e_1e_2\in A_3} \frac{\lambda^2_{\overline{\Gamma^{++}}}(e_1)+\lambda^2_{\overline{\Gamma^{++}}}(e_2)}{\lambda_{\overline{\Gamma^{++}}}(e_1)\lambda_{\overline{\Gamma^{++}}}(e_2)}\\
&= \sum_{xy\notin E(\Gamma)} \frac{(s+m-1-2\lambda_\Gamma(x))^2+(s+m-1-2\lambda_\Gamma(y))^2}{(s+m-1-2\lambda_\Gamma(x))(s+m-1-2\lambda_\Gamma(y))}\\
&\quad + \sum_{xe\in A_2} \frac{(s+m-1-2\lambda_\Gamma(x))^2+(s+m-3)^2}{(s+m-1-2\lambda_\Gamma(x))(s+m-3)}\\
&\quad + \sum_{e_1e_2\in A_3} \frac{(s+m-3)^2+(s+m-3)^2}{(s+m-3)(s+m-3)}\\
&= I_1+I_2+I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \sum_{xy\notin E(\Gamma)} \frac{(s+m-1-2\lambda_\Gamma(x))^2+(s+m-1-2\lambda_\Gamma(y))^2}{(s+m-1-2\lambda_\Gamma(x))(s+m-1-2\lambda_\Gamma(y))}\\
&= \frac{2(s+m-1)^2(\binom{n}{2}-m)+4\overline{F}(\Gamma)-4(s+m-1)\overline{M_1}(\Gamma)}{(s+m-1-2\delta(\Gamma))^2},
\end{aligned}$$

a similar argument of Theorem (2.4).

$$\begin{aligned}
I_2 &= \sum_{xe\in A_2} \frac{(s+m-1-2\lambda_\Gamma(x))^2+(s+m-3)^2}{(s+m-1-2\lambda_\Gamma(x))(s+m-3)}\\
&= \sum_{xe\in A_2} \frac{(s+m-1)^2+4\lambda^2_\Gamma(x)-4(s+m-1)\lambda_\Gamma(x)+(s+m-3)^2}{(s+m-1-2\lambda_\Gamma(x))(s+m-3)}\\
&\le \sum_{x\in V(\Gamma)} (m-\lambda_\Gamma(x))\Big(\frac{(s+m-1)^2+4\lambda^2_\Gamma(x)-4(s+m-1)\lambda_\Gamma(x)+(s+m-3)^2}{(s+m-1-2\delta(\Gamma))(s+m-3)}\Big)\\
&= \frac{m\Big(s(s+m-1)^2+4M_1(\Gamma)-8m(s+m-1)+s(s+m-3)^2\Big)}{(s+m-1-2\delta(\Gamma))(s+m-3)}\\
&\quad -\frac{\Big(2m(s+m-1)^2+4F(\Gamma)-4(s+m-1)M_1(\Gamma)+2m(s+m-3)^2\Big)}{(s+m-1-2\delta(\Gamma))(s+m-3)}\\
&= \frac{4(2m+s-1)M_1(\Gamma)-4F(\Gamma)}{(s+m-1-2\delta(\Gamma))(s+m-3)}\\
&\quad +\frac{m(s-2)(s+m-1)^2+m(s-2)(s+m-3)^2-8m^2(s+m-1)}{(s+m-1-2\delta(\Gamma))(s+m-3)}.
\end{aligned}$$

$$I_3 = \sum_{e_1e_2 \in E(\Gamma)} \frac{2(s+m-3)^2}{(s+m-3)^2} = 2\binom{m}{2} = m(m-1).$$

From I_1, I_2 and I_3, we get the desired result.

Theorem 3.2. *Let Γ be a (s, m)-graph. Then $SDD(\overline{\Gamma^{+-}}) = s(s-1) + m(m-3) + \frac{2m((s-1)^2+(m+1)^2)}{(s-1)(m+1)}$.*

Proof. Partition the edge set of $\overline{\Gamma^{+-}}$ into three subsets, namely, $A_1 = \{xy | xy \notin E(\Gamma)\}$, $A_2 = \{xe|$the vertex x is incident to the edge e in $\Gamma\}$ and $A_3 = \{e_1e_2 | \; e_1, e_2 \in E(\Gamma)\}$. One can check that $|A_1| = \binom{s}{2} - m$, $|A_2| = 2m$ and $|A_3| = \binom{m}{2}$. By the structure of the graph $\overline{\Gamma^{+-}}$, if $x \in V(\Gamma)$ then $\lambda_{\overline{\Gamma^{+-}}}(x) = s-1$ and if $e \in E(\Gamma)$ then $\lambda_{\overline{\Gamma^{+-}}}(e) = m+1$. Therefore

$$SDD(\overline{\Gamma^{+-}}) = \sum_{xy \in E(\overline{\Gamma^{+-}})} \frac{\lambda^2_{\overline{\Gamma^{+-}}}(x) + \lambda^2_{\overline{\Gamma^{+-}}}(y)}{\lambda_{\overline{\Gamma^{+-}}}(x)\lambda_{\overline{\Gamma^{+-}}}(y)}$$

$$= \sum_{xy \in A_1} \frac{\lambda^2_{\overline{\Gamma^{+-}}}(x) + \lambda^2_{\overline{\Gamma^{+-}}}(y)}{\lambda_{\overline{\Gamma^{+-}}}(x)\lambda_{\overline{\Gamma^{+-}}}(y)} + \sum_{xe \in A_2} \frac{\lambda^2_{\overline{\Gamma^{+-}}}(x) + \lambda^2_{\overline{\Gamma^{+-}}}(e)}{\lambda_{\overline{\Gamma^{+-}}}(x)\lambda_{\overline{\Gamma^{+-}}}(e)} + \sum_{e_1e_2 \in A_3} \frac{\lambda^2_{\overline{\Gamma^{+-}}}(e_1) + \lambda^2_{\overline{\Gamma^{+-}}}(e_2)}{\lambda_{\overline{\Gamma^{+-}}}(e_1)\lambda_{\overline{\Gamma^{+-}}}(e_2)}$$

$$= 2\Big(\binom{n}{2} - m\Big) + 2m\Big(\frac{(s-1)^2 + (m+1)^2}{(s-1)(m+1)}\Big) + 2\binom{m}{2}$$

$$= s(s-1) + m(m-3) + \frac{2m((s-1)^2 + (m+1)^2)}{(s-1)(m+1)}.$$

Theorem 3.3. *Let Γ be a (s, m)-graph. Then $SDD(\overline{\Gamma^{-+}}) = m(m+1) + \frac{m(s-2)(m^2+(s+m-3)^2)}{m(s+m-3)}$.*

Proof. Partition the edge set of $\overline{\Gamma^{-+}}$ into three subsets, namely, $A_1 = \{xy | xy \in E(\Gamma)\}$, $A_2 = \{xe|$the vertex x is not incident to the edge e in $\Gamma\}$ and $A_3 = \{e_1e_2 | e_1, e_2 \in E(\Gamma)\}$. One can check that $|A_1| = m$, $|A_2| = m(s-2)$ and $|A_3| = \binom{m}{2}$. By the structure of the graph $\overline{\Gamma^{-+}}$, if $x \in V(\Gamma)$ then $\lambda_{\overline{\Gamma^{-+}}}(x) = m$ and if $e \in E(\Gamma)$ then $\lambda_{\overline{\Gamma^{-+}}}(e) = s+m-3$. Therefore

$$SDD(\overline{\Gamma^{-+}}) = \sum_{xy \in E(\overline{\Gamma^{-+}})} \frac{\lambda^2_{\overline{\Gamma^{-+}}}(x) + \lambda^2_{\overline{\Gamma^{-+}}}(y)}{\lambda_{\overline{\Gamma^{-+}}}(x)\lambda_{\overline{\Gamma^{-+}}}(y)}$$

$$= \sum_{xy \in A_1} \frac{\lambda^2_{\overline{\Gamma^{-+}}}(x) + \lambda^2_{\overline{\Gamma^{-+}}}(y)}{\lambda_{\overline{\Gamma^{-+}}}(x)\lambda_{\overline{\Gamma^{-+}}}(y)} + \sum_{xe \in A_2} \frac{\lambda^2_{\overline{\Gamma^{-+}}}(x) + \lambda^2_{\overline{\Gamma^{-+}}}(e)}{\lambda_{\overline{\Gamma^{-+}}}(x)\lambda_{\overline{\Gamma^{-+}}}(e)} + \sum_{e_1e_2 \in A_3} \frac{\lambda^2_{\overline{\Gamma^{-+}}}(e_1) + \lambda^2_{\overline{\Gamma^{-+}}}(e_2)}{\lambda_{\overline{\Gamma^{-+}}}(e_1)\lambda_{\overline{\Gamma^{-+}}}(e_2)}$$

$$= 2m + m(s-2)\Big(\frac{m^2 + (s+m-3)^2}{m(s+m-3)}\Big) + 2\binom{m}{2}$$

$$= m(m+1) + \frac{m(s-2)(m^2+(s+m-3)^2)}{m(s+m-3)}.$$

Theorem 3.4. *Let Γ be a (s,m)-graph. Then* $SDD(\overline{\Gamma^{--}}) = SDD(\Gamma) + m(m-1) + \frac{4M_1(\Gamma)+s(m+1)^2}{2(m+1)}$.

Proof. Partition the edge set of $\overline{\Gamma^{--}}$ into three subsets, namely, $A_1 = \{xy|xy \in E(\Gamma)\}$, $A_2 = \{xe|\text{the vertex } x \text{ is incident to the edge } e \text{ in } \Gamma\}$ and $A_3 = \{e_1e_2|e_1, e_2 \in E(\Gamma)\}$. One can check that $|A_1| = m$, $|A_2| = 2m$ and $|A_3| = \binom{m}{2}$. By the structure of the graph $\overline{\Gamma^{--}}$, if $x \in V(\Gamma)$ then $\lambda_{\overline{\Gamma^{--}}}(x) = 2\lambda_\Gamma(x)$ and if $e \in E(\Gamma)$ then $\lambda_{\overline{\Gamma^{--}}}(e) = m+1$. Therefore

$$SDD(\overline{\Gamma^{--}}) = \sum_{xy\in E(\overline{\Gamma^{--}})} \frac{\lambda^2_{\overline{\Gamma^{--}}}(x) + \lambda^2_{\overline{\Gamma^{--}}}(y)}{\lambda_{\overline{\Gamma^{--}}}(x)\lambda_{\overline{\Gamma^{--}}}(y)}$$

$$= \sum_{xy\in A_1} \frac{\lambda^2_{\overline{\Gamma^{--}}}(x) + \lambda^2_{\overline{\Gamma^{--}}}(y)}{\lambda_{\overline{\Gamma^{--}}}(x)\lambda_{\overline{\Gamma^{--}}}(y)} + \sum_{xe\in A_2} \frac{\lambda^2_{\overline{\Gamma^{--}}}(x) + \lambda^2_{\overline{\Gamma^{--}}}(e)}{\lambda_{\overline{\Gamma^{--}}}(x)\lambda_{\overline{\Gamma^{--}}}(e)} + \sum_{e_1e_2\in A_3} \frac{\lambda^2_{\overline{\Gamma^{--}}}(e_1) + \lambda^2_{\overline{\Gamma^{--}}}(e_2)}{\lambda_{\overline{\Gamma^{--}}}(e_1)\lambda_{\overline{\Gamma^{--}}}(e_2)}$$

$$= \sum_{xy\in E(\Gamma)} \frac{4\lambda^2_\Gamma(x) + 4\lambda^2_\Gamma(y)}{4\lambda_\Gamma(x)\lambda_\Gamma(y)} + \sum_{xe\in A_2} \frac{4\lambda^2_\Gamma(x) + (m+1)^2}{2\lambda_\Gamma(x)(m+1)} + \sum_{e_1e_2\in A_3} \frac{2(m+1)^2}{(m+1)^2}$$

$$= \sum_{xy\in E(\Gamma)} \frac{\lambda^2_\Gamma(x) + \lambda^2_\Gamma(y)}{\lambda_\Gamma(x)\lambda_\Gamma(y)} + \sum_{x\in V(\Gamma)} \frac{\lambda_\Gamma(x)\left(4\lambda^2_\Gamma(x) + (m+1)^2\right)}{2d_\Gamma(x)(m+1)} + \sum_{e_1e_2\in A_3} (2)$$

$$= SDD(\Gamma) + m(m-1) + \frac{4M_1(\Gamma) + s(m+1)^2}{2(m+1)}.$$

4. Conclusion

This article, we obtained the expressions for symmetric division deg invariant of some derived graph and its complements. and also evinced the some interesting results .

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