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CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH $q-$ANALOGUE OF BESSEL FUNCTIONS

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Abstract: In this paper we consider various subclasses of bi-univalent functions defined by the Horadam polynomials associated with q-analogue of Bessel functions. Further, we obtain coefficient estimates and Fekete-Szegö inequalities for the defined classes.

Keywords and Phrases: Univalent functions, bi-univalent functions, bi-convex functions, bi-starlike functions, Fekete-Szegö inequality, $q-$derivative operator, Horadam polynomials, Bessel functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in Δ.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both the function f and its inverse f^{-1} are univalent in Δ. Let σ denote the class of bi-univalent functions in Δ given by (1.1).

In 2010, Srivastava et al. [35] revived the study of bi-univalent functions by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in the very recent investigations (see, for example, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [13], [14], [16], [19], [20], [22], [23], [25], [26], [27], [28], [29], [31], [32], [33], [34], [36], [37]) and including the references therein. The afore-cited all these papers on the subject were actually motivated by the work of Srivastava et al. [35]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \cdots)$ for functions $f \in \sigma$ is still an open problem.

For analytic functions f and g in Δ, f is said to be subordinate to g if there exists an analytic function w such that (see, for example, [12], [24])

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)).$$

This subordination will be denoted here by

$$f \prec g$$

or, conventionally, by

$$f(z) \prec g(z).$$

In particular, when g is univalent in Δ,

$$f \prec g \qquad (z \in \Delta) \;\Leftrightarrow\; f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The Bessel function of the first kind of order ν is defined by the infinite series (see [21])

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}, \qquad (z \in \mathbb{C},\ \nu \in \mathbb{R}) \tag{1.3}$$

where Γ stands for the Gamma function. Recently, Szasz and Kupan [30] investigated the univalence of the normalized Bessel function of the first kind $\kappa_\nu : \Delta \to \mathbb{C}$ defined by

$$\begin{aligned} \kappa_\nu(z) :=& 2^\nu \Gamma(\nu+1) z^{1-\nu/2} J_\nu(z^{1/2}) \\ =& z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} z^n, \qquad (z \in \Delta,\ \nu \in \mathbb{R}) \end{aligned} \tag{1.4}$$

For $0 < q < 1$, El-Deeb and Bulboaca [15] defined the $q-$derivateive operator for κ_ν as follows:

$$\begin{aligned} \partial_q \kappa_\nu(z) =& \partial_q \left[z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} z^n \right] := \frac{\kappa_\nu(qz) - \kappa_\nu(z)}{z(q-1)} \\ =& 1 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} [n,q] z^{n-1}, \qquad (z \in \Delta) \end{aligned} \tag{1.5}$$

where

$$[n,\ q] := \frac{1-q^n}{1-q} = 1 + \sum_{j=1}^{n-1} q^j, \qquad [0,\ q] := 0. \tag{1.6}$$

Using (1.6), we will define the next two products:

1. For any nonnegative integer n, the $q-$shifted factorial is given by

$$[n,\ q] := \begin{cases} 1, & \text{if } n = 0 \\ [1,\ q][2,\ q]\dots[k,\ q] & \text{if } n \in \mathbb{N}. \end{cases} \tag{1.7}$$

2. For any positive number r, the $q-$ generalized Pochhammer symbol is defined by

$$[r,\ q]_n := \begin{cases} 1, & \text{if } n = 0 \\ [r,\ q][r+1,\ q]\dots[r+k-1,\ q] & \text{if } n \in \mathbb{N}. \end{cases} \tag{1.8}$$

For $\nu > 0$, $\lambda > -1$ and $0 < q < 1$, El-Deeb and Bulboacă [15] defined the function $\mathcal{J}^{\lambda}_{\nu,\, q} : \Delta \to \mathbb{C}$ by (see [14], [16])

$$\mathcal{J}^{\lambda}_{\nu,\, q}(z) := z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \, \frac{[n,q]!}{[\lambda+1,q]_{n-1}} z^n, \qquad (z \in \Delta). \tag{1.9}$$

A simple computation shows that

$$\mathcal{J}^{\lambda}_{\nu,\, q}(z) * \mathcal{M}_{q,\, \lambda+1}(z) = z\partial_q \kappa_\nu(z), \qquad (z \in \Delta), \tag{1.10}$$

where the function $\mathcal{M}_{q,\, \lambda+1}(z)$ is given by

$$\mathcal{M}_{q,\, \lambda+1}(z) := z + \sum_{n=2}^{\infty} \frac{[\lambda+1,q]_{n-1}}{[n-1,q]!} z^n, \qquad (z \in \Delta). \tag{1.11}$$

Using the definition of $q-$derivative along with the idea of convolutions, El-Deeb and Bulboacă [15] introduced the linear operator $\mathcal{N}^{\lambda}_{\nu,\, q} : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} \mathcal{N}^{\lambda}_{\nu,\, q} f(z) :=& \mathcal{J}^{\lambda}_{\nu,\, q} * f(z) \\ =& z + \sum_{n=2}^{\infty} \psi_n a_n z^n, \qquad (\nu > 0,\ \lambda > -1,\ 0 < q < 1,\ z \in \Delta), \end{aligned} \tag{1.12}$$

where

$$\psi_n := \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \, \frac{[n,q]!}{[\lambda+1,q]_{n-1}} \tag{1.13}$$

Remark 1.1. [15] *From the definition relation (1.12), we can easily verify that the next relations hold for all* $f \in \mathcal{A}$ *:*

$$[\lambda+1,\ q]\mathcal{N}^{\lambda}_{\nu,\, q} f(z) = [\lambda,\ q]\mathcal{N}^{\lambda+1}_{\nu,\, q} f(z) + q^{\lambda} z\partial_q \left([\lambda+1,\ q]\mathcal{N}^{\lambda+1}_{\nu,\, q} f(z)\right), \qquad z \in \Delta \tag{1.14}$$

and

$$\begin{aligned} \lim_{q\to 1^-} \mathcal{N}^{\lambda}_{\nu,\, q} f(z) =& \mathcal{J}^{\lambda}_{\nu,\, 1} f(z) := \mathcal{J}^{\lambda}_{\nu} f(z) \\ =& z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \, \frac{n!}{(\lambda+1)_{n-1}} a_n z^n, \quad (z \in \Delta). \end{aligned} \tag{1.15}$$

The Horadam polynomials $h_n(x,\ a,\ b;\ p,\ q)$*, or briefly* $h_n(x)$ *are given by the following recurrence relation (see [17], [18])):*

$$h_1(x) = a, \quad h_2(x) = bx \quad \text{and} \quad h_n(x) = \quad px h_{n-1}(x) + q h_{n-2}(x) \qquad (n \geq 3) \tag{1.16}$$

for some real constants a, b, p and q.

The generating function of the Horadam polynomials $h_n(x)$ (see [18]) is given by

$$\Pi(x,\ z) := \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}, \qquad 1 - pxz - qz^2 \neq 0, \qquad \forall z \in \Delta. \tag{1.17}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is, $x \neq \Re(z)$.

Note that for particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ leads to various polynomials, among those, we list a few cases here (see, [17], [18] for more details):

1. For $a = b = p = q = 1$, we have the Fibonacci polynomials $F_n(x)$.
2. For $a = 2$ and $b = p = q = 1$, we obtain the Lucas polynomials $L_n(x)$.
3. For $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n(x)$.
4. For $a = b = p = 2$ and $q = 1$, we attain the Pell-Lucas polynomials $Q_n(x)$.
5. For $a = b = 1,\ p = 2$ and $q = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind
6. For $a = 1,\ b = p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $U_n(x)$ of the second kind.

Abirami et al. [1] considered bi- Mocanu - convex functions and bi-$\mu-$ starlike functions to discuss initial estimations of Taylor-Macularin series which is associated with Horadam polynomials, Abirami et al. [2] discussed coefficient estimates for the classes of $\lambda-$bi-pseudo-starlike and bi-Bazilevič functions using Horadam polynomial, Alamoush [3], [4] defined subclasses of bi-starlike and bi-convex functions involving the Poisson distribution series involving Horadam polynomials and a class of bi-univalent functions associated with Horadam polynomials respectively and obtained initial coefficient estimates, Altınkaya and Yalçın [7], [8] obtained coefficient estimates for Pascu-type bi-univalent functions and for the class of linear combinations of bi-univalent functions by means of (p, q)-Lucas polynomials respectively, Aouf et al. [10] discussed initial coefficient estimates for general class of pascu-type bi-univalent functions of complex order defined by $q-$Sălăgean operator and associated with Chebyshev polynomials, Awolere and Oladipo [11] found initial coefficients of bi-univalent functions defined by sigmoid functions involving pseudo-starlikeness associated with Chebyshev polynomials, Naeem et al. [22] considered a general class of bi-Bazilevič type functions associated with Faber polynomial to discuss n-th coefficients estimates, Magesh and Bulut [23] discussed Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, Orhan et al. [25] discussed initial estimates and Fekete-Szegö bounds for bi-Bazilevič functions related to shell-like curves, Sakar and Aydogan [28] obtained initial bounds for the class of generalized Sălăgean type bi-$\alpha-$ convex functions of complex order associated with the Horadam polynomials, Singh et al. [31] found coefficient estimates for bi-α-convex functions defined by generalized Sãlãgean operator related to shell-like curves connected with Fibonacci numbers, Srivastava et al. [32] introduced a technique by defining a new class of bi-univalent functions associated with the Horadam polynomials to discuss the coefficient estimates, Srivastava et al. [34] gave a direction to study the Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Srivastava et al. [36] obtained general coefficient $|a_n|$ for a general class analytic and bi-univalent functions defined by using differential subordination and a certain fractional derivative operator associated with Faber polynomial, Wanas and Alina [37] discussed applications of Horadam polynomials on Bazilevič bi-univalent functions by means of subordination and found initial bounds. Motivated in these lines, estimates on initial coefficients of the Taylor-Maclaurin series expansion (1.1) and Fekete-Szegö inequalities for certain classes of bi-univalent functions defined by means of Horadam polynomials are obtained. The classes introduced in this paper are motivated by the corresponding classes investigated in [19], [14].

2. Coefficient Estimates and Fekete-Szegö Inequalities

Definition 2.1. *A function $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{M}_\sigma$ (α, λ, ν, q, x) for $\alpha \geq 0$, $\nu > 0$, $\lambda > -1$, $0 < q < 1$, and $z,\ w \in \Delta$, if the following conditions are satisfied:*

$$\alpha\left(1+\frac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}\right)+(1-\alpha)\frac{1}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}\prec \Pi(x,\ z)+1-a$$

and for g the analytic extension (continuation) of f^{-1} given by (1.2)

$$\alpha\left(1+\frac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}\right)+(1-\alpha)\frac{1}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}\prec \Pi(x,\ w)+1-a,$$

where the real constant a is as in (1.16).

Remark 2.1. *Putting $q \to 1^-$, we obtain that*

$$\lim_{q\to 1^-} \mathcal{M}_\sigma(\alpha,\ \lambda,\ \nu,\ q,\ x) =: \mathcal{M}_\sigma(\alpha,\ \lambda,\ \nu,\ x),$$

where $f \in \sigma$,

$$\alpha\left(1+\frac{z\left(\mathcal{J}_\nu^\lambda f(z)\right)''}{\left(\mathcal{J}_\nu^\lambda f(z)\right)'}\right)+(1-\alpha)\frac{1}{\left(\mathcal{J}_\nu^\lambda f(z)\right)'} \prec \Pi(x,\ z)+1-a$$

and for g the analytic extension (continuation) of f^{-1} given by (1.2)

$$\alpha\left(1+\frac{w\left(\mathcal{J}_\nu^\lambda g(w)\right)''}{\left(\mathcal{J}_\nu^\lambda g(w)\right)'}\right)+(1-\alpha)\frac{1}{\left(\mathcal{J}_\nu^\lambda g(w)\right)'} \prec \Pi(x,\ w)+1-a,$$

where the real constant a is as in (1.16).

For functions in the class $\mathcal{M}_\sigma(\alpha,\ \lambda,\ \nu,\ q,\ x)$, the following coefficient estimates and Fekete-Szegö inequality are obtained.

Theorem 2.1. *Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{M}_\sigma(\alpha,\ \lambda,\ \nu,\ q,\ x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|(9\,\alpha-3)\,\psi_3 b^2x^2-\left[(8\,\alpha-4)\,b^2x^2+4\,\left(px^2b+qa\right)(2\,\alpha-1)^2\right]{\psi_2}^2\right|}},$$

$$|a_3| \le \frac{|bx|}{(9\,\alpha-3)\psi_3}+\frac{b^2x^2}{4\,(2\,\alpha-1)^2\,\psi_2^2}$$

and for $\mu \in \mathbb{R}$

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{|bx|}{(9\alpha-3)\psi_3} \\ \quad \text{if } |\mu-1| \le \dfrac{\left|(9\,\alpha-3)\,\psi_3 b^2x^2-\left[(8\,\alpha-4)\,b^2x^2+4\,\left(px^2b+qa\right)(2\,\alpha-1)^2\right]{\psi_2}^2\right|}{b^2x^2\,(9\alpha-3)\,\psi_3} \\[2ex] \dfrac{|bx|^3\,|\mu-1|}{\left|(9\,\alpha-3)\,\psi_3 b^2x^2-\left[(8\,\alpha-4)\,b^2x^2+4\,\left(px^2b+qa\right)(2\,\alpha-1)^2\right]{\psi_2}^2\right|} \\ \quad \text{if } |\mu-1| \ge \dfrac{\left|(9\,\alpha-3)\,\psi_3 b^2x^2-\left[(8\,\alpha-4)\,b^2x^2+4\,\left(px^2b+qa\right)(2\,\alpha-1)^2\right]{\psi_2}^2\right|}{b^2x^2\,(9\alpha-3)\,\psi_3}. \end{cases}$$

Proof. Let $f \in \mathcal{M}_\sigma(\alpha,\ \lambda,\ \nu,\ q,\ x)$ be given by the Taylor-Maclaurin expansion (1.1). Then, there are analytic functions $r(z)$ and $s(w)$ such that

$$r(0) = 0; \quad s(0) = 0, \quad |r_n| < 1 \quad \text{and} \quad |s_n| < 1 \quad (\forall\ z,\ w \in \Delta),$$

and we can write

$$\alpha\left(1 + \frac{z\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)''}{\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)'}\right) + (1-\alpha)\frac{1}{\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)'} = \Pi(x,\ r(z)) + 1 - a \tag{2.1}$$

and

$$\alpha\left(1 + \frac{w\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)''}{\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)'}\right) + (1-\alpha)\frac{1}{\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)'} = \Pi(x,\ s(w)) + 1 - a. \tag{2.2}$$

Equivalently,

$$\alpha\left(1 + \frac{z\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)''}{\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)'}\right) + (1-\alpha)\frac{1}{\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)'}$$
$$= 1 + h_1(x) - a + h_2(x)r(z) + h_3(x)[r(z)]^2 + \cdots \tag{2.3}$$

and

$$\alpha\left(1 + \frac{w\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)''}{\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)'}\right) + (1-\alpha)\frac{1}{\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)'}$$
$$= 1 + h_1(x) - a + h_2(x)s(w) + h_3(x)[s(w)]^2 + \cdots . \tag{2.4}$$

From (2.3) and (2.4) and in view of (1.17), we obtain

$$\alpha\left(1 + \frac{z\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)''}{\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)'}\right) + (1-\alpha)\frac{1}{\left(\mathcal{N}^\lambda_{\nu,\, q} f(z)\right)'}$$
$$= 1 + h_2(x)r_1 z + [h_2(x)r_2 + h_3(x)r_1^2]z^2 + \cdots \tag{2.5}$$

and

$$\alpha\left(1 + \frac{w\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)''}{\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)'}\right) + (1-\alpha)\frac{1}{\left(\mathcal{N}^\lambda_{\nu,\, q} g(w)\right)'}$$
$$= 1 + h_2(x)s_1 w + [h_2(x)s_2 + h_3(x)s_1^2]w^2 + \cdots . \tag{2.6}$$

If

$$r(z) = \sum_{n=1}^{\infty} r_n z^n \qquad \text{and} \qquad s(w) = \sum_{n=1}^{\infty} s_n w^n,$$

then it is well known that

$$|r_n| \leq 1 \qquad \text{and} \qquad |s_n| \leq 1 \qquad (n \in \mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (2.5) and (2.6), we have

$$2\psi_2 (2\alpha - 1) a_2 = h_2(x) r_1 \tag{2.7}$$

$$(9\alpha - 3) \psi_3 a_3 - 4 (2\alpha - 1) \psi_2^2 a_2^2 = h_2(x) r_2 + h_3(x) r_1^2 \tag{2.8}$$

$$-2 (2\alpha - 1) \psi_2 a_2 = h_2(x) s_1 \tag{2.9}$$

and

$$[(18\alpha - 6) \psi_3 - 4(2\alpha - 1)\psi_2^2] a_2^2 - 3 (3\alpha - 1) \psi_3 a_3 = h_2(x) s_2 + h_3(x) s_1^2. \tag{2.10}$$

From (2.7) and (2.9), we can easily see that

$$r_1 = -s_1, \qquad \text{provided} \qquad h_2(x) = bx \neq 0 \tag{2.11}$$

and

$$8\, a_2^2 \,(2\,\alpha - 1)^2\, \psi_2^2 = (h_2\,(x))^2 \left({r_1}^2 + {s_1}^2\right)$$

$$a_2^2 = \frac{(h_2\,(x))^2\,({r_1}^2 + {s_1}^2)}{8\,(2\,\alpha - 1)^2\,\psi_2^2}\ . \tag{2.12}$$

If we add (2.8) to (2.10), we get

$$\left((18\,\alpha - 6)\,\psi_3 - 2\ (8\,\alpha - 4)\,{\psi_2}^2\right) {a_2}^2 = (r_2 + s_2)\, h_2\,(x) + h_3\,(x) \left({r_1}^2 + {s_1}^2\right). \tag{2.13}$$

By substituting (2.12) in (2.13), we obtain

$$a_2^2 = \frac{(r_2 + s_2)\,(h_2\,(x))^3}{\left[(18\,\alpha - 6)\,\psi_3 - (16\,\alpha - 8)\,{\psi_2}^2\right] (h_2\,(x))^2 - 8\,h_3\,(x)\,(2\,\alpha - 1)^2\,{\psi_2}^2} \tag{2.14}$$

and by taking $h_2(x) = bx$ and $h_3(x) = bpx^2 + qa$ in (2.14), it further yields

$$|a_2| \leq \frac{|bx|\,\sqrt{|bx|}}{\sqrt{\left|(9\,\alpha - 3)\,\psi_3 b^2 x^2 - \left[(8\,\alpha - 4)\,b^2 x^2 + 4\ (px^2 b + qa)\,(2\,\alpha - 1)^2\right] {\psi_2}^2\right|}}. \tag{2.15}$$

By subtracting (2.10) from (2.8) we get

$$-6\ (3\,\alpha-1)\,\psi_3\left(a_2{}^2-a_3\right)=(r_2-s_2)\,h_2\,(x)+\left(r_1{}^2-s_1{}^2\right)h_3\,(x)$$

In view of (2.11) , we obtain

$$a_3=\frac{(r_2-s_2)\,h_2\,(x)}{(18\,\alpha-6)\psi_3}+a_2{}^2. \tag{2.16}$$

Then in view of (2.12), (2.16) becomes

$$a_3 \;=\; \frac{(r_2-s_2)\,h_2\,(x)}{(18\,\alpha-6)\psi_3}+\frac{(h_2\,(x))^2\,(r_1{}^2+s_1{}^2)}{8\,(2\,\alpha-1)^2\,\psi_2^2}.$$

Applying (1.16), we deduce that

$$|a_3| \;\leq\; \frac{|bx|}{(9\,\alpha-3)\psi_3}+\frac{b^2x^2}{4\,(2\,\alpha-1)^2\,\psi_2^2}.$$

From (2.16), for $\mu\in\mathbb{R}$, we write

$$a_3-\mu a_2^2=\frac{h_2(x)\,(r_2-s_2)}{(18\,\alpha-6)\psi_3}+(1-\mu)\,a_2^2. \tag{2.17}$$

By substituting (2.14) in (2.17), we have

$$\begin{aligned} a_3-\mu a_2^2 =&\frac{h_2(x)\,(r_2-s_2)}{(18\,\alpha-6)\psi_3}\\ &+\left(\frac{(1-\mu)\,(r_2+s_2)\,(h_2\,(x))^3}{\left[(18\,\alpha-6)\,\psi_3-(16\,\alpha-8)\,\psi_2{}^2\right](h_2\,(x))^2-8\,h_3\,(x)\,(2\,\alpha-1)^2\,\psi_2{}^2}\right)\\ =&h_2(x)\left\{\left(\Lambda(\mu,\ x)+\frac{1}{(18\,\alpha-6)\psi_3}\right)r_2\right.\\ &\left.+\left(\Lambda(\mu,\ x)-\frac{1}{(18\,\alpha-6)\psi_3}\right)s_2\right\}, \end{aligned} \tag{2.18}$$

where

$$\Lambda(\mu,\ x)=\frac{(1-\mu)\,[h_2(x)]^2}{\left[(18\,\alpha-6)\,\psi_3-(16\,\alpha-8)\,\psi_2{}^2\right](h_2\,(x))^2-8\,h_3\,(x)\,(2\,\alpha-1)^2\,\psi_2{}^2}.$$

Hence, we conclude that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{|h_2(x)|}{(9\alpha-3)\psi_3} & ; 0 \leq |\Lambda(\mu,\ x)| \leq \dfrac{1}{(18\,\alpha-6)\psi_3} \\[2ex] 2\,|h_2(x)|\,|\Lambda(\mu,\ x)| & ; |\Lambda(\mu,\ x)| \geq \dfrac{1}{(18\,\alpha-6)\psi_3} \end{cases}$$

and in view of (1.16), it evidently completes the proof of Theorem 2.1.

Definition 2.2. *A function $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{L}_\sigma(\lambda,\ \nu,\ q,\ x)$ for $\nu > 0,\ \lambda > -1,\ 0 < q < 1,$ and $z,\ w \in \Delta$, if the following conditions are satisfied:*

$$\frac{1 + \dfrac{z\left(\mathcal{N}^{\lambda}_{\nu,\,q} f(z)\right)''}{\left(\mathcal{N}^{\lambda}_{\nu,\,q} f(z)\right)'}}{\dfrac{z\left(\mathcal{N}^{\lambda}_{\nu,\,q} f(z)\right)'}{\mathcal{N}^{\lambda}_{\nu,\,q} f(z)}} \prec \Pi(x,\ z) + 1 - a$$

and for g the analytic extension (continuation) of f^{-1} given by (1.2)

$$\frac{1 + \dfrac{w\left(\mathcal{N}^{\lambda}_{\nu,\,q} g(w)\right)''}{\left(\mathcal{N}^{\lambda}_{\nu,\,q} g(w)\right)'}}{\dfrac{w\left(\mathcal{N}^{\lambda}_{\nu,\,q} g(w)\right)'}{\mathcal{N}^{\lambda}_{\nu,\,q} g(w)}} \prec \Pi(x,\ w) + 1 - a,$$

where the real constant a is as in (1.16).

Remark 2.2. *Putting $q \to 1^-$, we obtain that*

$$\lim_{q\to 1^-} \mathcal{L}^*_\sigma(\lambda,\ \nu,\ q,\ x) =: \mathcal{M}_\sigma(\lambda,\ \nu,\ x),$$

where $f \in \sigma$,

$$\frac{1 + \dfrac{z\left(\mathcal{J}^{\lambda}_{\nu} f(z)\right)''}{\left(\mathcal{J}^{\lambda}_{\nu} f(z)\right)'}}{\dfrac{z\left(\mathcal{J}^{\lambda}_{\nu} f(z)\right)'}{\mathcal{J}^{\lambda}_{\nu} f(z)}} \prec \Pi(x,\ z) + 1 - a$$

and for g the analytic extension (continuation) of f^{-1} given by (1.2)

$$\frac{1+\dfrac{w\left(\mathcal{J}_{\nu}^{\lambda}g(w)\right)''}{\left(\mathcal{J}_{\nu}^{\lambda}g(w)\right)'}}{\dfrac{w\left(\mathcal{J}_{\nu}^{\lambda}g(w)\right)'}{\mathcal{J}_{\nu}^{\lambda}g(w)}} \prec \Pi(x,\ w)+1-a,$$

where the real constant a is as in (1.16).

For functions in the class $\mathcal{L}_{\sigma}(\lambda,\ \nu,\ q,\ x)$, the following coefficient estimates and Fekete-Szegö inequality are obtained.

Theorem 2.2. *Let $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{L}_{\sigma}(\lambda,\ \nu,\ q,\ x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|4\,b^2x^2\psi_3-(4\,b^2x^2+px^2b+qa)\,{\psi_2}^2\right|}}, \qquad \text{and} \qquad |a_3| \leq \frac{|bx|}{4\psi_3}+\frac{b^2x^2}{\psi_2^2}$$

and for $\mu\in\mathbb{R}$

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{4\psi_3} \\ \qquad \text{if} \qquad |\mu-1| \leq \dfrac{\left|4\,b^2x^2\psi_3-\left(4\,b^2x^2+px^2b+qa\right)\psi_2{}^2\right|}{4b^2x^2\psi_3} \\ \dfrac{|bx|^3\,|\mu-1|}{\left|4\,b^2x^2\psi_3-(4\,b^2x^2+px^2b+qa)\,\psi_2{}^2\right|} \\ \qquad \text{if} \qquad |\mu-1| \geq \dfrac{\left|4\,b^2x^2\psi_3-\left(4\,b^2x^2+px^2b+qa\right)\psi_2{}^2\right|}{4b^2x^2\psi_3}. \end{cases}$$

Proof. Let $f\in\mathcal{L}_{\sigma}(\lambda,\ \nu,\ q,\ x)$ be given by the Taylor-Maclaurin expansion (1.1). Then, there are analytic functions $r(z)$ and $s(w)$ such that

$$r(0)=0; \quad s(0)=0, \quad |r_n|<1 \quad \text{and} \quad |s_n|<1 \quad (\forall\ z,\ w\in\Delta),$$

and we can write

$$\frac{1+\dfrac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}}{\dfrac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}{\mathcal{N}_{\nu,\,q}^{\lambda}f(z)}} = \Pi(x,\ r(z))+1-a \tag{2.19}$$

and

$$\frac{1+\dfrac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}}{\dfrac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}{\mathcal{N}_{\nu,\,q}^{\lambda}g(w)}}=\Pi(x,\ s(w))+1-a. \tag{2.20}$$

Equivalently,

$$\frac{1+\dfrac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}}{\dfrac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}{\mathcal{N}_{\nu,\,q}^{\lambda}f(z)}}=1+h_1(x)-a+h_2(x)r(z)+h_3(x)[r(z)]^2+\cdots \tag{2.21}$$

and

$$\frac{1+\dfrac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}}{\dfrac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}{\mathcal{N}_{\nu,\,q}^{\lambda}g(w)}}=1+h_1(x)-a+h_2(x)s(w)+h_3(x)[s(w)]^2+\cdots. \tag{2.22}$$

From (2.21) and (2.22) and in view of (1.17), we obtain

$$\frac{1+\dfrac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}}{\dfrac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}{\mathcal{N}_{\nu,\,q}^{\lambda}f(z)}}=1+h_2(x)r_1z+[h_2(x)r_2+h_3(x)r_1^2]z^2+\cdots \tag{2.23}$$

and

$$\frac{1+\dfrac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}}{\dfrac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}{\mathcal{N}_{\nu,\,q}^{\lambda}g(w)}}=1+h_2(x)s_1w+[h_2(x)s_2+h_3(x)s_1^2]w^2+\cdots. \tag{2.24}$$

If

$$r(z)=\sum_{n=1}^{\infty}r_nz^n \qquad \text{and} \qquad s(w)=\sum_{n=1}^{\infty}s_nw^n,$$

then it is well known that

$$|r_n| \leq 1 \qquad \text{and} \qquad |s_n| \leq 1 \qquad (n \in \mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (2.23) and (2.24), we have

$$\psi_2 a_2 = h_2(x) r_1 \tag{2.25}$$

$$4\left(a_3\psi_3 - a_2^2\psi_2^2\right) = h_2(x) r_2 + h_3(x) r_1^2 \tag{2.26}$$

$$-\psi_2 a_2 = h_2(x) s_1 \tag{2.27}$$

and

$$\left(8\,{a_2}^2 - 4\,a_3\right)\psi_3 - 4\,{a_2}^2{\psi_2}^2 = h_2(x) s_2 + h_3(x) s_1^2. \tag{2.28}$$

The results of this theorem now follow from (2.25)-(2.28) by applying the procedure as in Theorem 2.1 with respect to (2.7)-(2.10).

Definition 2.3. *A function $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{P}_\sigma\ (\gamma,\ \lambda,\ \nu,\ q,\ x)$ for $0 \leq \gamma \leq 1$, $\nu > 0$, $\lambda > -1$, $0 < q < 1$, and $z,\ w \in \Delta$, if the following conditions are satisfied:*

$$\frac{z\left(\mathcal{N}_{\nu,\, q}^{\lambda} f(z)\right)' + \gamma z^2\left(\mathcal{N}_{\nu,\, q}^{\lambda} f(z)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\, q}^{\lambda} f(z)\right) + \gamma z\left(\mathcal{N}_{\nu,\, q}^{\lambda} f(z)\right)'} \prec \Pi(x,\ z) + 1 - a$$

and for g the analytic extension (continuation) of f^{-1} given by (1.2)

$$\frac{w\left(\mathcal{N}_{\nu,\, q}^{\lambda} g(w)\right)' + \gamma w^2\left(\mathcal{N}_{\nu,\, q}^{\lambda} g(w)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\, q}^{\lambda} g(w)\right) + \gamma w\left(\mathcal{N}_{\nu,\, q}^{\lambda} g(w)\right)'} \prec \Pi(x,\ w) + 1 - a,$$

where the real constant a is as in (1.16).

Remark 2.3. *Putting $q \to 1^-$, we obtain that*

$$\lim_{q \to 1^-} \mathcal{P}_\sigma(\gamma,\ \lambda,\ \nu,\ q,\ x) =: \mathcal{P}_\sigma(\gamma,\ \lambda,\ \nu,\ x),$$

where $f \in \sigma$,

$$\frac{z\left(\mathcal{J}_{\nu}^{\lambda} f(z)\right)' + \gamma z^2\left(\mathcal{J}_{\nu}^{\lambda} f(z)\right)''}{(1-\gamma)\left(\mathcal{J}_{\nu}^{\lambda} f(z)\right) + \gamma z\left(\mathcal{J}_{\nu}^{\lambda} f(z)\right)'} \prec \Pi(x,\ z) + 1 - a$$

and for g the analytic extension (continuation) of f^{-1} given by (1.2)

$$\frac{w\left(\mathcal{J}_{\nu}^{\lambda}g(w)\right)' + \gamma w^2\left(\mathcal{J}_{\nu}^{\lambda}g(w)\right)''}{(1-\gamma)\left(\mathcal{J}_{\nu}^{\lambda}g(w)\right) + \gamma w\left(\mathcal{J}_{\nu}^{\lambda}g(w)\right)'} \prec \Pi(x,\ w) + 1 - a,$$

For functions in the class $\mathcal{P}_{\sigma}(\gamma,\ \lambda,\ \nu,\ q,\ x)$, the following coefficient estimates and Fekete-Szegö inequality are obtained.

Theorem 2.3. *Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{P}_{\sigma}(\gamma,\ \lambda,\ \nu,\ q,\ x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|2\,b^2x^2\,(2\gamma+1)\,\psi_3 - (1+\gamma)^2\,(b^2x^2+px^2b+qa)\,{\psi_2}^2\right|}},$$

$$|a_3| \leq \frac{|bx|}{2\psi_3\,(2\,\gamma+1)} + \frac{b^2x^2}{(1+\gamma)^2\,{\psi_2}^2}$$

and for $\mu \in \mathbb{R}$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|bx|}{2\psi_3\,(2\,\gamma+1)} \\ \qquad if \quad |\mu-1| \leq \dfrac{\left|2\,b^2x^2\,(2\gamma+1)\,\psi_3 - (1+\gamma)^2\left(b^2x^2+px^2b+qa\right){\psi_2}^2\right|}{2b^2x^2\psi_3\,(2\,\gamma+1)} \\ \dfrac{|bx|^3\,|\mu-1|}{\left|2\,b^2x^2\,(2\gamma+1)\,\psi_3 - (1+\gamma)^2\,(b^2x^2+px^2b+qa)\,{\psi_2}^2\right|} \\ \qquad if \quad |\mu-1| \geq \dfrac{\left|2\,b^2x^2\,(2\gamma+1)\,\psi_3 - (1+\gamma)^2\left(b^2x^2+px^2b+qa\right){\psi_2}^2\right|}{2b^2x^2\psi_3\,(2\,\gamma+1)}. \end{cases}$$

Proof. Let $f \in \mathcal{P}_{\sigma}(\gamma,\ \lambda,\ \nu,\ q,\ x)$ be given by the Taylor-Maclaurin expansion (1.1). Then, there are analytic functions $r(z)$ and $s(w)$ such that

$$r(0)=0; \quad s(0)=0, \quad |r_n|<1 \quad \text{and} \quad |s_n|<1 \quad (\forall\ z,\ w \in \Delta),$$

and we can write

$$\frac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)' + \gamma z^2\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right) + \gamma z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'} = \Pi(x,\ r(z)) + 1 - a \tag{2.29}$$

and

$$\frac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'+\gamma w^{2}\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)+\gamma w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}=\Pi(x,\ s(w))+1-a. \tag{2.30}$$

Equivalently,

$$\frac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'+\gamma z^{2}\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)+\gamma z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}$$
$$=1+h_1(x)-a+h_2(x)r(z)+h_3(x)[r(z)]^2+\cdots \tag{2.31}$$

and

$$\frac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'+\gamma w^{2}\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)+\gamma w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}$$
$$=1+h_1(x)-a+h_2(x)s(w)+h_3(x)[s(w)]^2+\cdots. \tag{2.32}$$

From (2.31) and (2.32) and in view of (1.17), we obtain

$$\frac{z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'+\gamma z^{2}\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)+\gamma z\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(z)\right)'}$$
$$=1+h_2(x)r_1z+[h_2(x)r_2+h_3(x)r_1^2]z^2+\cdots \tag{2.33}$$

and

$$\frac{w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'+\gamma w^{2}\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{(1-\gamma)\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)+\gamma w\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'}$$
$$=1+h_2(x)s_1w+[h_2(x)s_2+h_3(x)s_1^2]w^2+\cdots. \tag{2.34}$$

If

$$r(z)=\sum_{n=1}^{\infty}r_nz^n \qquad \text{and} \qquad s(w)=\sum_{n=1}^{\infty}s_nw^n,$$

then it is well known that

$$|r_n|\leq 1 \qquad \text{and} \qquad |s_n|\leq 1 \qquad (n\in\mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (2.33) and (2.34), we have

$$(1+\gamma)\psi_2a_2=h_2(x)r_1 \tag{2.35}$$

$$2(1+2\gamma)\psi_3 a_3 - (1+\gamma)^2\psi_2^2 a_2^2 = h_2(x)r_2 + h_3(x)r_1^2 \tag{2.36}$$

$$-(1+\gamma)\psi_2 a_2 = h_2(x)s_1 \tag{2.37}$$

and

$$\left((8\,\gamma+4)\,\psi_3 - {\psi_2}^2\,(1+\gamma)^2\right) {a_2}^2 - 2\,a_3\,(2\gamma+1)\,\psi_3 = h_3\,(x)\,{s_1}^2 + h_2\,(x)\,s_2 \tag{2.38}$$

The results of this theorem now follow from (2.35)-(2.38) by applying the procedure as in Theorem 2.1 with respect to (2.7)-(2.10).

3. Corollaries and Consequences

Taking $\gamma = 0$ in Theorem (2.3), we have following corollary.

Corollary 3.1. *Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_\sigma(\lambda,\ \nu,\ q,\ x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|2\,b^2x^2\psi_3 - (b^2x^2 + px^2b + qa)\,{\psi_2}^2\right|}}, \quad |a_3| \le \frac{|bx|}{2\psi_3} + \frac{b^2x^2}{{\psi_2}^2}$$

and for $\mu \in \mathbb{R}$

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{|bx|}{2\psi_3} \\ \qquad \text{if} \qquad |\mu - 1| \le \dfrac{\left|2\,b^2x^2\psi_3 - \left(b^2x^2 + px^2b + qa\right)\psi_2{}^2\right|}{2b^2x^2\psi_3} \\ \dfrac{|bx|^3\,|\mu-1|}{\left|2\,b^2x^2\psi_3 - (b^2x^2 + px^2b + qa)\,{\psi_2}^2\right|} \\ \qquad \text{if} \qquad |\mu - 1| \ge \dfrac{\left|2\,b^2x^2\psi_3 - \left(b^2x^2 + px^2b + qa\right)\psi_2{}^2\right|}{2b^2x^2\psi_3}. \end{cases}$$

Taking $\alpha = 1$ in Theorem 2.1 or $\gamma = 1$ in Theorem 2.3, we have following corollary.

Corollary 3.2. *Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{K}_\sigma(\lambda,\ \nu,\ q,\ x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|6\,b^2x^2\psi_3 - 4\,(b^2x^2 + px^2b + qa)\,{\psi_2}^2\right|}}, \qquad |a_3| \le \frac{|bx|}{6\psi_3} + \frac{b^2x^2}{4{\psi_2}^2}$$

and for $\mu \in \mathbb{R}$

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{6\psi_3} \\ \qquad \textit{if} \qquad |\mu - 1| \leq \dfrac{\left|3\, b^2x^2\psi_3 - 2\left(b^2x^2 + px^2b + qa\right)\psi_2{}^2\right|}{3b^2x^2\psi_3} \\ \dfrac{|bx|^3\,|\mu - 1|}{\left|6\, b^2x^2\psi_3 - 4\left(b^2x^2 + px^2b + qa\right)\psi_2{}^2\right|} \\ \qquad \textit{if} \qquad |\mu - 1| \geq \dfrac{\left|3\, b^2x^2\psi_3 - 2\left(b^2x^2 + px^2b + qa\right)\psi_2{}^2\right|}{3b^2x^2\psi_3}. \end{cases}$$

4. Conclusion

One could find initial coefficient estimates for the classes defined in Remarks 2.1, 2.2 and 2.3. We leave those to interested readers.

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