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ON NORMALISATION OF HALF-INTEGRAL WEIGHT MODULAR FORMS

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Abstract: In this paper, we derive the algebraic nature of the Fourier coefficients of the Hecke eigenform f of weight $k + 1/2$ for $\Gamma_0(4N)$, where $k \geq 2$ and N is an odd and square-free integer.

Keywords and Phrases: Modular forms, Hecke eigenforms, Operators.

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1. Introduction

Let $k \geq 2$ be an integer. Let N be an odd and square - free integer. Let f be a cusp form in Kohnen plus space of weight $k + 1/2$ for $\Gamma_0(4N)$ as defined in [3], [4] so that $a_f(n) = 0$ whenever, $(-1)^k n \equiv 2,\ 3\ (mod\ 4)$. Let F be a cusp form and a normalized newform of weight $2k$, level N. Then it is known that the Fourier coefficients $a_f(n)$ can be taken as real and algebraic numbers whenever f is an Hecke eigenform which corresponds to F via Shimura - Kohnen lifts. In this note, we present a proof of this fact and also derive the same fact for a Hecke eigenform f which is in the old classes under the assumption that f is an eigenform under all the w - operators w_p (see the definition in [4]) for various prime p dividing N and the Hecke operators T_{n^2}, $(n, N) = 1$.

2. Notations

Throughout this paper, the letters k, m, M, N stand for natural numbers and $2|k$. ($k > 1, m \equiv 1 (mod\, 4)$ is a square-free odd integer). Let N be a square- free integer, $(m, N) = 1$. Let τ be an element of $\mathbb{H}$, the complex upper half-plane. Let $\mathbb{C}$ and $\mathbb{Z}$ respectively denote the complex plane and the ring of integers.
For a complex number z, we write $\sqrt{z}$ for the square root with argument in $(-\pi, \pi]$ and we set $z^{a/2} = (\sqrt{z})^a$ for any $a \in \mathbb{Z}$.
For integers a, b, let $\left(\frac{a}{b}\right)$ denote the generalized quadratic residue symbol. Let $d(c)$ denote $d(mod\ c)$, $c, d \in \mathbb{Z}$.

The space of modular forms of weight $2k$ and level N is denoted as $M_{2k}(N)$ and its sub space of all the cusp forms by $S_{2k}(N)$. For cusp forms f, g in the space $S_{2k}(N)$, we denote their Petersson scalar product by $< f, g >$.
We write the Fourier expansion of a modular form f as

$$f(\tau) = \sum_{n \geq 0} a_f(n) e^{2\pi i n \tau}.$$

For the details of modular forms of weight $2k$ level N, we refer to [8].

3. Definitions

Definition 3.1. *Modular forms of half-integral weight [2]*
Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \gamma z = \frac{az+b}{cz+d}$. *In the transformation rule* $f(\gamma z) = (cz+d)^k f(z)$ *the term* $(cz + d)^k$ *is called the automorphy factor. It depends on* γ *and on* z. *It is denoted as* $J(\gamma, z)$ *for a non-zero function* f *and has the property that* $f(\gamma z) = J(\gamma, z) f(z)$ *for* $z \in \mathbb{H}$ *and* γ *in some matrix group.*
Let G *denote the four-sheeted covering of* $GL_2^+(\mathbb{Q})$ *defined as the set of all ordered pairs* $(\alpha, \phi(\tau))$, *where* $\alpha \left(= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \in GL_2^+(\mathbb{Q})$ *and* $\phi(z)$ *is a holomorphic function on* $\mathbb{H}$ *such that* $\phi^2(z) = t\frac{cz+d}{\sqrt{det\alpha}}$ *for some* t *with* $t = 1, -1, i, -i$. *Then* G *is a group with the following multiplication rule.*

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)).$$

For a complex valued function f *defined on the upper half-plane* $\mathbb{H}$ *and an element* $(\alpha, \phi(z)) \in G$, *define the stroke operator by*

$$f\ |_{k+1/2}\ (\alpha, \phi(z))(z) = \phi(z)^{-2k-1} f(\alpha z).$$

If $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, *we always let* $j(\alpha, z) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(cz + d)^{1/2}$ *so that* $(\alpha, j(\alpha, z)) \in \mathbb{G}$.

Definition 3.2. *Hecke operators for half-integral weight*
For n a positive integer and $f \in M_k(\Gamma)$ (Γ is a congruence subgroup of $\Gamma_0(4)$) we can define $f|T_n$ as follows. Let Δ^n be the set of all 2×2 matrices with integer entries and determinant n. For any double coset $\Gamma\alpha\Gamma \subset \Delta^n$, where $\alpha \in \Delta^n$, we define $f|[\Gamma\alpha\Gamma]_k = \sum f|[\alpha\gamma_j]_k$, where the sum is over all right cosets $\Gamma\alpha\gamma_j \subset \Gamma\alpha\Gamma$; equivalently, γ_j runs through a complete set of right coset representatives of Γ modulo $\alpha^{-1}\Gamma\alpha \cap \Gamma$. Then

$$f|T_n \underset{def}{=} n^{(k/2)-1} \sum f|[\Gamma\alpha\Gamma]_k,$$

where the sum is over all double cosets of Γ in Δ^n.
A modular form $f(z) \in M_k(\Gamma)$ is called a Hecke eigenform if for every positive integer m there exists $\lambda_m \in \mathbb{C}$ with $T_m(f) = \lambda_m(f)$.

Definition 3.3. *Let $S_{k+1/2}(4N)$ denote the space of cusp forms of weight $k+1/2$ for $\Gamma_0(4N)$. It contains all the holomorphic functions on $\mathbb{H}$ with complex values and the functions are holomorphic at all the rational points and each of them satisfies the transformation law: $f|(A, j(A,\tau)) = f$ for all $A \in \Gamma_0(4N)$.*
Let $S^+_{k+1/2}(4N)$ denote the Kohnen plus space in $S_{k+1/2}(4N)$ and let $S^{+,new}_{k+1/2}(4N)$ the space of newforms in the plus space. For this we refer to [5].

Let T_n denote the Hecke operator on the space $S_{2k}(N)$ and $T_{n^2}, (n,N) = 1$ denote the Hecke operator on the space $S^+_{k+1/2}(4N)$. For a prime p, we denote the Hecke operators by T_{p^2} when $(p,N) = 1$ and by U_{p^2} when $p|N$ on $S^{+,new}_{k+1/2}(4N)$. Let $f \in S^+_{k+1/2}(4N)$ be a Hecke eigenform equivalent to a normalised newform $F \in S^{new}_{2k}(N)$ with

$$f|T_{p^2} = a_F(p)f, \quad (p \nmid N)$$

For $f \in S_k(N)$, we define U_p as

$$f|U_p = p^{k/2-3/4} \sum_{\nu (mod\ p)} f\left| \left(\begin{pmatrix} 1 & \nu \\ 0 & p \end{pmatrix}, p^{k/2+1/4} \right) \right.$$

and if $p|N$, there exists $\lambda_p \in \mathbb{C}$ with $\lambda_{p^2} = 1$ and we have,

$$f|U_{p^2} = -p^{k-1}\lambda_p f$$

In the following Lemma 4.1, we find the value of the constant λ_p explicitly.

Definition 3.4. *Waldspurger formula (see [5]) If f, F are the Hecke eigenforms as above, $(D,N) = 1$ with $(-1)^k D > 0$ is a fundamental discriminant, then we have*

$$\frac{a_f(|D|)^2}{< f, f >} = \frac{2^{\nu_N}(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(F,D,k)}{< F, F >}$$

where ν_N denotes the number of distinct prime divisors of N.

Definition 3.5. *For each prime divisor p of N we put*

$$w_p = p^{-k/2+1/4} U_p W_p$$

where W_p is the $W-$ operator on $S_{k+1/2}(4M)$; $M|N$ we define

$$W_p = \left(\begin{pmatrix} pa & b \\ 4Mc & p \end{pmatrix}, \; p^{-1/4}(4Mc\tau + p)^{1/2} \right)$$

where a, b, c are integers such that $b \equiv 1 \pmod p$ and $p^2a - 4Mpc = p$.
The definition given here is same as defined by Kohnen in [4], but slightly differs by a constant α with $\alpha^2 = 1$.

4. Properties of w_p operators (refer [6])

- $f|T_{p^2} = f|U_{p^2} + p^{k-1} f|w_p, \quad (p \nmid N)$
- For $p|N$, the $W-$ operator w_p acts as the identity operator on $S^+_{k+1/2}(4N)$.
- The space $S^{+,new}_{k+1/2}(4N)$ has a basis of eigenforms with respect to the Hecke operators $T_{p^2}, \; p \nmid N$, or $U_{p^2}, \; p|N$. Further, these are eigenforms with respect to the $W-$ operators $w_p, \; p|N$.

Lemma 4.1. *If f is a newform in $S^+_{k+1/2}(4N)$, then for a prime p, $f|w_p = -\left(\frac{D}{p}\right) p^{k-1} f$, where $(-1)^k D > 0$ is a fundamental discriminant, $(D, N) = 1$ and $a_f(|D|) \neq 0$.*
Proof. For the proof we use equation (9) of [4].

$$\begin{aligned} f|w_p &= f\Big| \left(p^{-\frac{k}{2}+\frac{1}{4}} U_p W_p\right) \\ &= p^{-1/2}\left(\frac{-4}{p}\right)^{k+1/2} \sum_{\alpha(p^*)} f\Big| \left(\begin{pmatrix} p & \alpha \\ 0 & p \end{pmatrix} \left(\frac{-\alpha}{p}\right)\right) + p^{-1/2} f \Big| \left(\begin{pmatrix} 1 & v_0 \\ 0 & p \end{pmatrix}, p^{1/4}\right) W_p. \end{aligned}$$

Thus,

$$f|w_p = \sum_{n \geq 1} \left(\frac{(-1)^k n}{p}\right) a_f(n) q^n + p^{-1/2} f \Big| \left(\begin{pmatrix} 1 & v_0 \\ 0 & p \end{pmatrix}, p^{1/4}\right) W_p$$

where v_0 is an integer with $a + 4\frac{M}{p}v_0 c \equiv 0 \ (mod\ p)$
Now,

$$\left(\begin{pmatrix} 1 & v_0 \\ 0 & p \end{pmatrix}, p^{1/4}\right) W_p = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \left(\frac{-4}{p}\right)^{1/2}\right) C^* W_p \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}\right)$$

where, $C \in \Gamma_0(4M)$. [refer pg. 41, [4]]
Hence,

$$f|w_p = \sum_{n\geq 1} \left(\frac{(-1)^k n}{p}\right) a_f(n) e^{2\pi i n\tau} + \lambda f|W_p \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}\right).$$

Let $f|w_p = \lambda_p f$.
Substituting this in the above we get,

$$\lambda_p f = \sum_{n\geq 1} \left(\frac{(-1)^k n}{p}\right) a_f(n) e^{2\pi i n\tau} + \lambda f|W_p \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}\right).$$

Comparing the n^{th} Fourier coefficients on both sides where $p \nmid N$, we get

$$\lambda_p a_f(n) = \left(\frac{(-1)^k n}{p}\right) a_f(n), \quad p \nmid N.$$

Since $p \nmid n$ and $f|W_p$ is invariant under $\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right)$ the second term $\lambda f|W_p \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}\right)$ has zero as its n^{th} Fourier coefficient whenever $(n,p) = 1$. Hence, if we select n such that $a_f(n) \neq 0$ with $(n,p) = 1$ we get,

$$\lambda_p = \left(\frac{(-1)^k n}{p}\right).$$

If $(D,p) = 1$, set $(-1)^k D = n > 0$, then the eigenvalue is $\left(\frac{D}{p}\right)$.

If f is a newform as above in $S^+_{k+1/2}(4N)$, then, we have the following theorem.

Theorem 4.2. *We normalise f by letting $a_f(n)$ to be real and algebraic.*
Proof. Let us consider for a prime p the k_p operator studied by Serre and Stark [10] which maps $\sum_{n\geq 1} a_f(n) e^{2\pi i n\tau}$ into $\sum_{n\geq 1} \overline{a_f(n)} e^{2\pi i n\tau}$. In that, they proved that k_p maps

$S_{k+1/2}(4N)$ to $S_{k+1/2}(4N)$. But, using the definition of plus space they concluded that, it also maps

$$S^{+}_{k+1/2}(4N) \mapsto S^{+}_{k+1/2}(4N)$$

Moreover, it commutes with T_{p^2} and U_{p^2}. Hence, $f|k_p$ and f have same eigenvalues under all the Hecke operators. The multiplicity one result (proved in [4]) shows that $f|k_p = \lambda f$. Since, k_p^2 equals the identity on $S^{+}_{k+1/2}(4N)$, $\lambda = \pm 1$. Therefore, we take either f or if and we assume that Fourier coefficients are all real.

Thus, we let $f \in S^{+}_{k+1/2}(4N)$ to be a Hecke eigenform whose Fourier coefficients are all real. In order to prove that they are all algebraic we use the following two results.

If D is a fundamental discriminant with $(-1)^k D > 0$ and $n \geq 1$ we have

$$a_f(|D|n^2) = a_f(|D|) \sum_{d|n} \mu(d) d^{k-1} \left(\frac{D}{d}\right) a_F(n/d).$$

If ν_N denotes the number of different prime divisors of N, then we have

$$\frac{a_f(|D|)^2}{< f, f >} = \frac{2^{\nu_N}(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(F, D, k)}{< F, F >}$$

Due to these two results it is enough to prove the algebraic nature for $a_f(|D|)$ whenever D is a fundamental discriminant with $(-1)^k D > 0$. The above formula due to Waldspurger is the same for both f and $-if$. Using the result of [5]

$$|D|^{-1/2} \pi^{-k} \frac{L(F, D, k)}{\omega_{(-1)^{k-1}}}$$

is algebraic and real and using $< F, F >= \omega_{(-1)^{k-1}} \omega_{(-1)^k}$, which is a product of two positive real constants and selecting f such that $< f, f >= \omega_{(-1)^k}$, we get $a_f(|D|)^2$ is real, positive and algebraic. This proves that $a_f(|D|)$ is real and algebraic.

Thus, we have the following:

Theorem 4.3. *If f is in the old class and f is the Hecke eigenform and eigenform under all W operators then, $a_f(n)$ are real and algebraic.*

Proof. Let $g \in S^{new}_{k+1/2}(4M)$, $(M|N)$ be a non-zero Hecke eigenform.

Let f be an eigenform in the space $S^{+,old}_{k+1/2}(4N)$ and generated by a newform $g \in S^{+,new}_{k+1/2}(4M)$, $M|N$, under all $W-$ operators w_p, $(p \mid N)$, where M is a proper divisor of N. Thus, using g is an Hecke eigenform under all Hecke operators we conclude that $a_g(n)$ are algebraic and real. Moreover, its eigenvalue under the W operator for a prime $p|N$ is $\left(\frac{D}{p}\right)$. We write

$$f = g\Bigg|\left(\sum_{d|N/M}\left(\frac{D}{d}\right)w_d\right),$$

We see that f is an eigenform under all $w-$ operators w_p, $p|N$ and f is an eigenform under all Hecke operators T_{p^2}, $(p\nmid M)$. Also, by using

$$p^{k-1}g|w_p = g|T_{p^2} - g|U_{p^2}$$

which was derived in [6] such that $p\nmid M$ and $p|\frac{N}{M}$ and from the fact that the Fourier coefficients of g are real and algebraic, the result is immediate by the Lemma.

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