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VERTEX-EDGE NEIGHBORHOOD PRIME LABELING OF SOME TREES

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, $N_V(u) = \{w \in V(G)/uw \in E(G)\}$ and $N_E(u) = \{e \in E(G)/e = uv$, for some $v \in V(G)\}$. A bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ is said to be a vertex-edge neighborhood prime labeling, if for $u \in V(G)$ with $deg(u) = 1$, gcd $\{f(w), f(uw)/w \in N_V(u)\} = 1$; for $u \in V(G)$ with $deg(u) > 1$, $gcd\{f(w)/w \in N_V(u)\} = 1$ and $gcd\{f(e)/e \in N_E(u)\} = 1$. A graph which admits vertex-edge neighborhood prime labeling is called a vertex-edge neighborhood prime graph. In this paper we investigate vertex-edge neighborhood prime labeling for some trees namely coconut tree, double coconut tree, spider graph, olive tree, comb graph and $F(n, 2)$-firecrackers.

Keywords and Phrases: Neighborhood-prime labeling, total neighborhood prime labeling, vertex-edge neighborhood prime labeling.

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1. Introduction and Definitions

In this paper we consider simple, finite, connected, undirected graph G with $V(G)$ as vertex set and $E(G)$ as edge set. For various notations and terminology of graph theory, we follow Gross and Yellen [3] and for some results of number theory, we follow Burton [1].

For a graph G with n vertices, a bijective function $f : V(G) \rightarrow \{1, 2, 3, \ldots, n\}$ is said to be a **neighborhood-prime labeling** if for every vertex u in $V(G)$ with $deg(u) > 1$, gcd $\{f(p)/p \in N(u)\} = 1$, where $N(u) = \{w \in V(G)/uw \in E(G)\}$. A graph which admits a neighborhood-prime labeling is called a neighborhood-prime graph.

The notion of neighborhood-prime labeling was introduced by Patel and Shrimali [7]. In [8] they proved that union of some graphs are neighborhood-prime graphs. They also proved that product of some graphs are neighborhood-prime [9]. For further list of results regarding neighborhood-prime graph reader may refer [2].

For a graph G, a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ is said to be **total neighborhood prime labeling**, if for each vertex in G having degree greater than 1, the gcd of the labels of its neighborhood vertices is 1 and the gcd of the labels of its incident edges is 1. A graph which admits total neighborhood prime labeling is called a total neighborhood prime graph.

Motivated by neighborhood-prime labeling, Rajesh and Methew [4] introduced the total neighborhood prime labeling. In the total neighborhood prime labeling conditions are applied on neighborhood vertices as well as incident edges of each vertex of degree greater than 1. They proved that path, cycle C_{4k} and comb graph admit total neighborhood prime labeling. Shrimali and Pandya proved comb, disjoint union of paths, disjoint union of sunlet graphs, disjoint union of wheel graphs, graph obtained by one copy of path P_n and n copies of $K_{1,m}$ and joining i^{th} vertex of P_n with an edge to fix vertex in the i^{th} copy of $K_{1,m}$, corona product of cycle with m copies of K_1 and subdivision of bistar are total neighborhood prime graphs [6].

In the total neighborhood prime labeling vertex of degree 1 is not considered. Shrimali and pandya [5] extended the condition on vertex of degree 1 and they defined vertex-edge neighborhood prime labeling which is nothing but an extension of total neighborhood prime labeling.

Let G be a graph. For $u \in V(G)$, $N_E(u) = \{e \in E(G)/e = uv,\ \text{for some } v \in V(G)\}$ and $N_V(u) = \{w \in V(G)/uw \in E(G)\}$. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ is said to be a **vertex-edge neighborhood prime labeling**, if for $u \in V(G)$ with $deg(u) = 1$, gcd $\{f(w), f(uw)/w \in N_V(u)\} = 1$ and for $u \in V(G)$ with $deg(u) > 1$, $gcd\ \{f(w)/w \in N_V(u)\} = 1$ and $gcd\{f(e)/e \in N_E(u)\} = 1$. A graph which admits a vertex-edge neighborhood prime labeling is called a vertex-edge neighborhood prime graph.

In [5], Shrimali and Pandya proved that path, helm, sunlet, bistar, central edge subdivision of bistar, subdivision of edges of bistar admit a vertex-edge neighborhood prime labeling.

Shrimali and Rathod proved that generalized web graph, generalized web graph without central vertex, splitting graph of path, splitting graph of star, graph obtained by switching of a vertex in path, graph obtained by switching of a vertex in cycle and middle graph of path are vertex-edge neighborhood prime graphs [10].

A **coconut tree** $CT(m,n)$ is graph obtained by identifying the central vertex of star graph $K_{1,m}$ with a pendant vertex of path P_n.

A **double coconut tree** $D(n,r,m)$ is graph obtained from path P_r by identifying two pendant vertices of path P_r with apex vertex of star graphs $K_{1,n}$ and $K_{1,m}$ respectively.

An **olive tree** T_k is a rooted tree consisting of k branches such that i^{th} branch is a path of length i.

A **spider tree** is a tree that has at most one vertex (called the center) of degree greater than two.

Let G_1, G_1,...,G_n be the disjoint copies of star graph $K_{1,m}$. Let v_i be the pendant vertex of G_i, $1 \leq i \leq n$. The tree which contains all the stars and a path joining $v_1, v_2, \ldots, v_n$ is called a $F(n,m)$-**firecrackers graph**.

In this paper, we prove that coconut tree, double coconut tree, spider graph, olive tree, comb graph and $F(n,2)$-fire cracker graph are vertex-edge neighborhood prime graphs.

2. Main Results

Theorem 2.1. *The coconut tree $CT(m,n)$ is vertex-edge neighborhood prime graph.*

Proof. Let G be coconut tree $CT(m,n)$. In G, we denote consecutive vertices of path P_n by $u_1, u_2, \ldots, u_n$. Let $u_1^{'}, u_2^{'}, \ldots, u_m^{'}$ be the pendant vertices and u_0 be the apex vertex of star graph $K_{1,m}$. We identify u_1 with u_0 and denote the identifying vertex with u_1. Thus $V(G) = \{u_1, u_2, \ldots u_n, u_1', u_2', \ldots u_m'\}$ Let $e_i = u_iu_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $e_i' = u_1u_i'$ for $i = 1, 2, \ldots, m$ be the edges of G. So, $|V(G)| = m+n$ and $|V(G)| = m+n-1$.

Now we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ as follows.

$f(u_i) = \dfrac{i+1}{2}$ for i is odd

$f(u_i^{'}) = 2n - 1 + i$ for $1 \leq i \leq m$

$f(e_i^{'}) = 2n + m - 1 + i$ for $1 \leq i \leq m$

Consider the following two cases.

Case 1. n is even

$f(u_i) = \dfrac{3n}{2} - 1 + \dfrac{i}{2}$ for i is even

$f(e_i) = \dfrac{3n}{2} - i$ for $1 \leq i \leq n-1$

Case 2. n is odd

$f(u_i) = \dfrac{n+1+i}{2}$ for i is even
$f(e_i) = 2n - i$ for $1 \leq i \leq n-1$
We claim that G is a vertex-edge neighborhood prime graph. Let w be an arbitrary vertex with degree 1. One can observe that gcd $\{f(v), f(vw)\} = 1$. For any vertex w with degree greater than 1, $\{f(v)/v \in N_V(w)\}$ and $\{f(e)/e \in N_E(w)\}$ contain at least two consecutive numbers or consecutive odd numbers or 1. So, gcd $\{f(v)/v \in N_V(w)\} =$ gcd $\{f(e)/e \in N_E(w)\} = 1$.
Hence, G is a vertex-edge neighborhood prime graph.

Illustration 1. Vertex-edge neighborhood prime labeling of $CT(5,6)$ is shown in Figure 1.

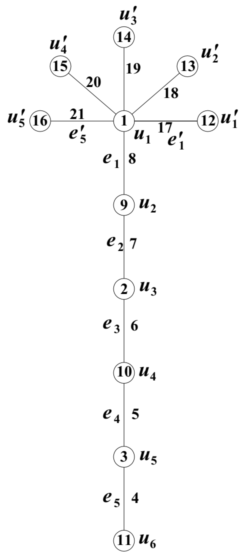

Figure 1: Vertex-edge neighborhood prime labeling of $CT(5,6)$.

Theorem 2.2. *Double Coconut tree $D(m,n,m)$ is vertex-edge neighborhood prime graph.*
Proof. Let $u_1, u_2, u_3, \ldots, u_n$ be the consecutive vertices of the path P_n. Let $v_0, v_1, v_2, v_3, ..., v_m$ and $w_0, w_1, w_2, w_3, ..., w_m$ be the vertices of two copies of star graph $K_{1,m}$, respectively where v_0 and w_0 are the apex vertices. To obtain double coconut tree $G = D(m,n,m)$, u_1 and u_n are identified with v_0 and w_0 respectively. Let $d_i = u_1v_i$, $g_i = u_nw_i$ where $i = 1, 2, ..., m$ and $e_i = u_iu_{i+1}, i = 1, 2, ..., n-1$ be the edges of G.
Here, vertex set $V(G) = \{u_1, u_2, u_3, \ldots, u_n\} \cup \{v_1, v_2, v_3, \ldots, v_m\} \cup \{ w_1, w_2, w_3, \ldots, w_m\}$ and $|V(G)| = 2m + n$. Edge set $E(G) = \{e_1, e_2, \ldots, e_{n-1}\} \cup \{d_1, d_2, \ldots,$

$d_m \} \cup \{g_1, g_2, \ldots, g_m\}$ and $|E(G)| = 2m + n - 1$.
Now, we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ as follows.
$f(u_1) = 1,\ f(u_n) = 2,$
For $i \neq 1, n$

$$f(u_i) = \begin{cases} \lceil \frac{n+2}{2} \rceil + \frac{i-1}{2} & \text{for } i \text{ is odd} \\ \frac{i}{2} + 2, & \text{for } i \text{ is even} \end{cases}$$

For each $1 \leq i \leq m$

$$f(w_i) = \begin{cases} n + 2i & \text{for } n \text{ is odd} \\ n + 2i - 1 & \text{for } n \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} n + 2i - 1 & \text{for } n \text{ is odd} \\ n + 2i & \text{for } n \text{ is even} \end{cases}$$

$$f(g_i) = \begin{cases} n + 2m + 2i & \text{for } n \text{ is odd} \\ n + 2m + 2i - 1 & \text{for } n \text{ is even} \end{cases}$$

$$f(d_i) = \begin{cases} n + 2m + 2i - 1 & \text{for } n \text{ is odd} \\ n + 2m + 2i & \text{for } n \text{ is even} \end{cases}$$

$f(e_i) = 2(n + 2m) - i \quad \text{for } 1 \leq i \leq n - 1$

Now we will show that f satisfies both the conditions of vertex-edge neighborhood prime labeling. Let w be an arbitrary vertex of G. If w is adjacent to u_1 with degree 1 then gcd $\{f(u_1), f(u_1 w)\} = 1$ because $f(u_1) = 1$. If w is adjacent to u_n with degree 1 then gcd $\{f(u_n), f(u_n w)\} = 1$ because $f(u_n) = 2$ and $f(u_n w)$ are odd numbers. Now for the vertex w with degree greater than 1, $\{f(v)/v \in N_V(w)\}$ and $\{f(e)/e \in N_E(w)\}$ contain at least two consecutive numbers or consecutive odd numbers or 1. Therefore, gcd $\{f(v)/v \in N_V(w)\} = 1$ and gcd $\{f(e)/e \in N_E(w)\} = 1$.

Hence, f is a vertex-edge neighborhood prime labeling of G and G is a vertex-edge neighborhood prime graph.

Illustration 2. Vertex-edge neighborhood prime labeling of $D(5, 13, 5)$ is shown in Figure 2.

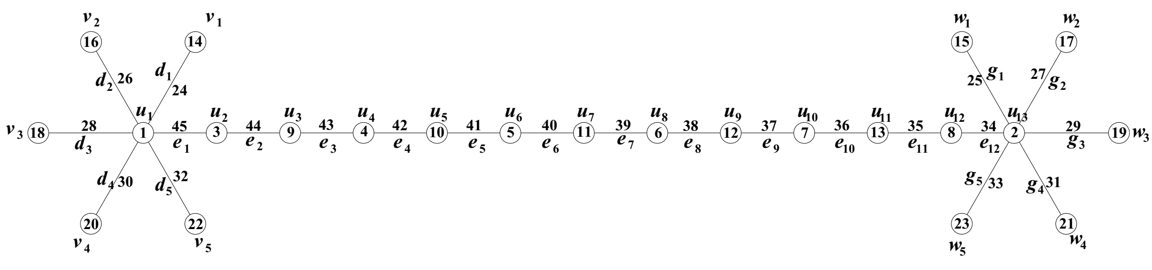

Figure 2: Vertex-edge neighborhood prime labeling of $D(5, 13, 5)$.

Theorem 2.3. *Spider graph with k legs of equal length n is vertex-edge neighborhood prime graph.*

Proof. Let G be a spider graph with k legs of equal length n. Let $u, u_{i,1}, u_{i,2}, \ldots, u_{i,n}$ be the consecutive vertices of i^{th} leg for $1 \leq i \leq k$, where u is the common vertex in each leg. Let $e_{i,1} = uu_{i,1}$ and $e_{i,j} = u_{i,j-1}u_{i,j}$ for $2 \leq j \leq n$ and $1 \leq i \leq k$.
Vertex set $V(G) = \{u, u_{i,j}/1 \leq i \leq k$ and $1 \leq j \leq n\}$ and edge set $E(G) = \{e_{i,j}/1 \leq i \leq k$ and $1 \leq j \leq n\}$. So, $|V(G)| = nk + 1$ and $|E(G)| = nk$.
Now we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ in the following two cases.

Case (i): n is odd

$f(u) = 1$

$$f(u_{1,j}) = \begin{cases} 2n + 2 - \dfrac{j+1}{2} & \text{for } j \text{ is odd} \\ 1 + \dfrac{j}{2} & \text{for } j \text{ is even} \end{cases}$$

For each $2 \leq i \leq k$,

$$f(u_{i,j}) = \begin{cases} 1 + 2n(i-1) + \dfrac{j+1}{2} & \text{for } j \text{ is odd} \\ 1 + 2n(i-1) + \dfrac{n+1}{2} + \dfrac{j}{2} & \text{for } j \text{ is even} \end{cases}$$

$$f(e_{1,j}) = \frac{3(n+1)}{2} - j \quad \text{for } 1 \leq j \leq n$$

For each $2 \leq i \leq k$,

$$f(e_{i,j}) = 2(ni+1) - j \quad \text{for } 1 \leq j \leq n$$

Case (ii): n is even

$f(u) = 1$

For $2 \leq i \leq k$,

$$f(u_{i,j}) = \begin{cases} 1 + 2n(i-1) + \dfrac{j+1}{2} & \text{for } j \text{ is odd} \\ 2n(i-1) + \dfrac{3n+2}{2} + \dfrac{j}{2} & \text{for } j \text{ is even} \end{cases}$$

Consider the subcases as follows.

Subcase(ii) a. $n \equiv 0(mod 4)$

$$f(u_{1,j}) = \begin{cases} \dfrac{j+5}{2} & \text{for } j \text{ is odd} \\ \dfrac{3n+2+j}{2} & \text{for } j \text{ is even} \end{cases}$$

$f(e_{1,j}) = 2 \quad \text{for } j = 1.$

$$f(e_{1,j}) = \frac{3n+4}{2} - j \quad \text{for } 2 \leq j \leq n$$

For each $2 \leq i \leq k$,

$f(e_{i,j}) = 2n(i-1) + \dfrac{3n+4}{2} - j \quad \text{for } 1 \leq j \leq n$
Subcase(ii) b. $n \equiv 2(mod 4)$

$$f(u_{1,j}) = \begin{cases} \dfrac{n+3+j}{2} & \text{for } j \text{ is odd} \\ 1 + \dfrac{j}{2} & \text{for } j \text{ is even} \end{cases}$$

$f(e_{1,j}) = 2n + 2 - j \quad \text{for } 1 \leq j \leq n$
For each $2 \leq i \leq k$,
$f(e_{i,j}) = 2n(i-1) + \dfrac{3n+4}{2} - j \quad \text{for } 1 \leq j \leq n$
Let w be an arbitrary vertex of G. For any pendant vertex w, $f(v)$ and $f(vw)$ are consecutive numbers, so we are done. Let w be any vertex with degree greater than 1 and $w \neq u$. In this case $\{f(v)/v \in N_V(w)\}$ and $\{f(e)/e \in N_E(w)\}$ contain at least two consecutive numbers or 1. So the conditions of the labeling are satisfied. Now for $w = u$, we consider following two cases.
Case(i): n is odd.
$\{f(v)/v \in N_V(u)\}$ contains consecutive numbers.
$e_{2,1}$ and $e_{3,1}$ are in $N_E(u)$. Since $f(e_{2,1}) = nk_1 + 1$ and $f(e_{3,1}) = n(k_1+2)+1$, gcd $\{f(e_{2,1}), f(e_{3,1})\} = \gcd\{nk_1+1, n(k_1+2)+1\} = 1$. So, gcd $\{f(e)/e \in N_E(u)\} = 1$.
Case(ii): n is even.
Subcase (ii) a: $n \equiv 0(mod 4)$
$N_V(u) = \{u_{i,1}/i = 1, 2, \ldots, k\}$.
$f(u_{1,1}) = 3$ and $f(u_{i,1}) = n[2 + 2(i-2)] + 2,$ where $i = 2, 3, 4, \ldots, k$
Since gcd $\{f(u_{1,1}), f(u_{2,1}), f(u_{3,1})\} = \{3, 2n+2, 4n+2\} = 1$, gcd $\{f(v)/v \in N_V(u)\} = 1$. $\{f(e)/e \in N_E(u)\}$ contains 2 and some odd numbers. Thus, gcd $\{f(e)/e \in N_E(u)\} = 1$.
Subcase (ii) b: $n \equiv 2(mod 4)$
$N_V(u) = \{u_{i,1}/i = 1, 2, \ldots, k\}$
$f(u_{1,1}) = \dfrac{n+4}{2}$, which is odd because $n \equiv 2(mod 4)$
$f(u_{i,1}) = n[2 + 2(i-2)] + 2,$ where $i = 2, 3, 4, \ldots, k$
Since gcd $\{f(u_{1,1}), f(u_{2,1}), f(u_{3,1})\} = \left\{\dfrac{n+4}{2}, 2n+2, 4n+2\right\} = 1$, gcd $\{f(v)/v \in N_V(u)\} = 1$. By similar argument gcd $\{f(e)/e \in N_E(u)\} = 1$.
So, f satisfies all the conditions of vertex-edge neighborhood prime labeling. Hence G is a vertex-edge neighborhood prime graph.

Illustration 3. Vertex-edge neighborhood prime labeling of spider graph with 6 legs of length 7 is shown in Figure 3.

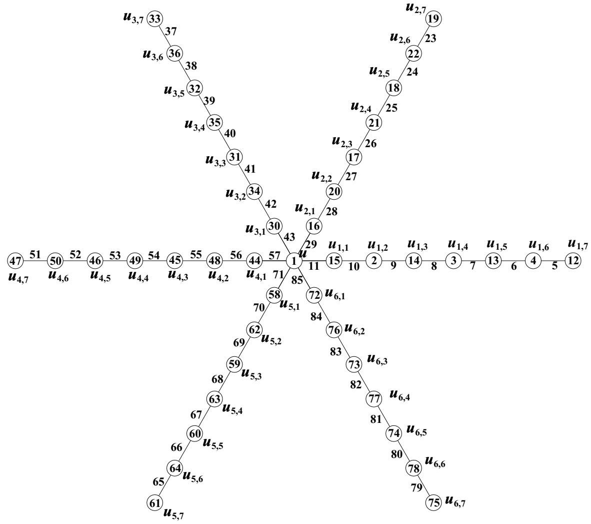

Figure 3: Vertex-edge neighborhood prime labeling of spider graph with 6 legs of length 7 .

Theorem 2.4. *An Olive tree T_k is vertex-edge neighborhood prime graph.*

Proof. Let $G = T_k$. Let $u, u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,i}$ be the consecutive vertices of i^{th} path of length i. u is the common end vertex of paths in a graph G. Let $e_{i,1}, e_{i,2}, e_{i,3}, \ldots, e_{i,i}$ be the edges of i^{th} path where $e_{i,1}$ is an edge between u and $u_{i,1}$ and $e_{i,j}$ is an edge between $u_{i,j-1}$ and $u_{i,j}$.

Here, vertex set $V(G) = \{u, u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,i}/i = 1, 2, \ldots, k\}$ and $|V(G)| = \dfrac{k^2+k+2}{2}$. Edge set $E(G) = \{e_{i,1}, e_{i,2}, e_{i,3}, \ldots, e_{i,i}/i = 1, 2, \ldots, k\}$ and $|E(G)| = \dfrac{k^2+k}{2}$.

Now we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ as follows.

$f(u) = 1$

Consider the following two cases.

Case(i): i is odd

$$f(u_{i,j}) = \begin{cases} \lfloor \frac{i}{2} \rfloor + 1 + i^2 + \frac{j+1}{2} & \text{for } j \text{ is odd} \\ i(i-1) + 1 + \frac{j}{2} & \text{for } j \text{ is even} \end{cases}$$

Case(ii): i is even

$$f(u_{i,j}) = \begin{cases} i(i-1) + 1 + \frac{j+1}{2} & \text{for } j \text{ is odd} \\ \frac{i(2i+1)}{2} + 1 + \frac{j}{2} & \text{for } j \text{ is even} \end{cases}$$

$f(e_{i,j}) = \lfloor \frac{i}{2} \rfloor + 2 + i^2 - j$ for $1 \le j \le i$ and $1 \le i \le k$

Let w be an arbitrary vertex of G. For a vertex w with degree 1, $f(v)$ and $f(vw)$

are consecutive numbers. So, gcd $\{f(v), f(vw)\} = 1$. Let w be any vertex with degree greater than 1. Since $\{f(v)/v \in N_V(w)\}$ and $\{f(e)/e \in N_E(w)\}$ contain at least two consecutive numbers or consecutive odd numbers or 1, gcd $\{f(v)/v \in N_V(w)\} = 1$ and gcd $\{f(e)/e \in N_E(w)\} = 1$.
Hence, f is a vertex-edge neighborhood prime labeling and G is a vertex-edge neighborhood prime graph.

Illustration 4. Vertex-edge neighborhood prime labeling of T_6 is shown in Figure 4.

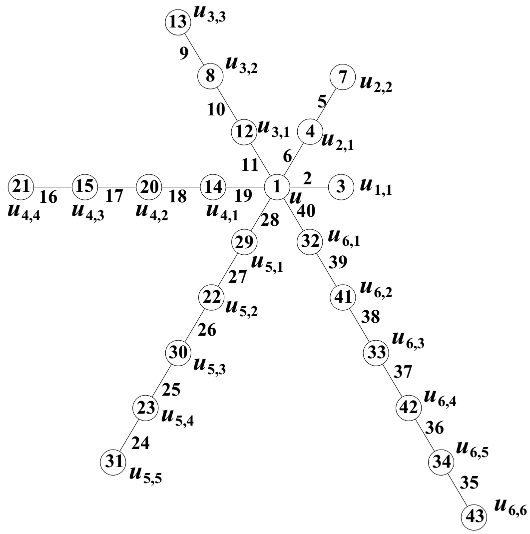

Figure 4: Vertex-edge neighborhood prime labeling of T_6.

Theorem 2.5. *The Comb graph $P_n \odot K_1$ is vertex-edge neighborhood prime graph.*

Proof. Let $G = P_n \odot K_1$. Let $u_1, u_2, \ldots, u_n$ be the consecutive vertices, e_1, e_2, $\ldots$, e_{n-1} be the consecutive edges and $u_1', u_2', \ldots, u_n'$ are pendant vertices in G, where u_i and u_i' are adjacent. Denote edge between u_i and u_i' by e_i' for each i. Here, vertex set $V(G) = \{u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n'\}$ and $|V(G)| = 2n$. Edge set $E(G) = \{e_1, e_2, \ldots, e_{n-1}, e_1', e_2' \ldots, e_n'\}$ and $|E(G)| = 2n - 1$. Now we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ in the following two cases.
Case(i): n is odd

$$f(u_i) = \begin{cases} i & \text{for } i \text{ is odd }, i \neq n \\ n-1 & \text{for } i = n \\ 2n+1+i & \text{for } i \text{ is even} \end{cases}$$

$$f(u_i^{'}) = \begin{cases} 2n+1 & \text{for } i = 1 \\ 3n+i & \text{for } 2 \leq i \leq n-1 \\ 3n+1 & \text{for } i = n \end{cases}$$

$f(e_i) = 2n - i \quad \text{for } 1 \leq i \leq n-1$

$$f(e_i^{'}) = \begin{cases} 2n & \text{for } i = 1 \\ i-1 & \text{for } i \text{ is odd }, i \neq 1, n \\ 2n+i & \text{for } i \text{ is even} \\ n & \text{for } i = n \end{cases}$$

Case(ii) : n is even

$$f(u_i) = \begin{cases} i & \text{for } i = 1, 2 \\ i+1 & \text{for } i \text{ is even }, i \neq 2, n \\ 3n+2-i & \text{for } i \text{ is odd }, i \neq 1 \\ n & \text{for } i = n \end{cases}$$

$$f(u_i^{'}) = \begin{cases} 3n+i & \text{for } 1 \leq i \leq n-1 \\ 2n+2 & \text{for } i = n \end{cases}$$

$f(e_i) = 2n + 1 - i \quad \text{for } 1 \leq i \leq n-1$

$$f(e_i^{'}) = \begin{cases} 2n+1 & \text{for } i = 1 \\ 3 & \text{for } i = 2 \\ 3n+3-i & \text{for } i \text{ is odd }, i \neq 1 \\ i & \text{for } i \text{ is even }, i \neq 2, n \\ n+1 & \text{for } i = n \end{cases}$$

It is easy to verify that gcd $\{f(v), f(vw)\} = 1$ for any vertex w with degree 1 and gcd $\{f(v)/v \in N_V(w)\} = 1$ and gcd $\{f(e)/e \in N_E(w)\} = 1$ for any vertex w with degree greater than 1. So, f defines a vertex-edge neighborhood prime labeling on G. Hence, G is a vertex-edge neighborhood prime graph.

Illustration 5. Vertex-edge neighborhood prime labeling of $P_9 \odot K_1$ is shown in Figure 5.

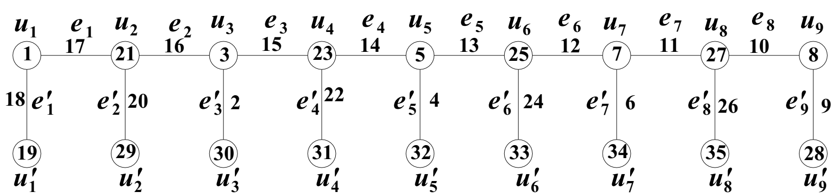

Figure 5: Vertex-edge neighborhood prime labeling of $P_9 \odot K_1$.

Theorem 2.6. *$F(n, 2)$-firecrackers graph is vertex-edge neighborhood prime graph.*
Proof. Let $G = F(n, 2)$ be a firecrackers graph obtained from disjoints copies

$G_1, G_2, \ldots, G_n$ of star graph $K_{1,2}$. Let $u_1, u_2, \ldots, u_n$ be the apex vertices of star graphs $G_1, G_2, \ldots, G_n$ respectively. $u_{i,1}$ and $u_{i,2}$ are the pendant vertices in G_i for each i. Without loss of generality we obtain G by joining $u_{1,1}, u_{2,1}, \ldots, u_{n,1}$ to form a path. Let $e_{i,j} = u_i u_{i,j}$, where $i = 1, 2, \ldots, n$, $j = 1, 2$ and $d_i = u_{i,1}u_{i+1,1}$, where $i = 1, 2, \ldots, n-1$. Here, vertex set $V(G) = \{u_i, u_{i,j}/i = 1, 2, 3, \ldots, n \text{ and } j = 1, 2\}$ and $|V(G)| = 3n$. Edge set $E(G) = \{e_{i,j}/i = 1, 2, 3, \ldots, n \text{ and } j = 1, 2\} \cup \{d_i/i = 1, 2, \ldots, n-1\}$ and $|E(G)| = 3n - 1$. Now we define $f : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ as follows.

$$f(u_i) = \begin{cases} 2 & \text{for } i = 1 \\ 7 & \text{for } i = 2 \\ 5i - 4 & \text{for } 3 \leq i \leq n \end{cases}$$

$$f(u_{i,1}) = \begin{cases} 6 & \text{for } i = 1 \\ 1 & \text{for } i = 2 \\ 5i & \text{for } 3 \leq i \leq n-1 \\ 6n - 1 & \text{for } i = n \end{cases} \qquad f(u_{i,2}) = \begin{cases} 5i & \text{for } i = 1, 2 \\ 5i - 1 & \text{for } 3 \leq i \leq n-1 \\ 6n - 2 & \text{for } i = n \end{cases}$$

$$f(e_{i,1}) = \begin{cases} 5i - 1 & \text{for } i = 1, 2 \\ 5i - 2 & \text{for } 3 \leq i \leq n \end{cases} \qquad f(e_{i,2}) = \begin{cases} 5i - 2 & \text{for } i = 1, 2 \\ 5i - 3 & \text{for } 3 \leq i \leq n \end{cases}$$

$f(d_i) = 6n - 2 - i \quad \text{for } 1 \leq i \leq n-1$

It is easy to verify that gcd $\{f(v), f(vw)\} = 1$ for any vertex w with degree 1 and gcd $\{f(v)/v \in N_V(w)\} = 1$ and gcd $\{f(e)/e \in N_E(w)\} = 1$ for any vertex w with degree greater than 1. So, f defines a vertex-edge neighborhood prime labeling on G. Hence, G is a vertex-edge neighborhood prime graph.

Illustration 6. Vertex-edge neighborhood prime labeling of $F(7, 2)$ is shown in Figure 6.

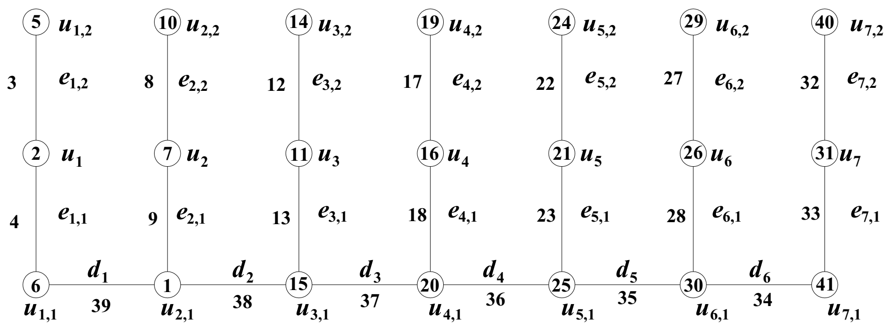

Figure 6: Vertex-edge neighborhood prime labeling of $F(7, 2)$.

3. Acknowledgment

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