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A NOTE ON RAMANUJAN'S GENERAL THETA FUNCTION

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Abstract: In this paper, Ramanujan's general theta function has been generalized and its properties have been discussed.

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1. Introduction

Jacobi in 1829 [3] defined following four functions which are called Jacobi's theta functions,

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)z, \tag{1.1}$$

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)z, \tag{1.2}$$

$$\theta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz \tag{1.3}$$

and

$$\theta_4(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{1.4}$$

For $z = 0$, (1.1) - (1.4) yield,

$$\theta_1(q) = 0 \tag{1.5}$$

$$\theta_2(q) = 2\sum_{n=0}^{\infty} q^{n^2+n+\frac{1}{4}}$$

$$= 2q^{1/4}\sum_{n=0}^{\infty} q^{n^2+n} = q^{1/4}\sum_{n=-\infty}^{\infty} q^{n^2+n}$$

Applying Jacobi's triple product identity [2; App. II (II.28)],

$$\theta_2(q) = 2q^{1/4}(q^2;q^2)_\infty(-q^2;q^2)^2_\infty. \tag{1.6}$$

$$\theta_3(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{n^2}$$

By an appeal of triple product identity [2; App. II (II.28)],

$$\theta_3(q) = (q^2;q^2)_\infty(-q;q^2)^2_\infty. \tag{1.7}$$

$$\theta_4(q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2},$$

Applying triple product identity [2; App. II (II.28)] we find,

$$\theta_4(q) = (q^2;q^2)_\infty(q;q^2)^2_\infty. \tag{1.8}$$

Large number of fascinating identities are available in the literature involving $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$, out of which most celebrated one is

$$\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q). \tag{1.9}$$

Motivated with these remarkable results involving $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$, Ramanujan defined a general theta function as,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.10}$$

[1; (1.1.5), p. 11]

which by an appeal of Jacobi's triple product identity [2; App. II (II. 28)] yields,

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}=\sum_{n=-\infty}^{\infty}(ab)^{\frac{n^2}{2}}\left(\frac{a^{1/2}}{b^{1/2}}\right)^n,$$

$$=(ab;ab)_\infty(-a;ab)_\infty(-b;ab)_\infty,\quad |ab|<1. \tag{1.11}$$

Further, Ramanujan defined following functions as the special cases of (1.11).

$$\Phi(q)=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=(q^2;q^2)_\infty(-q;q^2)_\infty, \tag{1.12}$$

$$\Psi(q)=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.13}$$

and

$$f(-q)=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty. \tag{1.14}$$

[1; (1.1.6), (1.1.7) and (1.1.8), p. 11]

Making use of these functions, Ramanujan has established large number of identities in his second and 'Lost' notebooks [4, 5].

2. Notations and Definitions

Here and the sequel we employ the customary $q-$ product notation given as below.
For arbitrary number α and q, $|q|<1$, let

$$(a;q)_n=(1-a)(1-aq)...(1-aq^{n-1}),\quad n\in\{1,2,3,...\}, \tag{2.1}$$

$$(a;q)_\infty=\prod_{r=0}^{\infty}(1-aq^r) \tag{2.2}$$

and for brevity we write,

$$(a_1,a_2,..,a_r;q)_n=(a_1;q)_n(a_2;q)_n...(a_r;q)_n. \tag{2.3}$$

3. Further Generalization of Ramanujan's Theta Function

In this section, we give following generalized Ramanujan's theta function.

$$f(a,b,z)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}z^n$$

$$= \sum_{n=-\infty}^{\infty} (ab)^{n^2/2} \left(\frac{a^{1/2}z}{b^{1/2}}\right)^n . \tag{3.1}$$

By an appeal of Jacobi's triple product identity [2; App. II (II. 28)] we have

$$f(a,b,z) = (ab, -az, -b/z; ab)_\infty. \tag{3.2}$$

Following Ramanujan, we have

$$\Phi(q,z) = f(q,q,z) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2;q^2)_\infty(-zq, -q/z; q^2)_\infty. \tag{3.3}$$

$$\Psi(q,z) = f(q,q^3,z) = (q^4;q^4)_\infty(-zq, -q^3/z; q^4)_\infty. \tag{3.4}$$

$$f(-q,z) = f(-q,-q^2,z) = (q^3;q^3)_\infty(zq, q^2/z; q^3)_\infty. \tag{3.5}$$

Putting $z = e^{2i\theta}$ in (3.3) we get

$$\Phi(q, e^{2i\theta}) = (q^2;q^2)_\infty \prod_{r=0}^{\infty}(1 + 2q^{2r+1}\cos 2\theta + q^{4r+2}), \tag{3.6}$$

where as the partial $\Phi(q, e^{2i\theta})$ is expressed as,

$$\Phi_N(q, e^{2i\theta}) = (q^2;q^2)_\infty \prod_{r=0}^{N}(1 + 2q^{2r+1}\cos 2\theta + q^{4r+2}). \tag{3.7}$$

Putting zq for z in (3.4) and then replacing z by $e^{2i\theta}$ we have

$$\Psi(q, qe^{2i\theta}) = f(q, q^3, qe^{2i\theta})$$

$$= (q^4;q^4)_\infty \prod_{r=0}^{\infty}(1 + 2q^{4r+2}\cos 2\theta + q^{8r+4}). \tag{3.8}$$

Partial $\Psi(q, qe^{2i\theta})$ is represented by

$$\Psi_N(q, qe^{2i\theta}) = (q^4;q^4)_\infty \prod_{r=0}^{N}(1 + 2q^{4r+2}\cos 2\theta + q^{8r+4}). \tag{3.9}$$

From (3.6) and (3.8) we have,

$$\Phi(q^2, e^{2iz}) = \Psi(q, qe^{2iz}). \tag{3.10}$$

4. Certain Properties of $\Phi(q, e^{2i\theta})$ and $\Psi(q, qe^{2i\theta})$

Putting $\theta = 0$ in (3.6) we get,

$$\Phi(q,1) = \Phi(q) = (q^2;q^2)_\infty \prod_{r=0}^{\infty}(1+q^{2r+1})^2 = (q^2;q^2)_\infty(-q;q^2)_\infty^2. \tag{4.1}$$

For $\theta = \pi/2$, (3.6) yields

$$\Phi(q, e^{i\pi}) = (q^2;q^2)_\infty(q;q^2)_\infty^2. \tag{4.2}$$

Putting $\theta = \pi/4$ in (3.6) we have

$$\Phi(q, e^{i\pi/2}) = (q^2;q^2)_\infty(-q^2;q^4)_\infty. \tag{4.3}$$

Differentiating both sides of (3.6) with respect to θ and then putting $\theta = \pi/4$ we get,

$$\frac{d}{d\theta}\Phi(q, e^{2i\theta}) = -4(q^2;q^2)_\infty(-q^2;q^4)_\infty \sum_{r=0}^{\infty} \frac{q^{2r+1}}{1+q^{4r+2}}. \tag{4.4}$$

(For $\theta = \pi/4$)
From (3.3) we have,

$$\Phi(q, e^{2i\theta}) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2in\theta}. \tag{4.5}$$

Differentiating (4.5) with respect to θ we have

$$\frac{d}{d\theta}\Phi(q, e^{2i\theta}) = \sum_{n=-\infty}^{\infty} i2nq^{n^2} e^{2in\theta}. \tag{4.6}$$

Putting $\theta = \pi/4$ in (4.6) we have,

$$\frac{d}{d\theta}\Phi(q, e^{2i\theta}) = 2i \sum_{n=-\infty}^{\infty} nq^{n^2} i^n$$

For $\theta = \pi/4$

$$= -2 \sum_{n=-\infty}^{\infty} (-1)^{(n-1)/2} nq^{n^2}. \tag{4.7}$$

Equating (4.4) and (4.7) we have

$$\sum_{n=-\infty}^{\infty} n(-1)^{\frac{n-1}{2}} q^{n^2} = 2(q^2;q^2)_\infty(-q^2;q^4)_\infty \sum_{r=0}^{\infty} \frac{q^{2r+1}}{1+q^{4r+2}}. \tag{4.8}$$

Identity (4.8) can be put as,

$$\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}=2(q^2;q^2)_\infty(-q^2;q^4)_\infty\sum_{r=0}^{\infty}\frac{q^{2r+1}}{1+q^{4r+2}}. \tag{4.9}$$

Similar other interesting results can also be scored.

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