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NEW GENERALIZED $\alpha - \psi -$GERAGHTY CONTRACTION TYPE MAPS AND FIXED POINTS

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Abstract: In this paper, we introduce the notion of new generalized $\alpha - \psi -$ Geraghty contraction type maps in the context of metric space and establish some fixed point theorems for such maps. This new contraction map is motivated by the different Geraghty contraction type maps introduced by many authors over the years. An example is also given to illustrate our result.

Keywords and Phrases: Metric space, fixed point, generalized $\alpha -$Geraghty contraction type map, generalized $\alpha - \psi -$Geraghty contraction type map, extended generalized $\alpha - \psi -$Geraghty contraction type map, new generalized $\alpha - \psi -$Geraghty contraction type map.

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1. Introduction

In 1973, Geraghty [5] generalized the Banach contraction principle in the setting of a complete metric space by considering an auxiliary function. This important result of Geraghty was further generalized and improved upon by the works of many authors namely Amini-Harandi & Emami [1], Caballero et al. [3] and Gordji et al. [6] etc. In 2012, Samet et al.[17] defined the notion of $\alpha-\psi-$contractive mappings and obtained remarkable fixed point results. Inspired by this notion of $\alpha-\psi-$contractive mappings, Karapinar & Samet [9] introduced the concept of generalized $\alpha-\psi-$contractive mappings and obtained fixed point results for such mappings. In 2013, Cho et al. [4] defined the concept of generalized $\alpha-$Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps. Further as generalizations of the type of maps defined by Cho et al. [4], Erdal Karapinar [10] introduced the concept of generalized $\alpha-\psi-$Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Cho et al.[4]. Recently, in 2014, Popescu [15] generalized the results of Cho et al. [4] and gave other conditions for the existence and uniqueness of a fixed point of $\alpha-$Geraghty contraction type maps. Then, very recently K. Anthony Singh [7] introduced extended generalized $\alpha-\psi-$ Geraghty contraction type maps and proved some fixed point results generalizing the results of Popescu [15].

In this paper, motivated by the different Geraghty contraction type maps introduced by many authors, we define new generalized $\alpha-\psi-$Geraghty contraction type maps in the setting of metric space and obtain the existence and uniqueness of a fixed point of such maps. We also give an example to illustrate our result.

2. Preliminaries

In this section, we recall some basic definitions and related results on the topic in the literature.

Let $\mathcal{F}$ be the family of all functions $\beta:[0,\infty)\to[0,1)$ which satisfies the condition

$$\lim_{n\to\infty}\beta(t_n)=1 \quad \text{implies} \quad \lim_{n\to\infty}t_n=0.$$

By using such a map, Geraghty proved the following interesting result.

Theorem 2.1. [5] *Let (X,d) be a complete metric space and T a mapping on X. Suppose that there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$d(Tx,Ty)\le\beta(d(x,y))d(x,y).$$

Then T has a unique fixed point $x_*\in X$ and $\{T^nx\}$ converges to x_* for each $x\in X$.

Definition 2.2. [17] *Let $T:X\to X$ be a map and $\alpha:X\times X\to\mathbb{R}$ be a function. Then T is said to be $\alpha-$admissible if $\alpha(x,y)\ge1$ implies $\alpha(Tx,Ty)\ge1$.*

Definition 2.3. [8] *A map $T:X\to X$ is said to be triangular $\alpha-$admissible if*

(T1) T is $\alpha-$admissible,
(T2) $\alpha(x,z)\geq 1$ and $\alpha(z,y)\geq 1$ imply $\alpha(x,y)\geq 1$.

Lemma 2.4. [8] *Let $T:X\to X$ be a triangular $\alpha-$admissible map. Assume that there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1}=Tx_n$. Then we have $\alpha(x_n,x_m)\geq 1$ for all $m,n\in\mathbb{N}$ with $n<m$.*
Cho et al. [4] introduced the following contraction and proved some interesting fixed point results generalizing many results in the existing literature.

Definition 2.5. [4] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized $\alpha-$Geraghty contraction type map if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)d(Tx,Ty)\leq\beta(M(x,y))M(x,y),$$

where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$.
Erdal Karapinar [10] defined the following class of auxiliary functions.
Let Ψ denote the class of functions $\psi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:
(a) ψ is nondecreasing;
(b) ψ is subadditive, that is, $\psi(s+t)\leq\psi(s)+\psi(t)$;
(c) ψ is continuous;
(d) $\psi(t)=0\Leftrightarrow t=0$.
Erdal Karapinar [10] also introduced the following contraction and proved some interesting fixed point results generalizing the results of Cho et al.[4].

Definition 2.6. [10] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized $\alpha-\psi-$Geraghty contraction type mapping if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty))\leq\beta(\psi(M(x,y)))\psi(M(x,y)),$$

where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$ and $\psi\in\Psi$.
Popescu [15] extended the notion of generalized $\alpha-$Geraghty contraction type map and gave the following definition.

Definition 2.7. [15] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized $\alpha-$Geraghty contraction type map if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)d(Tx,Ty)\leq\beta(M_T(x,y))M_T(x,y),$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.
K. Anthony Singh [7] further introduced the following contraction and proved some fixed point results generalizing the results of Popescu [15].

Definition 2.8. [7] *Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called an extended generalized $\alpha - \psi-$Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta(\psi(M_T(x,y)))\psi(M_T(x,y)),$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ *and* $\psi \in \Psi$.

3. Main Results

We now state and prove our main results. Here we introduce the following new definitions. The new contraction map defined below is motivated by the different Geraghty contraction type maps introduced by many authors as in the above section 2.
Let Ω be the family of all functions $\theta : [0,\infty) \to [0,1]$ which satisfy the following conditions
(1) $\theta(t) < 1$ for $t > 0$, and
(2) $\lim_{n\to\infty} \theta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Remark 3.1. *Here instead of the family $\mathcal{F}$ we are introducing a slightly extended family Ω.*

Definition 3.2. *Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Then the mapping $T : X \to X$ is called a new generalized $\alpha - \psi-$Geraghty contraction type map if there exists $\theta \in \Omega$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y)),$$

where $N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ *and* $\psi \in \Psi$.

Theorem 3.3. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is a new generalized $\alpha - \psi-$Geraghty contraction type map,
(ii) T is triangular $\alpha-$admissible,
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,
(iv) T is continuous.
Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.
Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$. We construct a sequence of

points $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is clearly a fixed point of T and the proof is complete. Hence, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.
By hypothesis, $\alpha(x_1, x_2) \geq 1$ and T is triangular $\alpha-$admissible. Therefore by Lemma 2.4., we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Then we have

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \leq \alpha(x_n, x_{n+1})\psi(d(Tx_n, Tx_{n+1})) \\ &\leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})) \quad \text{for all} \quad n \in \mathbb{N}. \quad (1) \end{aligned}$$

Here we have

$$\begin{aligned} N(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})}{2}, \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n, x_{n+2})}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\right\} \end{aligned}$$

If $d(x_{n+1}, x_{n+2}) \geq d(x_n, x_{n+1})$, then $N(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$.
Now from (1) and the definition of θ, we have

$$\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_{n+1}, x_{n+2}),$$

which is a contradiction.
Therefore, we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing and also by then we have $N(x_n, x_{n+1}) = d(x_n, x_{n+1})$.
Now, we prove that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.
It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence which is bounded from below. Therefore there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We show that $r = 0$. We suppose on the contrary that $r > 0$.
Then, we have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \leq \theta(\psi(d(x_n, x_{n+1}))) < 1.$$

Now by taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \theta(\psi(d(x_n, x_{n+1}))) = 1.$$

By the property of θ, we have $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ that is $r = 0$ which is a contradiction. Hence $r = 0$ that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2}$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let us suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that, for all positive integers k, there exist $m_k > n_k > k$ with

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \tag{3}$$

Let m_k be the smallest number satisfying the conditions above. Then we have

$$d(x_{m_k-1}, x_{n_k}) < \epsilon \tag{4}$$

By (3) and (4), we have

$$\begin{aligned}\epsilon &\leq d(x_{m_k}, x_{n_k})\\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k})\\ &< d(x_{m_k-1}, x_{m_k}) + \epsilon\end{aligned}$$

that is,

$$\epsilon \leq d(x_{m_k}, x_{n_k}) < \epsilon + d(x_{m_k-1}, x_{m_k}) \quad \text{for all} \quad k \in \mathbb{N}. \tag{5}$$

Then in view of (2) and (5), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{6}$$

Again, we have

$$\begin{aligned}d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{m_k-1})\\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{n_k-1}) + d(x_{m_k-1}, x_{n_k-1})\end{aligned}$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{n_k-1}, x_{n_k}) + d(x_{m_k}, x_{n_k}).$$

Taking limit as $k \to \infty$ and using (2) and (6), we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon. \tag{7}$$

Also, we have

$$|d(x_{n_k}, x_{m_k-1}) - d(x_{n_k}, x_{m_k})| \leq d(x_{m_k}, x_{m_k-1}).$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \epsilon.$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = \epsilon.$$

By Lemma 2.4., we get $\alpha(x_{n_k-1}, x_{m_k-1}) \geq 1$. Therefore, we have

$$\begin{aligned}\psi(d(x_{m_k}, x_{n_k})) &= \psi(d(Tx_{m_k-1}, Tx_{n_k-1}))\\ &\leq \alpha(x_{n_k-1}, x_{m_k-1})\psi(d(Tx_{n_k-1}, Tx_{m_k-1}))\\ &\leq \theta(\psi(N(x_{n_k-1}, x_{m_k-1})))\psi(N(x_{n_k-1}, x_{m_k-1})).\end{aligned}$$

Here we have

$$\begin{aligned}N(x_{n_k-1}, x_{m_k-1}) &= \max\Big\{d(x_{n_k-1}, x_{m_k-1}), \frac{d(x_{n_k-1}, Tx_{n_k-1}) + d(x_{m_k-1}, Tx_{m_k-1})}{2},\\ &\qquad \frac{d(x_{n_k-1}, Tx_{m_k-1}) + d(x_{m_k-1}, Tx_{n_k-1})}{2}\Big\}\\ &= \max\Big\{d(x_{n_k-1}, x_{m_k-1}), \frac{d(x_{n_k-1}, x_{n_k}) + d(x_{m_k-1}, x_{m_k})}{2},\\ &\qquad \frac{d(x_{n_k-1}, x_{m_k}) + d(x_{m_k-1}, x_{n_k})}{2}\Big\}\end{aligned}$$

And we see that

$$\lim_{k\to\infty} N(x_{n_k-1}, x_{m_k-1}) = \epsilon.$$

Now we have

$$\frac{\psi(d(x_{n_k}, x_{m_k}))}{\psi(N(x_{n_k-1}, x_{m_k-1}))} \leq \theta(\psi(N(x_{n_k-1}, x_{m_k-1}))) < 1.$$

By using (6) and taking limit as $k \to \infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \theta(\psi(N(x_{n_k-1}, x_{m_k-1}))) = 1.$$

So, $\lim_{k\to\infty} \psi(N(x_{n_k-1}, x_{m_k-1})) = 0 \Rightarrow \lim_{k\to\infty} N(x_{n_k-1}, x_{m_k-1}) = 0 = \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. As T is continuous, we have $Tx_n \to Tx^*$ that is $\lim_{n\to\infty} x_{n+1} = Tx^*$ and so $x^* = Tx^*$. Hence x^* is a fixed point of T.
Popescu [15] introduced the following two new concepts.

Definition 3.4. [15] *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be $\alpha-$orbital admissible if $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.*

Definition 3.5. [15] *A map $T : X \to X$ is said to be triangular $\alpha-$orbital admissible if (T1) T is $\alpha-$orbital admissible, (T2) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.*

Lemma 3.6. [15] *Let $T : X \to X$ be a triangular $\alpha-$orbital admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*
Obviously, every $\alpha-$admissible map is an $\alpha-$orbital admissible map and every triangular $\alpha-$admissible map is a triangular $\alpha-$orbital admissible map. If we replace the condition (ii) of Theorem 3.3. by a weaker condition "T is triangular $\alpha-$orbital admissible", we can still prove the theorem. Thus we have the following theorem:

Theorem 3.7. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is a new generalized $\alpha - \psi-$Geraghty contraction type map,
(ii) T is triangular $\alpha-$orbital admissible,
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,
(iv) T is continuous.
Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.
For the uniqueness of a fixed point of a new generalized $\alpha - \psi-$Geraghty contraction type map, we consider the following hypothesis:
(G) For any two fixed points x and y of T, there exists $z \in X$ such that $\alpha(x, z) \geq 1$, $\alpha(y, z) \geq 1$ and $\alpha(z, Tz) \geq 1$.

Theorem 3.8. *Adding condition (G) to the hypotheses of Theorem 3.3. (or Theorem 3.7.), we obtain that x^* is the unique fixed point of T.*
Proof. Due to Theorem 3.3. (or Theorem 3.7.), we obtain that $x^* \in X$ is a fixed point of T. Let $y^* \in X$ be another fixed point of T. Then by hypothesis (G), there

exists $z \in X$ such that
$\alpha(x^*, z) \geq 1$, $\alpha(y^*, z) \geq 1$ and $\alpha(z, Tz) \geq 1$.
Since T is triangular $\alpha-$admissible (or triangular $\alpha-$orbital admissible) we get $\alpha(x^*, T^n z) \geq 1$ and $\alpha(y^*, T^n z) \geq 1$ for all $n \in \mathbb{N}$.
Then we have

$$\begin{aligned}\psi(d(x^*, T^{n+1}z)) &\leq \alpha(x^*, T^n z)\psi(d(Tx^*, TT^n z)) \\ &\leq \theta(\psi(N(x^*, T^n z)))\psi(N(x^*, T^n z)), \quad \forall n \in \mathbb{N}.\end{aligned}$$

Here we have

$$\begin{aligned}N(x^*, T^n z) &= \max\left\{d(x^*, T^n z), \frac{d(x^*, Tx^*) + d(T^n z, TT^n z)}{2}, \frac{d(x^*, TT^n z) + d(T^n z, Tx^*)}{2}\right\} \\ &= \max\left\{d(x^*, T^n z), \frac{d(T^n z, T^{n+1}z)}{2}, \frac{d(x^*, T^{n+1}z) + d(T^n z, x^*)}{2}\right\}\end{aligned}$$

By Theorem 3.3. (or Theorem 3.7.) we deduce that the sequence $\{T^n z\}$ converges to a fixed point $z^* \in X$. Then taking limit $n \to \infty$ in the above equality, we get $\lim_{n\to\infty} N(x^*, T^n z) = d(x^*, z^*)$. And let us suppose that $z^* \neq x^*$. Then we have

$$\frac{\psi(d(x^*, T^{n+1}z))}{\psi(N(x^*, T^n z))} \leq \theta(\psi(N(x^*, T^n z))) < 1.$$

And taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(\psi(N(x^*, T^n z))) = 1.$$

Therefore we have $\lim_{n\to\infty} \psi(N(x^*, T^n z)) = 0 \Rightarrow \lim_{n\to\infty} N(x^*, T^n z) = 0$ that is $d(x^*, z^*) = 0$, which is a contradiction. Therefore we must have $z^* = x^*$. Similarly, we get $z^* = y^*$. Thus we have $y^* = x^*$. Hence x^* is the unique fixed point of T.
Here we give an example to illustrate Theorem 3.3.

Example 3.9. Let $X = \{1, 2, 3\}$ with the metric d defined as $d(1, 1) = d(2, 2) = d(3, 3) = 0$, $d(1, 2) = d(2, 1) = 1$, and $d(1, 3) = d(3, 1) = d(2, 3) = d(3, 2) = \frac{1}{2}$. Then (X, d) is a complete metric space. And let $\theta(t) = \frac{1}{1+2t}$ for all $t \geq 0$. Then $\theta \in \Omega$. Also let the function $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = \frac{t}{3}$. Then we have $\psi \in \Psi$.
Let a mapping $T : X \to X$ be defined by $T(1) = T(3) = 1, T(2) = 3$. And let a function $\alpha : X \times X \to \mathbb{R}$ be defined by

$$\alpha(x, y) = \begin{cases} 1 & (x = y) \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Then, T is triangular $\alpha-$admissible, which is condition (ii) of Theorem 3.3. Condition (iii) of Theorem 3.3. is satisfied with $x_1 = 1$. And condition (iv) of Theorem 3.3. is satisfied because T is continuous. We finally show that condition (i) is also satisfied, that is

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \theta(\psi(N(x, y)))\psi(N(x, y)).$$

If $(x, y) = (1, 1)$ or $(2, 2)$ or $(3, 3)$ then $d(Tx, Ty) = 0$. Therefore condition (i) is obviously satisfied.
If $(x, y) = (1, 3)$ or $(3, 1)$ then $d(Tx, Ty) = d(1, 1) = 0$. Therefore condition (i) is satisfied.
If $(x, y) = (1, 2)$ then we have

$$\alpha(x, y)\psi(d(Tx, Ty)) = \alpha(1, 2)\psi(d(T(1), T(2))) = \frac{1}{4}\frac{d(1, 3)}{3} = \frac{1}{24}.$$

And

$$\begin{aligned} N(x, y) = N(1, 2) &= \max\left\{d(1, 2), \frac{d(1, T(1)) + d(2, T(2))}{2}, \frac{d(1, T(2)) + d(2, T(1))}{2}\right\} \\ &= \max\left\{d(1, 2), \frac{d(1, 1) + d(2, 3)}{2}, \frac{d(1, 3) + d(2, 1)}{2}\right\} \\ &= \max\left\{1, \frac{1}{4}, \frac{3}{4}\right\} = 1. \end{aligned}$$

Therefore, $\theta(\psi(N(x, y)))\psi(N(x, y)) = \dfrac{\psi(N(x, y))}{1 + 2\psi(N(x, y))} = \dfrac{N(1, 2)/3}{1 + 2 \times N(1, 2)/3} = \dfrac{1/3}{1 + 2 \times 1/3} = \dfrac{1}{5}$. Thus condition (i) is satisfied. Similarly, we see that condition (i) is satisfied for $(x, y) = (2, 1)$. If $(x, y) = (2, 3)$, then $\alpha(x, y)\psi(d(Tx, Ty)) = \alpha(2, 3)\psi(d(T(2), T(3))) = \dfrac{1}{4}\dfrac{d(3, 1)}{3} = \dfrac{1}{24}$.
And

$$\begin{aligned} N(x, y) = N(2, 3) &= \max\left\{d(2, 3), \frac{d(2, T(2)) + d(3, T(3))}{2}, \frac{d(2, T(3)) + d(3, T(2))}{2}\right\} \\ &= \max\left\{d(2, 3), \frac{d(2, 3) + d(3, 1)}{2}, \frac{d(2, 1) + d(3, 3)}{2}\right\} \\ &= \max\left\{\frac{1}{2}, \frac{\frac{1}{2} + \frac{1}{2}}{2}, \frac{1 + 0}{2}\right\} = \max\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}. \end{aligned}$$

Therefore, $\theta(\psi(N(x,y)))\psi(N(x,y)) = \dfrac{\psi(N(x,y))}{1+2\psi(N(x,y))} = \dfrac{N(2,3)/3}{1+2\times N(2,3)/3} = \dfrac{1/6}{1+2\times 1/6} = \dfrac{1}{8}$.

Thus condition (i) is satisfied. Similarly, we see that condition (i) is satisfied for $(x,y)=(3,2)$. Hence all the conditions of Theorem 3.3. are satisfied and T has a unique fixed point $x^*=1$.

4. Conclusion

Recently, fixed-circle problem has been considered and studied by many authors as a geometric generalization of the fixed point theory in metric spaces and its generalizations. In some cases when we do not have uniqueness of the fixed point, such a map sometimes under certain conditions fixes a circle which we call a fixed-circle. Various fixed-circle theorems have been obtained using different approaches (see [11], [12], [13], [14], [18]). Also, in some papers, application of the obtained fixed-circle results was given to discontinuous activation functions on metric spaces. Therefore, it is becoming important and also interesting to investigate new fixed-circle results.

In closing, we want the readers to investigate, under what conditions, we can prove the results in this paper in fixed-circle. In general, we can always seek answer to the question: What is (are) the necessary and sufficient condition(s) for a self-mapping (two or more self-mappings) that make a given circle the fixed-circle (common fixed-circle)?

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