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INDUCED V_4- MAGIC LABELING OF SOME STAR AND PATH RELATED GRAPHS

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Abstract: Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0 and $G = (V(G), E(G))$, be the graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : V(G) \to V_4$ be a vertex labeling and $f^* : E(G) \to V_4$ denote the induced edge labeling of f defined by $f^*(uv) = f(u) + f(v)$ for all $uv \in E(G)$. Then f^* again induces a vertex labeling $f^{**} : V(G) \to V_4$ defined by $f^{**}(u) = \Sigma f^*(uv)$ where the summation is taken over all the vertices v which are adjacent to u. A graph $G = (V(G), E(G))$ is said to be an induced V_4-Magic graph if there exists a non zero vertex labeling $f : V(G) \to V_4$ such that $f \equiv f^{**}$. The function f, so obtained is called an induced V_4-Magic labeling of G. In this paper we discuss Induced V_4 magic labeling of some graphs and the Induced V_4 magic labeling of some star and path related graphs.

Keywords and Phrases: Klein-4-group, Induced V_4-magic graphs.

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1. Introduction

In this paper we consider simple, connected, finite and undirected graphs and the Klein 4-group is denoted by $V_4 = \{0, a, b, c\}$, which is a noncyclic abelian group of order 4 in which every nonidentity element has order 2. We refer to Frank Harary [2] for the standard terminology and notations related to graph theory. Let $G = (V(G), E(G))$, be the graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : V(G) \to V_4$ be a vertex labeling and $f^* : E(G) \to V_4$ denote the induced edge labeling of f defined by $f^*(uv) = f(u) + f(v)$ for all $uv \in E(G)$. Then this f^* induces another vertex labeling denoted by $f^{**} : V(G) \to V_4$ defined by $f^{**}(u) = \Sigma f^*(uv)$ where the summation is taken over all the vertices v which are adjacent to u. A graph $G = (V(G), E(G))$ is said to be an induced V_4-Magic graph denoted by IMV_4G or simply IMG if there exists a non zero labeling $f : V(G) \to V_4$ such that $f \equiv f^{**}$. The function f, so obtained is called a induced V_4-Magic labeling of G or simply induced Magic labeling of G and it is denoted by IMV_4L or simply IML. In this paper we discuss some Induced V_4 magic labeling of the graphs $C_n, K_n, K_{m,n}$, some star related graphs and some path related graphs which belong to the following categories:

(i) $\Gamma(V_4)$:= class of all induced V_4-magic graphs.

(ii) $\Gamma_k(V_4)$:= class of all induced V_4-magic graphs with induced magic labeling f satisfying $f(V(G)) = \{k\}$ for some $k \in V_4$.

(iii) $\Gamma_{k,0}(V_4)$:= class of all induced V_4-magic graphs with induced magic labeling f satisfying $f(V(G)) = \{k, 0\}$ for some $k \in V_4$.

Theorem 1.1. [5] *If f is an induced magic labeling of a graph G and u be a pendent vertex adjacent to a vertex v in G,then $f(v) = 0$.*

Corollary 1.2. [5] *If f is an induced magic labeling of a graph G and $wuvz$ is a path in G with w and z are pendent vertices in G,then $f^*(uv) = 0$.*

Theorem 1.3. [5] *Let f be any vertex labeling of a graph G and u, be a vertex in G with deg$(u) = m$. Then f is an induced magic labeling of G if and only if $(m-1)f(u)+\Sigma f(v) = 0$ where the summation is taken over all the vertices v which are adjacent to u.*

Theorem 1.4. [5] *$C_n \in \Gamma(A)$ if and only if $n \equiv 0$ $(mod\ 3)$, where A is an Abelian group.*

Theorem 1.5. [5] *$P_n \in \Gamma(A)$ if and only if $n \equiv 0$ $(mod\ 3)$, where A is an Abelian group.*

2. Main Results

Theorem 2.1. *For any graph G, $G \notin \Gamma_a(V_4)$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. If possible suppose $f(v_i) = a$ for $i = 1, 2, 3, \cdots, n$, then we have $f^*(v_jv_k) = f(v_j) + f(v_k) = 0$ for all $v_jv_k \in E(G)$. Thus $f^{**}(v_i) = \Sigma f^*(v_iv) = 0$ where the summation is taken over all the vertices v which are adjacent to v_i. Therefore $f^{**} \not\equiv f$, hence f is not an induced V_4 magic labeling and $G \notin \Gamma_a(V_4)$.

Corollary 2.2. *[Degree sum equation of a vertex]*
Let f be any vertex labeling of a graph G and u, be a vertex in G with $deg(u) = m$. Then f is an induced V_4magic labeling if and only if $f(u)+\Sigma f(v) = 0$ or $\Sigma f(v) = 0$ according as $deg(u) = m$ is even or odd, where the summation is taken over all the vertices v which are adjacent to u.
Proof. From Theorem 1.3, we have f is an induced V_4magic labeling of G if and only if $(m-1)f(u) + \Sigma f(v) = 0$, where v is adjacent to u, then the result follows directly from the fact that $f(u) \in V_4$.

3. Induced magic labeling of some graphs

Theorem 3.1. *The complete graph $K_n \in \Gamma(V_4)$ if and only if n is odd.*
Proof. Let $V(K_n) = \{v_1, v_2, v_3, \cdots, v_n\}$. Suppose n is odd. Define $f : V(K_n) \to V_4$ as :

$$f(v_i) = \begin{cases} 0 & \text{if} \quad i = 1 \\ a & \text{if} \quad i = 2, 3, \cdots, n \end{cases}$$

Then, f is an induced magic labeling of K_n. Conversely suppose n is an even number. Then $\deg v_i = n - 1$ is an odd number. Therefore by corollary 2.2 we have, f is an induced magic label if and only if $f$satisfies the following system equations:

$$\begin{aligned} f(v_2) + f(v_3) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ f(v_1) + f(v_3) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ f(v_1) + f(v_2) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ &\vdots \\ f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{n-2}) + f(v_n) &= 0 \\ f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{n-2}) + f(v_{n-1}) &= 0 \end{aligned}$$

Note that the above system of equations show that $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$ and which again implies that $(n-1)f(v_1) = 0$, that is $f(v_1) = 0$, thus $f \equiv 0$, which is a contradiction.

The following corollary follows directly from the proof of Theorem 3.1.

Corollary 3.2. $K_n \in \Gamma_{a,0}(V_4)$ *if and only if n is odd.*

Corollary 3.3. *$C_n \in \Gamma_{a,0}(V_4)$ if and only if $n \equiv 0$ $(mod\ 3)$.*
Proof. Consider C_n with vertex set $\{v_1, v_2, \ldots, v_{n-1}, v_n.\}$. Suppose $n \equiv 0$ $(mod\ 3)$. Define $f : V(C_n) \to V_4$ as follows:

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1\ (mod\ 3) \\ 0 & \text{if} \quad i \equiv 2\ (mod\ 3) \end{cases}$$

Then clearly f is Induced V_4 magic labeling of C_n. Converse part follows from Theorem 1.4.

Theorem 3.4. *For $m, n > 1$, the complete bipartite graph $K_{m,n} \in \Gamma(V_4)$ if and only if either m or n is odd.*
Proof. Let $V(K_{m,n}) = \{v_1, v_2, v_3, \cdots, v_m, u_1, u_2, u_3, \cdots, u_n\}$, where $v_iu_j \in E(K_{m,n})$ for $i = 1, 2, 3, \cdots m$, and $j = 1, 2, 3, \cdots, n$.

Case 1: m and n are odd

In this case define $f : V(K_{m,n}) \to V_4$ as follows:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_1, u_1 \\ a & \text{if} \quad v = v_2, v_3, v_4, \cdots, v_m \\ b & \text{if} \quad v = u_2, u_3, u_4, \cdots, u_n \end{cases}$$

Case 2: m is odd and n is even

In this case define $g : V(K_{m,n}) \to V_4$ as follows:

$$g(v) = \begin{cases} 0 & \text{if} \quad v = v_1, v_2, v_3, \cdots, v_m \\ a & \text{if} \quad v = u_1, u_2, u_3, \cdots, u_n \end{cases}$$

Case 3: m is even and n is odd

In this case define $h : V(K_{m,n}) \to V_4$ as follows:

$$h(v) = \begin{cases} 0 & \text{if} \quad v = u_1, u_2, u_3, \cdots, u_n \\ a & \text{if} \quad v = v_1, v_2, v_3, \cdots, v_m \end{cases}$$

Then in each case we can easily verify that the vertex labeling f, g and h are induced magic labeling of $K_{m,n}$. Thus $K_{m,n} \in \Gamma(V_4)$ if either m or n is odd. Now consider the following case:

Case 4: m and n are even

If possible suppose $f : V(K_{m,n}) \to V_4$ is an induced magic labeling of $K_{m,n}$. Then by corollary 2.2 f must satisfy the following system of equations:

$$\begin{aligned} f(v_i)+f(u_1)+f(u_2)+f(u_3)+\cdots+f(u_n) &= 0 \text{ for } i=1,2,3,\cdots,m \quad (1)\\ f(u_j)+f(v_1)+f(v_2)+f(v_3)+\cdots+f(v_m) &= 0 \text{ for } j=1,2,3,\cdots,n \quad (2)\end{aligned}$$

Note that the above equations in (1) imply that $f(v_1)=f(v_2)=f(v_3)=\cdots=f(v_n)$ and the equations in (2) imply that $f(u_1)=f(u_2)=f(u_3)=\cdots=f(u_n)$. Thus the above system reduces to:

$$\begin{aligned} f(v_1)+nf(u_1) &= 0\\ mf(v_1)+f(u_1) &= 0\end{aligned}$$

Since both m and n are even the above system implies that $f(v_1)=f(u_1)=0$. Thus $f\equiv 0$ and it is a contradiction to our assumption. Hence in this case $K_{m,n}\notin\Gamma(V_4)$.

4. Induced magic labeling of some star related graphs

Theorem 4.1. *For $n>1$, the star graph $K_{1,n}\in\Gamma(V_4)$.*
Proof. Let $V(K_{1,n})=\{v,v_1,v_2,v_3,\cdots,v_n,\}$, where $vv_i\in E(K_{1,n})$ for $i=1,2,3,\cdots n$.

Case 1: n is even

In this case define $f:V(K_{1,n})\to V_4$ as follows:

$$f(u)=\begin{cases} 0 & \text{if} \quad u=v\\ a & \text{if} \quad u=v_1,v_2,v_3,\cdots,v_n\end{cases}$$

Case 2: n is odd

In this case define $g:V(K_{1,n})\to V_4$ as follows:

$$g(u)=\begin{cases} 0 & \text{if} \quad u=v,v_1\\ a & \text{if} \quad u=v_2,v_3,v_4,\cdots,v_n\end{cases}$$

Then in each case we can easily verify that the vertex labeling f and g are induced magic labeling of $K_{1,n}$. Thus $K_{1,n}\in\Gamma(V_4)$. for all $n>1$.

Corollary 4.2. *For $n>1, K_{1,n}\in\Gamma_{a,0}(V_4)$.*

Definition 4.3. *The Bistar $B_{m,n}$ is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.*

Theorem 4.4. *For the Bistar $B_{m,n}\in\Gamma(V_4)$ for all m and n with $m+n>2$.*
Proof. Let $V(B_{m,n})=\{u,v,v_1,v_2,v_3,\cdots,v_m,u_1,u_2,u_3,\cdots,u_n\}$, where uv, vv_i, $uu_j\in E(B_{m,n})$ for $i=1,2,3,\cdots m$, and $j=1,2,3,\cdots,n$.

Case 1: m and n are even

In this case define $f : V(B_{m,n}) \to V_4$ as follows:

$$f(w) = \begin{cases} 0 & \text{if} \quad w = v, u \\ a & \text{if} \quad w = v_1, v_2, v_3, \cdots, v_m \\ b & \text{if} \quad w = u_1, u_2, u_3, \cdots, u_n \end{cases}$$

Case 2: m is odd and n is even

In this case define $g : V(B_{m,n}) \to V_4$ as follows:

$$g(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1 \\ a & \text{if} \quad w = v_2, v_3, v_4 \cdots, v_m \\ b & \text{if} \quad w = u_1, u_2, u_3, \cdots, u_n \end{cases}$$

Case 3: m is even and n is odd

In this case define $h : V(B_{m,n}) \to V_4$ as follows:

$$h(w) = \begin{cases} 0 & \text{if} \quad w = v, u, u_1 \\ a & \text{if} \quad w = v_1, v_2, v_3, \cdots, v_m \\ b & \text{if} \quad w = u_2, u_3, u_4 \cdots, u_n \end{cases}$$

Case 4: m and n are odd

In this case define $k : V(B_{m,n}) \to V_4$ as follows:

$$k(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1, u_1 \\ a & \text{if} \quad w = v_2, v_3, v_4, \cdots, v_m \\ b & \text{if} \quad w = u_2, u_3, u_4, \cdots, u_n \end{cases}$$

Then in each case we can easily verify that the vertex labeling f, g, h and k are induced magic labeling of $B_{m,n}$. Thus $B_{m,n} \in \Gamma(V_4)$, for all m and n.

Corollary 4.5. *For the Bistar $B_{m,n} \in \Gamma_{a,0}(V_4)$ for all m and n with $m + n > 2$.*

Proof. Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \cdots, v_m, u_1, u_2, u_3, \cdots, u_n\}$, where uv, vv_i, $uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \cdots m$, and $j = 1, 2, 3, \cdots, n$.

Case 1: m and n are even

In this case define $f : V(B_{m,n}) \to V_4$ as follows:

$$f(w) = \begin{cases} 0 & \text{if} \quad w = v, u \\ a & \text{if} \quad w = v_i, u_j, i = 1, 2, 3, \cdots m, j = 1, 2, 3, \cdots, n. \end{cases}$$

Case 2: m is odd and n is even

In this case define $g : V(B_{m,n}) \to V_4$ as follows:

$$g(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1 \\ a & \text{if} \quad w = v_i, u_j, i = 2, 3, \cdots m, j = 1, 2, 3, \cdots, n. \end{cases}$$

Case 3: m is even and n is odd

In this case define $h : V(B_{m,n}) \to V_4$ as follows:

$$h(w) = \begin{cases} 0 & \text{if} \quad w = v, u, u_1 \\ a & \text{if} \quad w = v_i, u_j, i = 1, 2, 3, \cdots m, j = 2, 3, \cdots, n. \end{cases}$$

Case 4: m and n are odd

In this case define $k : V(B_{m,n}) \to V_4$ as follows:

$$k(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1, u_1 \\ a & \text{if} \quad w = v_i, u_j, i = 2, 3, \cdots m, j = 2, 3, \cdots, n. \end{cases}$$

Then in each case we can easily verify that the vertex labeling f, g, h and k are induced magic labeling of $B_{m,n}$. Thus $B_{m,n} \in \Gamma_{a,0}(V_4)$, for all m and n.

Definition 4.6. [3] *Let $< K_{1,n} : m >$ denote the graph obtained by taking m disjoint copies of $K_{1,n}$, and joining a new vertex to the centers of the m copies of $K_{1,n}$.*

Theorem 4.7. *The graph $< K_{1,n} : m > \in \Gamma(V_4)$ for all m, n.*
Proof. Consider the graph $< K_{1,n} : m >$ with $\{v_i, v_{ij} : 1 \leq j \leq n\}$ as the vertex set of i^{th} copy of $K_{1,n}$ with central vertex v_i for $i = 1, 2, 3, \cdots m$ and let v be the unique vertex adjacent to the central vertices v_i in $< K_{1,n} : m >$. Then define $f : V(< K_{1,n} : m >) \to V_4$ as follows:

Case 1: m is odd

Subcase 1: n is odd
Define f as

$$f(u) = \begin{cases} a & \text{if} \quad u = v \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \; j = 1, 2, 3, \cdots, n. \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

Subcase 2: n is even

Define f as

$$f(u) = \begin{cases} a & \text{if} \quad u = v \\ 0 & \text{if} \quad u = v_{11} \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \ j = 2, 3, 4, \cdots, n \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

Case 2: m is even

Subcase 1: n is odd

Define f as

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v \\ 0 & \text{if} \quad u = v_{11} \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \ j = 2, 3, 4, \cdots, n. \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

Subcase 2: n is even

Define f as

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \ j = 1, 2, 3, \cdots, n. \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

One can easily verify that the vertex label f defined in all four cases are IML of $< K_{1,n} : m >$.

Definition 4.8. *The (n, k)-Banana tree $Bt(n, k)$ is the graph obtained by starting with n number of $k-$stars and connecting one end vertex from each to a new vertex.*

Theorem 4.9. *The (n, k)-Banana tree $Bt(n, k) \in \Gamma(V_4)$ for all n and k.*
Proof. Consider the graph $Bt(n, k)$. Let $V[Bt(n, k)]$=$\{v, v_i, v_{ij} : 1 \le i \le n \ , 1 \le j \le k\}$ and $E[Bt(n, k)]$=$\{vv_{i1}, v_iv_{ij}\} : 1 \le i \le n, 1 \le j \le k\}$.

Case 1 : k is odd

In this case define $f(V(Bt(n, k))) :\to V_4$ by

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v, v_i \text{ for } i = 1, 2, 3, \cdots n \\ 0 & \text{if} \quad u = v_{i1} \text{ for } i = 1, 2, 3, \cdots n \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots n, \ j = 2, 3, 4, \cdots, k. \end{cases}$$

Case 2 : k is even

In this case define $f(V(Bt(n,k))) :\to V_4$ by

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v, v_i \text{ for } i = 1,2,3,\cdots n \\ 0 & \text{if} \quad u = v_{i1}, v_{i2} \text{ for } i = 1,2,3,\cdots n, \\ a & \text{if} \quad u = v_{ij}, i = 1,2,3,\cdots n, j = 3,4,\cdots,k. \end{cases}$$

In both case, we can easily verify that, f is an IML of $Bt(n,k)$. Hence the Proof.

5. Induced magic labeling of some path related graphs

Theorem 5.1. *$P_n \in \Gamma_{a,0}(V_4)$ if and only if $n \equiv 0$ (mod 3).*
Proof. Consider the path P_n with vertex set $V = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ where $n \equiv 0$ (mod 3). Define $f : V(P_n) \to V_4$ as :

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0,1 \ (mod\ 3) \\ 0 & \text{if} \quad i \equiv 2 \ (mod\ 3) \end{cases}$$

Then, f is an induced magic labeling of P_n. Hence $P_n \in \Gamma_{a,0}(V_4)$.

Conversely if $n \not\equiv 0$ (mod 3) then by the Theorem 1.5 $P_n \notin \Gamma_{a,0}(V_4)$.

Definition 5.2. *The Corona $P_n \odot K_1$ is called the comb graph CB_n.*

Theorem 5.3. *The Comb graph $CB_n \notin \Gamma(V_4)$ for all n.*
Proof. Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of CB_n where $v_i (1 \leq i \leq n)$ are the pendent vertices adjacent to $u_i (1 \leq i \leq n)$. If possible suppose f is an IML of the graph CB_n. Then from the degree sum equation of the vertices u_i and v_i, we have $f(u_i) = f(v_i) = 0$. That is $f \equiv 0$, which is a contradiction.

Definition 5.4. [3] *A triangular snake graph TS_n is obtained from a path v_1, v_2, $\cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1,2,3, \cdots, n-1$.*

Theorem 5.5. *The triangular snake graph $TS_n \in \Gamma(V_4)$ for all n.*
Proof. Let $V(TS_n) = \{v_1, v_2, \cdots, v_n, w_1, w_2, w_3, \cdots, w_{n-1}\}$, where v_i's are the vertices of corresponding path P_n and f be an IML of TS_n with $f(v_i) = x_i$ and $f(w_j) = y_j$. Then the vertices v_i and w_j must satisfy the degree sum equation.

Note that the degree sum equation of v_i gives the following system of equations.

$$\begin{aligned} x_1 + x_2 + y_1 &= 0 \\ x_1 + x_2 + x_3 + y_1 + y_2 &= 0 \\ x_2 + x_3 + x_4 + y_2 + y_3 &= 0 \\ &\vdots \\ x_{n-2} + x_{n-1} + x_n + y_{n-2} + y_{n-1} &= 0 \\ x_{n-1} + x_n + y_{n-1} &= 0 \end{aligned}$$

Similarly the degree sum equation of w_j gives the following system of equations.

$$\begin{aligned} x_1 + x_2 + y_1 &= 0 \\ x_2 + x_3 + y_2 &= 0 \\ x_3 + x_4 + y_3 &= 0 \\ &\vdots \\ x_{n-2} + x_{n-1} + y_{n-2} &= 0 \\ x_{n-1} + x_n + y_{n-1} &= 0 \end{aligned}$$

On substituting the second system in the first system of equations we get

$$\begin{aligned} x_1 + x_2 + y_1 &= 0 \\ x_1 + y_1 &= 0 \\ x_2 + y_2 &= 0 \\ &\vdots \\ x_{n-2} + y_{n-2} &= 0 \\ x_{n-1} + x_n + y_{n-1} &= 0 \end{aligned}$$

From the above two system of equations one can easily conclude that

$$\begin{aligned} x_1 &= y_1 \\ x_2 = x_3 = x_4 = \cdots = x_{n-1} &= 0 \\ y_2 = y_3 = y_4 = \cdots = y_{n-2} &= 0 \\ x_n &= y_{n-1} \end{aligned}$$

Thus to get an IML of TS_n we need to define $f : V(TS_n) \to V_4$ as follows:

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, w_1 \\ 0 & \text{if} \quad v = v_i, \text{ for } i = 2, 3, 4, \cdots, n-1 \\ 0 & \text{if} \quad v = w_j, \text{ for } j = 2, 3, 4, \cdots, n-2 \\ b & \text{if} \quad v = v_n, w_{n-1} \end{cases}$$

Hence the proof.

Corollary 5.6. *The triangular snake graph $TS_n \in \Gamma_{a,0}(V_4)$ for all n.*
Proof. Define $f : V(TS_n) \to V_4$ as follows:

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, w_1 \\ 0 & \text{if} \quad v = v_i, \text{ for } i = 2, 3, 4, \cdots, n-1 \\ 0 & \text{if} \quad v = w_j, \text{ for } j = 2, 3, 4, \cdots, n-2 \\ a & \text{if} \quad v = v_n, w_{n-1} \end{cases}$$

Then from Theorem 5.5, f is an IML of TS_n. Hence the corollary follows.

Definition 5.7. [3] *A double triangular snake graph DTS_n consists of two triangular snake graphs that have a common path. That is, a double triangular snake is obtained from a path $v_1, v_2, ..., v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, ..., n-1$ and to a new vertex u_i for $i = 1, 2, ..., n-1$.*

Theorem 5.8. *The double triangular snake graph $DTS_n \in \Gamma(V_4)$ if and only if $n \equiv 0$ $(mod\ 3)$.*
Proof. Consider a double triangular snake graph DTS_n with vertex set $v_1, v_2, v_3, \cdots, v_n, w_1, w_2, w_3, \cdots, w_{n-1}, u_1, u_2, u_3, \cdots, u_{n-1}$, where $v_1, v_2, ..., v_n$ are the vertices of corresponding path and w_i, u_i are the vertices attached to v_i and v_{i+1} for $i = 1, 2, ..., n-1$. Suppose $n \equiv 0$ $(mod\ 3)$. Let $n = 3k$, then define $g : V(DTS_n) \to V_4$ as:

$$g(v) = \begin{cases} 0 & \text{if} \quad v = v_2, v_5, v_8, \cdots, v_{n-4}, v_{n-1} \\ a & \text{if} \quad v = v_1, v_3, v_4, v_6, v_7, \cdots, v_{n-3}, v_{n-2}, v_n \\ 0 & \text{if} \quad v = w_{3j}, \text{ for } j = 1, 2, 3, \cdots, k-1 \\ a & \text{if} \quad v = w_1, w_2, w_4, w_5, w_7, \cdots, w_{n-2}, w_{n-1} \\ 0 & \text{if} \quad v = u_{3j}, \text{ for } j = 1, 2, 3, \cdots, k-1 \\ a & \text{if} \quad v = u_1, u_2, u_4, u_5, u_7, \cdots, u_{n-2}, u_{n-1} \end{cases}$$

We can easily prove that g is an IML of DTS_n.
Conversely suppose that $n = 3k + 1$ or $n = 3k + 2$ for some integer k. If possible suppose f is an IML DTS_n. Then from the degree sum equation of the vertices v_i

we have:

$$\begin{aligned}
f(v_2)+f(u_1)+f(w_1) &= 0\\
f(v_1)+f(v_2)+f(v_3)+f(u_1)+f(u_2)+f(w_1)+f(w_2) &= 0\\
f(v_2)+f(v_3)+f(v_3)+f(u_2)+f(u_3)+f(w_2)+f(w_3) &= 0\\
f(v_3)+f(v_4)+f(v_5)+f(u_3)+f(u_4)+f(w_3)+f(w_4) &= 0\\
&\vdots\\
f(v_{n-2})+f(v_{n-1})+f(v_n)+f(u_{n-2})+f(u_{n-1})+f(w_{n-2})+f(w_{n-1}) &= 0\\
f(v_{n-1})+f(u_{n-1})+f(w_{n-1}) &= 0
\end{aligned}$$

Similarly from the degree sum equation of u_i and w_i we get the following system of equations:

$$\begin{aligned}
f(u_1)+f(v_1)+f(v_2) &= 0\\
f(u_2)+f(v_2)+f(v_3) &= 0\\
f(u_3)+f(v_3)+f(v_4) &= 0\\
&\vdots\\
f(u_{n-1})+f(v_{n-1})+f(v_n) &= 0
\end{aligned}$$

and

$$\begin{aligned}
f(w_1)+f(v_1)+f(v_2) &= 0\\
f(w_2)+f(v_2)+f(v_3) &= 0\\
f(w_3)+f(v_3)+f(v_4) &= 0\\
&\vdots\\
f(w_{n-1})+f(v_{n-1})+f(v_n) &= 0
\end{aligned}$$

On comparing the degree sum equations of u_i and w_i we get $f(u_i)=f(w_i)$, for $i=1,2,3,\cdots,n-1$, and using this in the degree sum equations of v_i we get

$$\begin{aligned}
f(v_2) &= 0\\
f(v_1)+f(v_2)+f(v_3) &= 0\\
f(v_2)+f(v_3)+f(v_3) &= 0\\
f(v_3)+f(v_4)+f(v_5) &= 0\\
&\vdots\\
f(v_{n-2})+f(v_{n-1})+f(v_n) &= 0\\
f(v_{n-1}) &= 0
\end{aligned}$$

On solving this we get $f(v_2) = f(v_5) = f(v_8) = \cdots = 0$, $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots$. On substituting this in the degree sum equation of w_i we get $f(w_3) = f(w_6) = f(w_9) = \cdots = 0, f(w_{n-3}) = f(w_{n-6}) = f(w_{n-9}) = \cdots = 0$ and $f(w_1) = f(w_2) = f(w_4) = f(w_{n-5}) \cdots = f(v_1) = f(w_{n-1}) = f(w_{n-2}) = f(w_{n-4}) = f(w_{n-5}) = \cdots = f(v_1)$

Case 1 : $n = 3k + 1$ for some integer k.

In this case, the above conclusion of the system of equations become, $f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{n-2}) = 0$ and $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = f(v_3) = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots = f(v_{n-1}) = f(v_n)$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots = f(v_4) = f(v_2) = f(v_1)$. Thus $f \equiv 0$.

Case 2 : $n = 3k + 2$ for some integer k.

In this case, the above conclusion of the system of equations become, $f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{n-3}) = f(v_n) = 0$ and $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = f(v_4) = f(v_1) = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots = f(v_{n-2}) = f(v_{n-1})$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots = f(v_3) = f(v_2)$. Thus $f \equiv 0$.

Hence in both cases we get $f \equiv 0$, which is a contradiction. Hence the proof.

Corollary 5.9. *The double triangular snake graph* $DTS_n \in \Gamma_{k,0}(V_4)$ *if and only if* $n \equiv 0 \ (mod\ 3)$.
Proof. Proof follows from Theorem 5.8.

Definition 5.10. *An open ladder* $O(L_n), n \geq 2$ *graph is obtained from two paths of length* $n-1$ *with* $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ *and* $E(G) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 2 \leq i \leq n-1\}$.

Theorem 5.11. *For* $n \geq 2$, *the open ladder,* $O(L_n) \in \Gamma(V_4)$ *for* $n \equiv 0 \ (mod\ 3)$.
Proof. Consider an open ladder $O(L_n), n \geq 2$, with vertex set $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E(G) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 2 \leq i \leq n-1\}$. Then for $n \equiv 0 \ (mod\ 3)$, define $f : V(O(L_n)) \to V_4$ as follows:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u_2, u_5, u_8, \cdots, u_{n-1} \\ 0 & \text{if} \quad v = v_2, v_5, v_8, \cdots, v_{n-1} \\ a & \text{if} \quad \text{otherwise.} \end{cases}$$

Then f is an IML of $O(L_n)$. Hence the proof follows.

Corollary 5.12. *For* $n \geq 2$, *the open ladder,* $O(L_n) \in \Gamma_{k,0}(V_4)$ *for* $n \equiv 0\ (mod\ 3)$.
Proof. Proof follows from Theorem 5.11.

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