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ON CERTAIN SUMMATION FORMULAE FOR q-HYPERGEOMETRIC SERIES

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Abstract: In this paper, making use of a transformation formula of basic bilateral q series due to Bailey, certain summation formulae of basic bilateral series have been established.

Keywords and Phrases: q-hypergeometric series, q-bilateral hypergeometric series, transformation formula, summation formula.

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1. Introduction, Notations and Definitions

Let q be a fixed complex parameter with $0 < |q| < 1$. The $q-$ shifted factorial is defined for any complex parameter 'a' by

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r), \quad (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where k is any integer.
For brevity, we write

$$(a_1, a_2, ..., a_r; q)_n = (a_1;q)_n(a_2;q)_n...(a_r;q)_n.$$

Further, recall the definition of basic hypergeometric series

$$_r\Phi_{r-1}\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_{r-1} \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_{r-1}; q)_n}, \tag{1.1}$$

and basic bilateral hypergeometric series

$$ {}_r\Psi_r\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_r \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(b_1, b_2, ..., b_r; q)_n}. \tag{1.2} $$

We shall make use of following results in our analysis. Bailey's transformation formula,

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha, \beta; q; z \\ \gamma, \delta \end{array}\right] = \frac{(\alpha z, \beta z, \gamma q/\alpha\beta z, \delta q/\alpha\beta z; q)_\infty}{(q/\alpha, q/\beta, \gamma, \delta; q)_\infty}\, {}_2\Psi_2\left[\begin{array}{l} \alpha\beta z/\gamma, \alpha\beta z/\delta; q; \frac{\gamma\delta}{\alpha\beta z} \\ \alpha z, \beta z \end{array}\right], \tag{1.3} $$

where $\left|\frac{\gamma\delta}{\alpha\beta}\right| < |z| < 1$.

[Andrews and Berndt 1;(12.4.1) p. 273]

Ramanujan's summation formula

$$ {}_1\Psi_1\left[\begin{array}{l} a; q; z \\ b \end{array}\right] = \frac{(az, q/az, q, b/a; q)_\infty}{(b, q/a, z, b/az; q)_\infty}. \tag{1.4} $$

[Gasper and Rahman 4; App. II (II.29), p. 357]

2. Summation Formulas

In this section we establish summation formulas for basic bilateral hypergeometric series.

(i) Putting $z = q/\alpha$ in (1.3) we find,

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha, \beta; q; q/\alpha \\ \gamma, \delta \end{array}\right] = \frac{(\beta q/\alpha, \gamma/\beta, \delta/\beta; q)_\infty (q; q)_\infty}{(q/\alpha, q/\beta, \gamma, \delta; q)_\infty}\, {}_2\Phi_1\left[\begin{array}{l} \beta q/\alpha, \beta q/\delta; q; \frac{\gamma\delta}{\beta q} \\ \beta q/\alpha \end{array}\right]. \tag{2.1} $$

(ii) Again, taking $\delta = \beta q$ in (2.1) we get the summation formula,

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha, \beta; q; q/\alpha \\ \gamma, \beta q \end{array}\right] = \frac{(\beta q/\alpha, \gamma/\beta; q)_\infty (q; q)_\infty^2}{(q/\alpha, q/\beta, \gamma, \beta q; q)_\infty}, \tag{2.2} $$

which is a known result due to Bhargava and Adiga [2].
Since

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha, \beta; q; 1/\alpha \\ \gamma, \beta q \end{array}\right] - \beta\, {}_2\Psi_2\left[\begin{array}{l} \alpha, \beta; q; q/\alpha \\ \gamma, \beta q \end{array}\right] = (1-\beta)\, {}_1\Psi_1\left[\begin{array}{l} \alpha; q; 1/\alpha \\ \gamma \end{array}\right]. $$

From (1.4) we find,

$$ {}_1\Psi_1\left[\begin{array}{l}\alpha;q;1/\alpha\\ \gamma\end{array}\right]=0. \quad \text{So,} \quad {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;1/\alpha\\ \gamma,\beta q\end{array}\right]=\beta\ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]. \quad (2.3) $$

Now, making use of (2.2), (2.3) yields the summation formula,
(iii)

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;1/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{\beta(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \quad (2.4) $$

Again, let us consider,

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]-\ \beta\ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^2/\alpha\\ \gamma,\beta q\end{array}\right] $$

$$ =\sum_{n=-\infty}^{\infty}\frac{(\alpha;q)_n(1-\beta)}{(\gamma;q)_n}\left(\frac{q}{\alpha}\right)^n=(1-\beta)\ {}_1\Psi_1\left[\begin{array}{l}\alpha;q;q/\alpha\\ \gamma\end{array}\right]. $$

From (1.4) we find,

$$ {}_1\Psi_1\left[\begin{array}{l}\alpha;q;q/\alpha\\ \gamma\end{array}\right]=0 \qquad \text{So,} \qquad {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^2/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{1}{\beta}\ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]. \quad (2.5) $$

Making use of (2.2) we have,
(iv)

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^2/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{1}{\beta}\frac{(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \quad (2.6) $$

Iterating this process, we have
(v)

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^{k+1}/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{1}{\beta^k}\frac{(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \quad (2.7) $$

where $|\gamma|<|q^k|<|\alpha|$.
Further, let us consider,

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]=\sum_{n=-\infty}^{\infty}\frac{(\alpha,\beta;q)_n(1/\alpha)^n}{(\gamma,\beta q;q)_n(1-c)}-\sum_{n=-\infty}^{\infty}\frac{(\alpha,\beta;q)_n(q/\alpha)^n c}{(\gamma,\beta q;q)_n(1-c)} $$

$$ =\frac{1}{(1-c)}\ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;1/\alpha\\ \gamma,\beta q\end{array}\right]-\frac{c}{(1-c)}\ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]. \quad (2.8) $$

(vi) Making use of (2.2) and (2.4) in (2.8) we get,

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \tag{2.9}$$

which is known summation [Exton 3; App. A (A.25), p. 305].
(vii) Again, if we consider,

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]-\beta\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]=(1-\beta)\,{}_2\Psi_2\left[\begin{array}{l}\alpha,cq;q;1/\alpha\\ \gamma,c\end{array}\right]. \tag{2.10}$$

Making use of (1.4) it is easy to show that

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,cq;q;1/\alpha\\ \gamma,c\end{array}\right]=0 \quad \text{So,}\quad {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta}\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]. \tag{2.11}$$

(viii) Making use of (2.9) in (2.11) we get the summation formula

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta}\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \tag{2.12}$$

which is also a known result [Exton 3; App. A (A.24), p. 305].
Proceeding as above

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]-\beta\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^2/\alpha\\ \gamma,\beta q,c\end{array}\right]=(1-\beta)\,{}_2\Psi_2\left[\begin{array}{l}\alpha,cq;q;q/\alpha\\ \gamma,c\end{array}\right]. \tag{2.13}$$

${}_2\Psi_2$ on the right hand side of (2.13) also vanishes, so;

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^2/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta}\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]. \tag{2.14}$$

(ix) Making use of (2.12) we have

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^2/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta^2}\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.15}$$

(x) Iterating the above process, we have

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^k/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta^k}\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.16}$$

Now, let us consider

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{1-d}\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c\end{array}\right]-\frac{d}{1-d}\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c\end{array}\right]. \tag{2.17}$$

Making use of (2.9) and (2.12) in (2.17) we get,
(xi)

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{(1-\beta/c)(1-\beta/d)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{\beta(1-1/c)(1-1/d)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.18}$$

Again,

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]-\beta\,{}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=(1-\beta)\,{}_3\Psi_3\left[\begin{array}{l}\alpha,cq,dq;q;\frac{1}{\alpha}\\ \gamma,c,d\end{array}\right]. \tag{2.19}$$

Using (1.4) it is easy to show that

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,cq,dq;q;\frac{1}{\alpha}\\ \gamma,c,d\end{array}\right]=0 \text{ So, } {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{\beta}\,{}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]. \tag{2.20}$$

Making use of (2.18) in (2.20) we get,
(xii)

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{\beta^2}\frac{(1-\beta/c)(1-\beta/d)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(1-1/d)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.21}$$

Iterating this process, we have

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q^k}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{\beta^{k+1}}\frac{(1-\beta/c)(1-\beta/d)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(1-1/d)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.22}$$

Also, proceeding as above, it is easy to show that,
(xiv)

$$ {}_5\Psi_5\left[\begin{array}{l}\alpha,\beta,cq,dq,eq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d,e\end{array}\right]=\frac{1}{\beta^2}\frac{\left(1-\frac{\beta}{c}\right)\left(1-\frac{\beta}{d}\right)\left(1-\frac{\beta}{e}\right)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{\left(1-\frac{1}{c}\right)\left(1-\frac{1}{d}\right)\left(1-\frac{1}{e}\right)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.23}$$

Iterating the process, we can show that,
(xv)

$$ {}_{r+3}\Psi_{r+3}\left[\begin{array}{l}\alpha,\beta,cq,c_1q,c_2q,...,c_rq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,c_1,c_2,...,c_r\end{array}\right]$$

$$= \frac{1}{\beta^r} \frac{\left(1-\frac{\beta}{c}\right)\left(1-\frac{\beta}{c_1}\right)\dots\left(1-\frac{\beta}{c_r}\right)(\beta q/\alpha, \gamma/\beta; q)_\infty (q;q)_\infty^2}{\left(1-\frac{1}{c}\right)\left(1-\frac{1}{c_1}\right)\dots\left(1-\frac{1}{c_r}\right)(q/\alpha, q/\beta, \gamma, \beta q; q)_\infty}. \tag{2.24}$$

3. Special Cases

In this section we shall deduce certain interesting summation formulae from the results established in previous section.

(i) Taking $\gamma = \alpha q$ in (2.2) we get,

$${}_2\Psi_2 \left[\begin{array}{l} \alpha, \beta; q; q/\alpha \\ \alpha q, \beta q \end{array} \right] = \frac{(\beta q/\alpha, \alpha q/\beta; q)_\infty (q;q)_\infty^2}{(q/\alpha, q/\beta, \alpha q, \beta q; q)_\infty}. \tag{3.1}$$

(ii) As $\alpha \to \infty$, (2.2) yields

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(\gamma;q)_n (1-\beta q^n)} = \frac{(\gamma/\beta; q)_\infty (q;q)_\infty^2}{(q/\beta, \gamma, \beta; q)_\infty}. \tag{3.2}$$

(iii) Taking $\gamma = q$ in (3.2) we find,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n (1-\beta q^n)} = \frac{(q;q)_\infty}{(\beta;q)_\infty}. \tag{3.3}$$

(iv) Replacing q by q^2 and then taking $\beta = q$ in (3.3) we get,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n (1-q^{2n+1})} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \Psi(q), \tag{3.4}$$

where $\Psi(q)$ is a function given in [Andrews, G.E. and Berndt, B.C. 1; (1.1.7), p. 11].

(v) Taking $\gamma = q$ in (2.4) we find,

$${}_2\Phi_1 \left[\begin{array}{l} \alpha, \beta; q; 1/\alpha \\ \beta q \end{array} \right] = \frac{\beta(\beta q/\alpha; q)_\infty (q;q)_\infty}{(q/\alpha, \beta q; q)_\infty}. \tag{3.5}$$

(vi) For $\alpha \to \infty$, (3.5) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n (1-\beta q^n)} = \frac{\beta(q;q)_\infty}{(\beta;q)_\infty}. \tag{3.6}$$

(vii) Taking $\gamma = \alpha q$ in (2.4) we find,

$$\sum_{n=-\infty}^{\infty} \frac{(1/\alpha)^n}{(1-\alpha q^n)(1-\beta q^n)} = \frac{\beta(\beta q/\alpha, \alpha q/\beta; q)_\infty (q;q)_\infty^2}{(q/\alpha, q/\beta, \alpha, \beta; q)_\infty}. \tag{3.7}$$

(viii) Taking $\alpha \to \infty$, $\gamma = 0$ in (2.2) we get,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1-\beta q^n)} = \frac{(q;q)_\infty^2}{(q/\beta,\beta;q)_\infty}. \tag{3.8}$$

(ix) Taking $\alpha \to \infty$ and $\gamma = 0$ in (2.4) we get

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1-\beta q^n)} = \frac{\beta(q;q)_\infty^2}{(q/\beta,\beta;q)_\infty}. \tag{3.9}$$

(x) Taking $\gamma = q$ in (2.9) we find,

$${}_3\Phi_2\left[\begin{array}{l} \alpha,\beta,cq;q;\frac{1}{\alpha} \\ \beta q, c \end{array}\right] = \frac{\left(1-\frac{\beta}{c}\right)(\beta q/\alpha;q)_\infty (q;q)_\infty}{\left(1-\frac{1}{c}\right)(q/\alpha,\beta q;q)_\infty}. \tag{3.10}$$

(xi) Taking $\alpha \to \infty$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}(1-cq^n)}{(q;q)_n(1-\beta q^n)} = \frac{(1-c)(1-/\beta/c)(q;q)_\infty}{(1-1/c)(\beta;q)_\infty}. \tag{3.11}$$

(xii) As $c \to \infty$, (3.10) yields

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n (q/\alpha)^n}{(q;q)_n(1-\beta q^n)} = \frac{(\beta q/\alpha,q;q)_\infty}{(q/\alpha,\beta;q)_\infty}. \tag{3.12}$$

(xiii) Taking q^8 for q and then $\alpha = \beta = q$ in (3.12) we get,

$$\sum_{n=0}^{\infty} \frac{(q;q^8)_n q^{7n}}{(q^8;q^8)_n(1-q^{8n+1})} = \frac{(q^8;q^8)_\infty^2}{(q,q^7;q^8)_\infty}. \tag{3.13}$$

(xiv) Taking q^8 for q and then $\alpha = \beta = q^3$ in (3.12) we have

$$\sum_{n=0}^{\infty} \frac{(q^3;q^8)_n q^{5n}}{(q^8;q^8)_n(1-q^{8n+3})} = \frac{(q^8;q^8)_\infty^2}{(q^3,q^5;q^8)_\infty}. \tag{3.14}$$

(xv) Taking the ratio of (3.13) and (3.14) and using the result [Andrew and Berndt 1; (6.2.38), p. 154] we get

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(q^3;q^8)_n q^{5n}}{(q^8;q^8)_n(1-q^{8n+3})}}{\displaystyle\sum_{n=0}^{\infty} \frac{(q;q^8)_n q^{7n}}{(q^8;q^8)_n(1-q^{8n+1})}} = \frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty} = \frac{1}{1+}\,\frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}. \tag{3.15}$$

(xvi) Replacing q by q^2 and then taking $\alpha = \beta = q$ in (3.12) we find,

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(q^2;q^2)_n(1-q^{2n+1})} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} = \Psi^2(q), \tag{3.16}$$

(xvii) Replacing q by q^6 and then $\alpha = \beta = q$ in (3.12) we get,

$$\sum_{n=0}^{\infty} \frac{(q;q^6)_n q^{5n}}{(q^6;q^6)_n(1-q^{6n+1})} = \frac{(q^6;q^6)_\infty^2}{(q,q^5;q^6)_\infty}. \tag{3.17}$$

(xviii) Replacing q by q^6 and then $\alpha = \beta = q^3$ in (3.12) we obtain

$$\sum_{n=0}^{\infty} \frac{(q^3;q^6)_n q^{3n}}{(q^6;q^6)_n(1-q^{6n+3})} = \frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty}. \tag{3.18}$$

(xv) Taking the ratio of (3.17) and (3.18) and using the result [Andrew and Berndt 1; (6.2.37), p. 154] we find

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(q^3;q^6)_n q^{3n}}{(q^6;q^6)_n(1-q^{6n+3})}}{\displaystyle\sum_{n=0}^{\infty} \frac{(q;q^6)_n q^{5n}}{(q^6;q^6)_n(1-q^{6n+1})}} = \frac{(q,q^5;q^6)_\infty}{(q^3;q^6)_\infty^2} = \frac{1}{1+}\,\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}. \tag{3.19}$$

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