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FEKETE-SZEGÖ TYPE COEFFICIENT INEQUALITIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract: In the present investigation, the authors obtain Fekete-Szegö inequality for certain subclasses of analytic functions on the open unit disk. For these classes, the Fekete-Szegö type defined through fractional derivatives is obtained.

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1. Introduction

Let $\mathcal{M}$ denote the class of all analytic functions which are analytic in the unit disk $\Delta = \{z : z \in \mathbb{C},\ |z| < 1\}$ of the form

$$f(z) = z + \sum_{j\geq 2} a_j z^j, \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{M}$ consisting of univalent functions. Let $\Phi(z)$ be an analytic function with positive real part on Δ with $\Phi(0) = 1$, $\Phi'(0) > 0$ and $Re\ \Phi(z) > 0$ $(z \in \Delta)$ which maps the unit disc Δ onto a starlike region with respect to 1 and is symmetric with respect to the real axis. Let $\mathcal{S}^*(\Phi)$ be the class of functions $f(z) \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \Phi(z), \quad (z \in \Delta)$$

and $\mathcal{C}(\Phi)$ be the class of functions $f(z) \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z), \quad (z \in \Delta).$$

These classes were introduced and studied by Ma and Minda [6]. For a brief history of the Fekete-Szegö problem for class of analytic functions, see the recent paper by Srivastava et al. [14].

It is well known that the n-th coefficient of a univalent function $f \in \mathcal{S}$ is bounded by n. In 1933, Fekete and Szego [4] obtained the sharp bound for $|a_3 - \eta a_2^2|$ as a function of the real parameter η and proved that

$$\left|a_3 - \eta a_2^2\right| \leq 1 + 2\exp\left(-\frac{2\eta}{1-\eta}\right) \quad (0 \leq \eta \leq 1)\,,$$

for functions in the class $\mathcal{S}$. The problem of finding sharp bound for the non-linear functional $|a_3 - \eta a_2^2|$ of any compact family of functions $f \in \mathcal{S}$ is identified as Fekete-Szegö problem. In the recent years many authors have considered the Fekete-Szegö problem for typical classes of univalent functions. For ready reference one can see [2], [3], [7], [8], [9], [10], [12], [13], [14], [19], [20], [5], [15], [16], [17], [18]. Here, we consider the following classes of functions,

$$\begin{aligned}
M_{\lambda,\mu}(\Phi) \ &: = \left\{ f \in \mathcal{M} : (1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{z}{f(z)}\right)^{\mu-1} \prec \Phi(z) \right\}, \\
N_{\lambda,\mu}(\Phi) \ &: = \left\{ f \in \mathcal{M} : (1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \Phi(z) \right\}, \\
L_{\gamma}(\Phi) \ &: = \left\{ f \in \mathcal{M} : \left[(1+e^{i\gamma})\left(\frac{zf'(z)}{f(z)}\right) - e^{i\gamma}\right] \prec \Phi(z) \right\},
\end{aligned}$$

where $\mu \geq 0$, $0 \leq \lambda < 1$ and $\gamma \geq 0$.
We note that

$(i) M_{\lambda,1}(\frac{1+(1-2\beta)z}{1-z}) = Q_{\lambda}(\beta)$(see [11]);
$(ii) N_{1,0}(\Phi) = S^*(\Phi)$ (see [6]).

For a function $f(z) \in \mathcal{S}$ given by (1.1), the $k-$th root transform is defined by:

$$F(z) = \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{j\geq 1} b_{kj+1} z^{kj+1}. \tag{1.2}$$

Remark 1. *Set $k = 1$, in (1.2) $F(z)$ reduce to the function f it self.*

In the present paper, we derive the Fekete-Szegö inequality for the classes $M_{\lambda,\mu}(\Phi)$, $N_{\lambda,\mu}(\Phi)$ and $L_\gamma(\Phi)$ which we define above. In order to derive our main results, we need the following lemmas.

Lemma 1. [1] *If*

$$w(z) = w_1 z + w_2 z^2 + \cdots \tag{1.3}$$

is an analytic function with positive real part in Δ. Then,

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t & \text{if } t \leq -1 \\ 1 & \text{if } -1 \leq t \leq 1 \\ t & \text{if } t \geq 1, \end{cases}$$

when $t < -1$ or $t > 1$, the equality holds if and only if $w(z)$ is z or one of its rotations. If $-1 < t < 1$, then the equality holds if and only if $w(z)$ is z^2 or one of its rotations. If $t = -1$, the equality holds if and only if $w(z) = z\frac{\nu+z}{1+\nu z}(0 \leq \nu \leq 1)$ or one of its rotations. If $t = 1$, the equality holds if and only if $w(z) = -z\frac{\nu+z}{1+\nu z}(0 \leq \nu \leq 1)$ or one of its rotations.

Lemma 2. [4] *(see also [10]) If $w(z)$ of the form* (1.3) *then, for any complex number t,*

$$\left|w_2 - t w_1^2\right| \leq max\left\{1;\ |t|\right\},$$

and the result is sharp for the functions defined by $w(z) = z^2$ or $w(z) = z$.

2. Fekete-Szegö Problem

Theorem 1. *Let $f(z)$ be defined by* (1.1). *Assume that*

$$\Phi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \dots\ . \tag{2.1}$$

If $f \in M_{\lambda,\mu}(\Phi)$, and F is the $k-$th root transformation of f given by (1.2), *then for any complex number γ*

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \leq \begin{cases} -\left(\frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k^2[\mu(1-2\lambda)+3\lambda]^2[2\lambda(2-\mu)+\mu]}\right) A_1^2 + \frac{A_2}{[2\lambda(2-\mu)+\mu]} & \text{if } \eta \leq \sigma_1, \\ \frac{A_1}{k[2\lambda(2-\mu)+\mu]} & \text{if } \sigma_1 \leq \eta \leq \sigma_2, \\ \left(\frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k^2[\mu(1-2\lambda)+3\lambda]^2[2\lambda(2-\mu)+\mu]}\right) A_1^2 - \frac{A_2}{[2\lambda(2-\mu)+\mu]} & \text{if } \eta \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 \ : \ = \frac{[\mu(1-2\lambda)+3\lambda]^2}{[2\lambda(2-\mu)+\mu]}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}-1\right)+\frac{(\mu-1)(\mu-4\lambda)}{[2\lambda(2-\mu)+\mu]}+\frac{(k-1)}{2k}\right],$$
$$\sigma_2 \ : \ = \frac{[\mu(1-2\lambda)+3\lambda]^2}{[2\lambda(2-\mu)+\mu]}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}+1\right)+\frac{(\mu-1)(\mu-4\lambda)}{[2\lambda(2-\mu)+\mu]}+\frac{(k-1)}{2k}\right],$$

and

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \le \frac{A_1}{k[2\lambda(2-\mu)+\mu]}\max\{1; \ |\upsilon|\}.$$

$$\upsilon = \frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k[\mu(1-2\lambda)+3\lambda]^2}-\frac{A_2}{A_1}.$$

Proof. If $f \in M_{\lambda,\mu}(\Phi)$, there exists a function $w(z)$ of the form (1.3) so that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{z}{f(z)}\right)^{\mu-1}=\Phi(w(z)). \tag{2.2}$$

Since

$$\left(\frac{f(z)}{z}\right)^{\mu}=1+\mu a_2 z+\mu\left(a_3+\frac{(\mu-1)}{2}a_2^2\right)z^2+\cdots \tag{2.3}$$

$$\left(\frac{z}{f(z)}\right)^{\mu-1}=1-(\mu-1)a_2z+(\mu-1)\left(\frac{\mu}{2}a_2^2-a_3\right)z^2+\cdots \tag{2.4}$$

$$\begin{aligned}&(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{z}{f(z)}\right)^{\mu-1}\\ = \ & 1+[\mu(1-2\lambda)+3\lambda]a_2z+\left[\frac{(\mu-1)(\mu-4\lambda)}{2}a_2^2+[2\lambda(2-\mu)+\mu]a_3\right]z^2+\cdots\end{aligned}$$

and

$$\Phi(w(z))=1+A_1w_1z+\left(A_1w_2+A_2w_1^2\right)z^2+\cdots.$$

From (2.2), we obtain

$$[\mu(1-2\lambda)+3\lambda]a_2=A_1w_1 \tag{2.5}$$

and

$$[2\lambda(2-\mu)+\mu]a_3=\left[A_1w_2+\left(A_2-\frac{(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2}A_1^2\right)w_1^2\right]. \tag{2.6}$$

For $f \in \mathcal{M}$ defined by (1.1), it is easy to show that

$$\left[f(z^k)\right]^{\frac{1}{k}} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{(k-1)}{k^2}a_2^2\right) z^{2k+1} + \cdots . \tag{2.7}$$

By using (1.2) and (27), we have got

$$b_{k+1} = \frac{1}{k}a_2, \tag{2.8}$$

also

$$b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{(k-1)}{k^2}a_2^2. \tag{2.9}$$

Using (2.5) and (2.6) in (2.8) and (2.9), we obtain

$$b_{k+1} = \frac{A_1 w_1}{k[\mu(1-2\lambda)+3\lambda]}$$

and

$$b_{2k+1} = \frac{1}{k[2\lambda(2-\mu)+\mu]}\left[A_1 w_2 + A_2 w_1^2\right.$$

$$+\left(\frac{-2k(\mu-1)(\mu-4\lambda)-(k-1)[2\lambda(2-\mu)+\mu]}{2k[\mu(1-2\lambda)+3\lambda]^2}\right)A_1^2 w_1^2],$$

and hence,

$$b_{2k+1} - \eta b_{k+1}^2 = \frac{A_1}{k[2\lambda(2-\mu)+\mu]}\left\{w_2 - \upsilon w_1^2\right\},$$

where

$$\upsilon = \frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k[\mu(1-2\lambda)+3\lambda]^2} - \frac{A_2}{A_1}.$$

The first part is obtained by applying Lemma 1.

If

$$[(\frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k[\mu(1-2\lambda)+3\lambda]^2})A_1 - \frac{A_2}{A_1}] \leq -1,$$

then,

$$\eta \leq \frac{1}{2[2\lambda(2-\mu)+\mu]}[2[\mu(1-2\lambda)+3\lambda]^2\frac{k}{A_1}(\frac{A_2}{A_1}-1)$$

$$-(2\lambda(2-\mu)+\mu)(k-1) - 2k(\mu-1)(\mu-4\lambda)] \quad i.e., \ \eta \leq \sigma_1,$$

and Lemma 1 gives:

$$\begin{aligned}\left|b_{2k+1}-\eta b_{k+1}^2\right| &\leq \left(\frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k^2[\mu(1-2\lambda)+3\lambda]^2[2\lambda(2-\mu)+\mu]}\right)A_1^2\\ &\quad -\frac{A_2}{[2\lambda(2-\mu)+\mu]}\quad .\end{aligned}$$

For

$$-1\leq [(\frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k[\mu(1-2\lambda)+3\lambda]^2})A_1-\frac{A_2}{A_1}]\leq 1,$$

we have got,

$$\begin{aligned}&\frac{1}{[2\lambda(2-\mu)+\mu]}[2[\mu(1-2\lambda)+3\lambda]^2\frac{k}{A_1}(\frac{A_2}{A_1}-1)\\ &-[2\lambda(2-\mu)+\mu](k-1)-2k(\mu-1)(\mu-4\lambda)]\\ \leq\ &\eta\\ \leq\ &\frac{1}{[2\lambda(2-\mu)+\mu]}[2[\mu(1-2\lambda)+3\lambda]^2\frac{k}{A_1}(\frac{A_2}{A_1}+1)\\ &-(2\lambda(2-\mu)+\mu)(k-1)-2k(\mu-1)(\mu-4\lambda)]\end{aligned}$$

and Lemma 1 yields:

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\leq\frac{A_1}{k[2\lambda(2-\mu)+\mu]}.$$

For

$$[(\frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k[\mu(1-2\lambda)+3\lambda]^2})A_1-\frac{A_2}{A_1}]\geq 1,$$

we have,

$$\begin{aligned}\eta\ &\geq\ \frac{1}{[2\lambda(2-\mu)+\mu]}[2[\mu(1-2\lambda)+3\lambda]^2\frac{k}{A_1}(\frac{A_2}{A_1}+1)\\ &\quad -(2\lambda(2-\mu)+\mu)(k-1)-2k(\mu-1)(\mu-4\lambda)\ ]\ \ i.e.,\\ \eta\ &\geq\ \sigma_2.\end{aligned}$$

Applying Lemma 1, we have got

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\leq -\left(\frac{2k(\mu-1)(\mu-4\lambda)+[(k-1)+2\eta][2\lambda(2-\mu)+\mu]}{2k^2[\mu(1-2\lambda)+3\lambda]^2[2\lambda(2-\mu)+\mu]}\right)A_1^2$$

$$+\frac{A_2}{[2\lambda(2-\mu)+\mu]}.$$

The second part follows by applying Lemma 2

$$\begin{aligned}\left|b_{2k+1}-\eta b_{k+1}^2\right| &= \frac{A_1}{k[2\lambda(2-\mu)+\mu]}\left|w_2-\upsilon w_1^2\right| \\ &\leq \frac{A_1}{k[2\lambda(2-\mu)+\mu]}\max\{1;\ |\upsilon|\}.\end{aligned}$$

Theorem 2. *Let* $f(z)$ *be given by* (1.1). *Assume that* $\Phi(z)$ *is defined by* (2.1). *If* $f\in N_{\lambda,\mu}(\Phi)$, *and* F *is defined by* (1.2), *then for any complex number* η

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \leq \begin{cases} \frac{(2\eta+k\mu-1)}{2k^2(\mu+\lambda)^2}A_1^2 - \frac{A_2}{k(\mu+2\lambda)} & \text{if } \eta\leq\sigma_1, \\ \frac{A_1}{k(\mu+2\lambda)} & \text{if } \sigma_1\leq\eta\leq\sigma_2, \\ \frac{-(2\eta+k\mu-1)}{2k^2(\mu+\lambda)^2}A_1^2 + \frac{A_2}{k(\mu+2\lambda)} & \text{if } \eta\geq\sigma_2 \end{cases}$$

where

$$\begin{aligned}\sigma_1 &:= \frac{(\mu+\lambda)^2}{(\mu+2\lambda)}\frac{k}{A_1}(\frac{A_2}{A_1}-1)-\mu k+1, \\ \sigma_2 &:= \frac{(\mu+\lambda)^2}{(\mu+2\lambda)}\frac{k}{A_1}(\frac{A_2}{A_1}+1)-\mu k+1,\end{aligned}$$

and

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \leq \frac{A_1}{k(\mu+2\lambda)}\max\left\{1;\ \left|\left(\frac{(\mu+2\lambda)}{2k(\mu+\lambda)^2}(1-k\mu-2\eta)A_1+\frac{A_2}{A_1}\right)\right|\right\}.$$

Proof. Let $f\in N_{\lambda,\mu}(\Phi)$, there exists a function $w(z)$ given by (1.3) so that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=\phi(w(z)). \tag{2.10}$$

Since

$$\left(\frac{f(z)}{z}\right)^{\mu}=1+\mu a_2 z+\mu\left(a_3+\frac{(\mu-1)}{2}a_2^2\right)z^2+\cdots \tag{2.11}$$

$$\left(\frac{f(z)}{z}\right)^{\mu-1}=1+(\mu-1)a_2 z+(\mu-1)\left(a_3+\frac{(\mu-2)}{2}a_2^2\right)z^2+\cdots \tag{2.12}$$

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \tag{2.13}$$
$$=1+(\mu+\lambda)a_2z+(\mu+2\lambda)\left(a_3+\frac{(\mu-1)}{2}a_2^2\right)z^2+\cdots$$

and

$$\phi(w(z))=1+A_1w_1z+\left(A_1w_2+A_2w_1^2\right)z^2+...,$$

by using (2.13), we have got

$$a_2=\frac{A_1w_1}{(\mu+\lambda)} \tag{2.14}$$

and

$$a_3=\frac{1}{(\mu+2\lambda)}[A_1w_2+(A_2-\frac{(\mu+2\lambda)(\mu-1)}{2(\mu+\lambda)^2}A_1^2)w_1^2]. \tag{2.15}$$

Using (2.14) and (2.15) in (2.8) and (2.9), it follows:

$$b_{k+1}=\frac{A_1w_1}{k(\mu+\lambda)},$$

and

$$b_{2k+1}=\frac{1}{k(\mu+2\lambda)}\left[A_1w_2+A_2w_1^2-\frac{(\mu+2\lambda)}{2(\mu+\lambda)^2}((\mu-1)+\frac{(k-1)}{k})A_1^2w_1^2\right],$$

and hence

$$b_{2k+1}-\eta b_{k+1}^2=\frac{A_1}{k(\mu+2\lambda)}\left\{w_2-\left(\frac{(\mu+2\lambda)}{2k(\mu+\lambda)^2}(k\mu+2\eta-1)A_1-\frac{A_2}{A_1}\right)w_1^2\right\}. \tag{2.16}$$

Applying Lemma 2 yields:

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|=\frac{A_1}{k(\mu+2\lambda)}\left|w_2-\left(\frac{(\mu+2\lambda)}{2k(\mu+\lambda)^2}(1-k\mu-2\eta)A_1-\frac{A_2}{A_1}\right)w_1^2\right|$$
$$\leq\frac{A_1}{k(\mu+2\lambda)}\max\left\{1;\ \left|\left(\frac{(\mu+2\lambda)}{2k(\mu+\lambda)^2}(1-k\mu-2\eta)A_1-\frac{A_2}{A_1}\right)\right|\right\}.$$

Theorem 3. *Let $f(z)$ be given by (1.1). Assume that $\phi(z)$ is defined by (2.1). If $f\in L_\gamma(\Phi)$, and F is defined by (1.2), then for any complex number η*

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\leq\frac{A_1}{2k\left|1+e^{i\gamma}\right|}\max\left\{1;\ \left|\frac{A_1}{k(1+e^{i\gamma})}(1-2\eta)-\frac{A_2}{A_1}\right|\right\}.$$

Proof. If f belongs to $L_\gamma(\Phi)$, there exists a function $w(z)$ of the form (1.3) so that

$$(1+e^{i\gamma})\left(\frac{zf'(z)}{f(z)}\right)-e^{i\gamma}=\Phi(w(z)).$$

Since

$$(1+e^{i\gamma})\left(\frac{zf'(z)}{f(z)}\right)-e^{i\gamma}=1+(1+e^{i\gamma})a_2z+(1+e^{i\gamma})(2a_3-a_2^2)z^2+\cdots, \quad (2.17)$$

and

$$\Phi(w(z))=1+A_1w_1z+\left(A_1w_2+A_2w_1^2\right)z^2+....$$

It follows from (2.17) that

$$a_2=\frac{A_1w_1}{(1+e^{i\gamma})}, \quad (2.18)$$

and

$$a_3=\frac{1}{2(1+e^{i\gamma})}\left[A_1w_2+A_2w_1^2+\frac{A_1^2w_1^2}{(1+e^{i\gamma})}\right]. \quad (2.19)$$

Using (2.18) and (2.19) in (2.8) and (2.9), we get

$$b_{k+1}=\frac{A_1w_1}{k(1+e^{i\gamma})},$$

and

$$b_{2k+1}=\frac{1}{2k(1+e^{i\gamma})}\left[A_1w_2+A_2w_1^2+\frac{A_1^2w_1^2}{k(1+e^{i\gamma})}\right],$$

and hence,

$$b_{2k+1}-\eta b_{k+1}^2=\frac{A_1}{2k(1+e^{i\gamma})}\left\{w_2-\left[\frac{A_1}{k(1+e^{i\gamma})}(2\eta-1)-\frac{A_2}{A_1}\right]w_1^2\right\},$$

applying Lemma 2 yields:

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\le\frac{A_1}{2k\left|1+e^{i\gamma}\right|}\max\left\{1;\ \left|\frac{A_1}{k(1+e^{i\gamma})}(1-2\eta)-\frac{A_2}{A_1}\right|\right\}.$$

3. Fekete-Szegö problem for z/f

In this section, the Fekete-Szegö type coefficient inequalities associated with the rational function Ψ of the form

$$\Psi(z)=z/f(z)=1+\sum_{j\ge1}d_jz^j, \quad (3.1)$$

where f belongs to the classes $M_{\lambda,\mu}(\Phi)$, $N_{\lambda,\mu}(\Phi)$ and $L_\gamma(\Phi)$ are derived.

Theorem 4. *Let $f(z)$ be given by (1.1). Assume that $\Phi(z)$ is defined by (2.1). If $f \in M_{\lambda,\mu}(\Phi)$, and Ψ is defined by (3.1), then for any complex number η*

$$\left|d_2 - \eta d_1^2\right| \leq \begin{cases} \left(\frac{[2\lambda(2-\mu)+\mu](1-\eta)+(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2[2\lambda(2-\mu)+\mu]}\right) A_1^2 - \frac{A_2}{[2\lambda(2-\mu)+\mu]} & \text{if } \eta \leq \sigma_1, \\ -\frac{A_1}{[2\lambda(2-\mu)+\mu]} \ , & \text{if } \sigma_1 \leq \eta \leq \sigma_2, \\ -\left(\frac{[2\lambda(2-\mu)+\mu](1-\eta)+(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2[2\lambda(2-\mu)+\mu]}\right) A_1^2 + \frac{A_2}{[2\lambda(2-\mu)+\mu]}, & \text{if } \eta \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 \ : \ = \frac{[\mu(1-2\lambda)+3\lambda]^2}{A_1}\left(\frac{A_2}{A_1} - 1\right) - [2\lambda(2-\mu)+\mu] - (\mu-1)(\mu-4\lambda),$$
$$\sigma_2 \ : \ = \frac{[\mu(1-2\lambda)+3\lambda]^2}{A_1}\left(\frac{A_2}{A_1} + 1\right) - [2\lambda(2-\mu)+\mu] - (\mu-1)(\mu-4\lambda),$$

and

$$\left|d_2 - \eta d_1^2\right| \leq \frac{A_1}{[2\lambda(2-\mu)+\mu]} \max\left\{1; \left|\left(\frac{[2\lambda(2-\mu)+\mu](1-\eta)+(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2}\right) A_1 - \frac{A_2}{A_1}\right|\right\}.$$

Proof. It is easy to see that

$$z/f(z) = 1 - a_2 z + \left(a_2^2 - a_3\right) z^2 + \ldots . \tag{3.2}$$

By using (3.1), (3.2), we have got

$$d_1 = -a_2 \tag{3.3}$$

and

$$d_2 = a_2^2 - a_3. \tag{3.4}$$

Using (2.5) and (2.6) in (3.3) and (3.4), we obtain

$$d_1 = \frac{-A_1 w_1}{[\mu(1-2\lambda)+3\lambda]},$$

and

$$d_2 = \frac{A_1^2 w_1^2}{[\mu(1-2\lambda)+3\lambda]^2} - \frac{1}{[2\lambda(2-\mu)+\mu]} \left[A_1 w_2 + \left(A_2 - \frac{(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2} A_1^2\right) w_1^2\right],$$

and hence,

$$d_2 - \eta d_1^2 = -\frac{A_1}{[2\lambda(2-\mu)+\mu]} \{w_2 - \left[\left(\frac{[2\lambda(2-\mu)+\mu](1-\eta)+(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2}\right) A_1 - \frac{A_2}{A_1}\right] w_1^2\bigg\}.$$

By applying Lemma 1, we have got the first part of the result and by Lemma 2, we get the second result:

$$\left|d_2 - \eta d_1^2\right| = \frac{A_1}{[2\lambda(2-\mu)+\mu]} |w_2 - \left[\left(\frac{[2\lambda(2-\mu)+\mu](1-\eta)+(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2}\right) A_1 - \frac{A_2}{A_1}\right] w_1^2\bigg|,$$

$$\left|d_2 - \eta d_1^2\right| \le \frac{A_1}{[2\lambda(2-\mu)+\mu]} \max\left\{1; \left|\left(\frac{[2\lambda(2-\mu)+\mu](1-\eta)+(\mu-1)(\mu-4\lambda)}{[\mu(1-2\lambda)+3\lambda]^2}\right) A_1 - \frac{A_2}{A_1}\right|\right\}.$$

Theorem 5. *Let* $f(z)$ *be given by* (1.1)*. Assume that* $\phi(z)$ *is defined by* (2.1)*. If* $f \in L_\gamma(\Phi)$*, and* Ψ *is defined by* (3.1)*, then for any complex number* η

$$\left|d_2 - \eta d_1^2\right| \le \frac{A_1}{2\left|1+e^{i\gamma}\right|} \max\left\{1; \left|\frac{A_1}{(1+e^{i\gamma})}(1-2\eta) - \frac{A_2}{A_1}\right|\right\}.$$

Proof. Using (2.18) and (2.19) in (3.3) and (3.4), we have

$$d_1 = \frac{-A_1 w_1}{(1+e^{i\gamma})}$$

and

$$d_2 = \frac{A_1^2 w_1^2}{(1+e^{i\gamma})^2} - \frac{1}{2(1+e^{i\gamma})} \left[A_1 w_2 + \left(A_2 + \frac{A_1^2}{(1+e^{i\gamma})}\right) w_1^2\right],$$

and hence,

$$d_2 - \mu d_1^2 = -\frac{A_1}{2(1+e^{i\gamma})}\left\{w_2 - \left[\frac{A_1}{(1+e^{i\gamma})}(1-2\eta) - \frac{A_2}{A_1}\right] w_1^2\right\}.$$

Lemma 2 gives:

$$\left|d_2 - \mu d_1^2\right| = \frac{A_1}{2\left|1+e^{i\gamma}\right|}\left|w_2 - \left[\frac{A_1}{(1+e^{i\gamma})}(1-2\eta) - \frac{A_2}{A_1}\right] w_1^2\right|,$$

$$\left|d_2 - \mu d_1^2\right| \leq \frac{A_1}{2\left|1+e^{i\gamma}\right|}\max\left\{1;\ \left|\frac{A_1}{(1+e^{i\gamma})}(1-2\eta) - \frac{A_2}{A_1}\right|\right\}.$$

For functions with non-negative derivative.

Theorem 6. *Let $f(z)$ be given by (1.1). Assume that $\Phi(z)$ is defined by (2.1). If $f \in N_{\lambda,\mu}(\Phi)$and Ψ is defined by (3.1), then for any complex number η*

$$\left|d_2 - \eta d_1^2\right| \leq \begin{cases} \frac{A_1^2}{(\mu+\lambda)^2}\left((1-\eta) + \frac{\mu(\mu-1)(\lambda+1)}{2(\mu+\lambda)}\right) - \frac{A_2}{(\mu+\lambda)}, & \text{if } \eta \leq \sigma_1, \\ \frac{-A_1}{(\mu+\lambda)}, & \text{if } \sigma_1 \leq \eta \leq \sigma_2, \\ -\frac{A_1^2}{(\mu+\lambda)^2}\left((1-\eta) + \frac{\mu(\mu-1)(\lambda+1)}{2(\mu+\lambda)}\right) + \frac{A_2}{(\mu+\lambda)}, & \text{if } \eta \geq \sigma_2 \end{cases},$$

where

$$\sigma_1 := 1 + \frac{\mu(\mu-1)(\lambda+1)}{2(\mu+\lambda)} + \frac{(\mu+\lambda)}{A_1} + \frac{(\mu+\lambda)A_2}{A_1^2},$$

$$\sigma_2 := 1 + \frac{\mu(\mu-1)(\lambda+1)}{2(\mu+\lambda)} - \frac{(\mu+\lambda)}{A_1} + \frac{(\mu+\lambda)A_2}{A_1^2},$$

and

$$\left|d_2 - \mu d_1^2\right| \leq \frac{A_1}{(\mu+\lambda)}\max\left\{1;\ \left|\frac{A_1}{(\mu+\lambda)}\left((1-\eta) + \frac{\mu(\mu-1)(\lambda+1)}{2(\mu+\lambda)}\right) - \frac{A_2}{A_1}\right|\right\}.$$

Proof. By using (2.14) and (2.15) in (3.3) and (3.4), we have got

$$d_1 = \frac{-A_1 w_1}{(\mu+\lambda)},\ \text{and } d_2 = \frac{A_1^2 w_1^2}{(\mu+\lambda)^2} - \frac{1}{(\mu+2\lambda)}\left[A_1 w_2 + \left(A_2 - \frac{(\mu+2\lambda)(\mu-1)}{2(\mu+\lambda)^2}A_1^2\right) w_1^2\right],$$

and hence,

$$d_2 - \eta d_1^2 = -\frac{A_1}{(\mu+2\lambda)}\left\{w_2 - \left[\frac{A_1}{(\mu+2\lambda)}\left((1-\eta) - \frac{(\mu+2\lambda)(\mu-1)}{2(\mu+\lambda)}\right) - \frac{A_2}{A_1}\right] w_1^2\right\}.$$

By applying Lemma 1, we have got the first part of the result and by Lemma 2, we get the second result:

$$\left|d_2 - \eta d_1^2\right| = \tfrac{A_1}{(\mu+2\lambda)} \left|w_2 - \left[\tfrac{A_1}{(\mu+2\lambda)} \left((1-\eta) - \frac{(\mu+2\lambda)(\mu-1)}{2\,(\mu+\lambda)}\right) - \tfrac{A_2}{A_1}\right] w_1^2\right|$$

$$\leq \tfrac{A_1}{(\mu+\lambda)} \max\left\{1;\ \left|\tfrac{A_1}{(\mu+\lambda)} \left((1-\eta) - \frac{(\mu+2\lambda)(\mu-1)}{2\,(\mu+\lambda)}\right) - \tfrac{A_2}{A_1}\right|\right\}.$$

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