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k-PARTITIONED FUZZY GRAPH

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Abstract: This paper has introduced a new concept which is partitioning of any fuzzy graph into k-partitioning fuzzy graphs by clustering the vertices of the fuzzy graph. The belongingness of the vertices is arranged as an array and partitioned into k disjoint subsets, such that all subsets have more or less equal sum. Some properties of k-partitioning fuzzy graph by order and size of partitioned fuzzy graph are discussed in this paper.

Keywords and Phrases: Fuzzy graph, 2-partitioned fuzzy graph, 3- partitioned fuzzy graph, 4-partitioned fuzzy graph, k-partitioned fuzzy graph, Strict k-partitioned fuzzy graph.

2010 Mathematics Subject Classification: 47H10.

1. Introduction

Uncertainty has quantified by Zadeh [34] through the concept of Fuzzy set theory in 1965 and Rosenfeld [25] developed the theory of fuzzy graph in 1975 using Kaufman's fuzzy relations. Rosenfeld's theoretical concepts have been studied by several researchers and utilized it for the applications. Connectedness of Fuzzy graphs have been studied by Yeh and Bang [33] and was published in the same year. Bhattacharya [5] discussed some properties of fuzzy graphs in 1987. In the year of 1989, fuzzy cut nodes and end nodes have been studied by Bhutani [6]. The operations on fuzzy graph was dealt with Moderson and Chang-Shyh [20] in 1994. In 2014, Pathinathan and Rosline [24] proposed a new fuzzy graph named double layered fuzzy graph. The proceeding researchers have introduced intuitionistic double layered fuzzy graph [23] in 2015, and they also extended the double layered fuzzy graph concept into the triple layered fuzzy graph [27]. They studied the relationship between the triple layered fuzzy graph and the parental fuzzy graph using order, size and degree of fuzzy graphs. Roseline and Pathinathan [26] constructed the structural core graph of double layered fuzzy graph using a new algorithm. Also they studied its diagrammatic properties. The concept of complement of a fuzzy graph has been modified and its properties have been studied by Sunitha and Vijaykumar [31] in 2012. Degree of vertices of fuzzy graphs has been studied by Nagoorgani and Radha [13] in 2009. In 1986, independent characterization of bipartite graph was studied by Eades and Mckay and Wormald [11]. Takao Asano [2] constructed a bipartite graph with maximum connectivity using degree in the year 1997. Jimmy Salvatore [28] studied on bipartite graph, sub graphs of bipartite graph, bipartite colorings and perfect matching in 2007.

Lakshimi and Ramakrishnan introduced bipartite fuzzy graphs using the concept of spanning subgraph. In 2011, Kittur [17] defined a complete matching of fuzzy bipartite graph and solved assignment problems using it. Poornima defined fuzzy bipartite graph while studying on the association between matching problem and bipartite graph in 2010. Nagoorgani and Rajalaxmi Subahashini [21] defined fuzzy bipartite graph using the concept of spanning fuzzy subgraph in 2014. Jahir Hussain and Kanzul Fathima [14] also used spanning concept to define fuzzy bipartite graph and also defined complete fuzzy bipartite graph in 2015. Mohamad and Suganthi discussed bipartite fuzzy graph in 2017 [19]. In 2017, bipartite fuzzy graph structure and its characteristics were introduced by Mathew Varkey and Shyla [18]. The chaotic number of the bipartite and tripartite graphs proposed by Chiang [7] in 2005.

In 1970, Kernighan and Lin [16] introduced an efficient heuristic procedure for partitioning graph using costs on its edges. George Steiner [30] studied to partition a graph as k-path partition in 2001. Computational studies on the ratio of upper bound to lower bound for the two-partition of a random graphs has been introduced by Donath and Hoffman [10] in 2003. An algorithm for allocating maximal bipartite is constructed by Bershtein and Dziouba [4] in 2000. In 2006, multilevel algorithms are designed for partitioning graph by Abou and Karypis [1]. This study identifies that the partitioning of graph plays a vital role in the applications of mathematics, complex networks, clustering data, image processing, parallel computing, VLSI design etc.

In this paper, we propose a new technique of partitioning a fuzzy graphs into a bi-partitioned fuzzy graph, tri-partitioned fuzzy graph and thus proceeding up to k-partitioned fuzzy graph. In our technique, node set of a fuzzy graph is partitioned into k disjoint subsets such that, each distinct subset has less than or equal sum of its memberships and they cover the vertex set of the crisp graph. We verify properties and theorems using degree, size and derive relevant proofs.

2. Preliminaries

Definition 2.1. [25] Let V be a non-empty subset. A fuzzy graph $\underset{\sim}{G}$ is a pair of functions $\underset{\sim}{G}(\sigma, \mu)$. $\underset{\sim}{G}$ is a set with two functions, $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ such that σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all u, v in V.

Definition 2.2. [12] *The order of a fuzzy graph $\underset{\sim}{G}(\sigma, \mu)$ are defined to be $O(\underset{\sim}{G}) = \sum_{x \in V} \sigma(x)$, where $x \in V$.*

Definition 2.3. [12] *The size of a fuzzy graph $\underset{\sim}{G}(\sigma, \mu)$. $\underset{\sim}{G}$ are defined to be $S(\underset{\sim}{G}) = \sum_{x,y \in V} \mu_{\underset{\sim}{G}}(xy)$, where $xy \in E$.*

Definition 2.4. [21] *The degree of a fuzzy graph for each vertex x in $\underset{\sim}{G}$ is defined as $d_{\underset{\sim}{G}}(x) = \sum_{\substack{y \neq x \\ x,y \in V}} \mu_{\underset{\sim}{G}}(xy)$, where $xy \in E$.*

Definition 2.5. [29] *A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, such that each edge joins a vertex in X and a vertex in Y, such a partition (X, Y) is called a bipartition of the graph. A bipartite graph is also known as a bigraph.*

Definition 2.6. [19] *A complete bipartite graph is a simple graph with bipartition (X, Y) in which each vertex of X is joined to each of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.*

3. k- Partitioned Fuzzy Graph (k-PFG)

We introduce the definition of k-partitioned fuzzy graph.

Definition 3.1. *Let $\underset{\sim}{G}(\sigma, \mu)$ be a fuzzy graph. We define a k-partitioned fuzzy graph of $\underset{\sim}{G}$. $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ as follows. The node set σ is partitioned into k disjoint subsets namely $\sigma_{X_1}, \sigma_{X_2}, ..., \sigma_{X_k}$, such that the sum of the membership of the nodes of the subsets is more or less equal to each other. i.e., the sum of the membership of nodes in σ_{X_i} satisfies the condition $|\sum \sigma_{X_i} - \sum \sigma_{X_j}| < \epsilon$ where $i, j = 1, 2, ...k$ and $i \neq j$. We have to partition σ such that an edge in $\mu_{k_p}(\underset{\sim}{G})$ originates at σ_{X_i} and ends edge in σ_{X_j}. And*

$$\mu_{k_p(\underset{\sim}{G})}(u_i v_j) = \begin{cases} \mu_{\underset{\sim}{G}}(u_i v_j)\,; u_i \in \sigma_{X_i} \text{ and } v_j \in \sigma_{X_j} \ \forall\ i \neq j \\ 0 \qquad\qquad ; otherwise \end{cases}$$

$\mu \in [0, 1]$. By definition $\mu_{k_p(\underset{\sim}{G})}(u_i v_j) \leq \sigma_{X_i}(u_i) \wedge \sigma_{X_j}(v_j)$. Hence $\mu_{k_p(\underset{\sim}{G})}$ is a fuzzy relation on the subsets $\sigma_{k_p(\underset{\sim}{G})}$.

Definition 3.2. *Let $\underset{\sim}{G}(\sigma,\mu)$ be a fuzzy graph. We define a strict k-partitioned fuzzy graph of $\underset{\sim}{G}$. $\underset{\sim}{G}_{k_{sp}}: (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ as follows. The node set σ is partitioned into k disjoint subsets namely $\sigma_{X_1}, \sigma_{X_2}, ..., \sigma_{X_k}$, such that the sum of the membership of the nodes of the subsets is more or less equal to each other. i.e., the sum of the membership of nodes in σ_{X_i} satisfies the condition $|\sum \sigma_{X_i} - \sum \sigma_{X_j}| < \epsilon$ where $i, j = 1, 2, ...k$ and $i \neq j$. We have to partition σ such that an edge in $\mu_{k_p}(\underset{\sim}{G})$ originates at σ_{X_i} and ends edge in σ_{X_j}. And*

$$\mu_{k_{sp}(\underset{\sim}{G})}(u_i v_j) = \left\{ \mu_{\underset{\sim}{G}}(u_i v_j); u_i \in \sigma_{X_i} \text{ and } v_j \in \sigma_{X_j} \ \forall \ i \text{ and } j \right.$$

$\mu \in [0,1]$. By definition
$\mu_{k_{sp}(\underset{\sim}{G})}(u_i v_j) \leq \sigma_{X_i}(u_i) \wedge \sigma_{X_j}(v_j)$. Hence $\mu_{k_{sp}(\underset{\sim}{G})}$ is a fuzzy relation on the subsets $\sigma_{k_{sp}(\underset{\sim}{G})}$.

Remark 3.3. *Any fuzzy graph with n vertices could be partitioned as strict k-partitioned fuzzy graph also as k-partitioned fuzzy graph. The node set σ is partitioned into k-subsets such that partition is allowed up to more or less equal to the maximum membership value of the node set of the fuzzy graph $\underset{\sim}{G}(\sigma,\mu)$.*

3.1. An example of a Fuzzy graph for partitioning

Any fuzzy graph is partitioned taking 'n' number of nodes. A fuzzy graph and its partitioned is examined for an even and also for an odd number of nodes in the following examples.

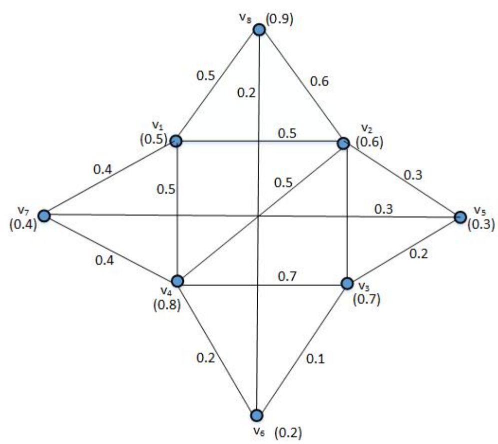

Figure 1: Fuzzy Graph $\underset{\sim}{G}$

An array of membership values is partitioned into k-disjoint subsets that all have more or less equal sum of node memberships and they completely covers the node

set of $\sigma_{k_p(\underset{\sim}{G})}$. Let the node set $\sigma(\underset{\sim}{G})$ of the fuzzy graph be $v_1 = 0.5, v_2 = 0.6, v_3 = 0.7, v_4 = 0.8, v_5 = 0.3, v_6 = 0.2, v_7 = 0.4$, and $v_8 = 0.9$ such that $\sum_i \sigma_{\underset{\sim}{G}}(v_i)$ is 4.4. Considering the above graph with $n = 8$ partitioning is as follows.
Case (i)(a) The node set σ can be partitioned into 2 subset.

$$\sigma_{X_1} = \{v_8, v_7, v_6, v_3\}$$
$$\sigma_{X_2} = \{v_4, v_2, v_1, v_5\}$$

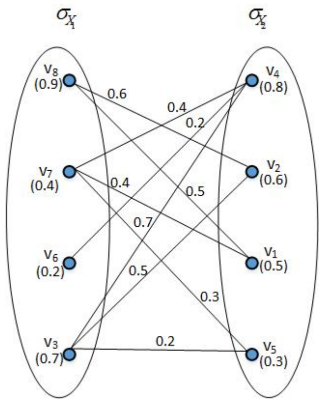

Figure 2: 2-Partitioned Fuzzy Graph $\underset{\sim}{G}$

From figure 2,
$\sum \sigma_{X_1} = \sum \sigma_{X_2}$. i.e.., the sum of the membership of the nodes of the subset is 2.2. The fuzzy graph $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a 2-partition fuzzy graph $\underset{\sim}{G}$.
Case (i)(b) According the definition of Strict k-partitioned fuzzy graph, we say $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a Strick 2-partitioned fuzzy graph $\underset{\sim}{G}$ for the same partitioned node set.

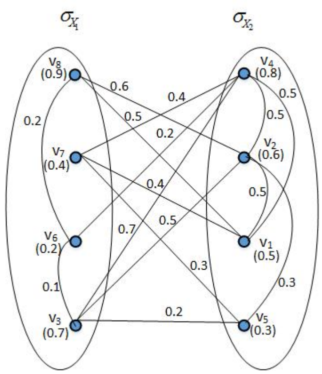

Figure 3: Strick 2-partitioned fuzzy graph $\underset{\sim}{G}$

Case (ii)(a) The node set σ can be partitioned into 3 subset.

$$\sigma_{X_1} = \{v_8, v_2\}$$
$$\sigma_{X_2} = \{v_4, v_7, v_5\}$$
$$\sigma_{X_3} = \{v_3, v_1, v_6\}$$

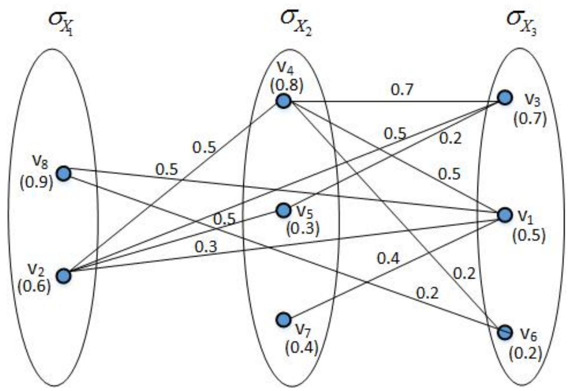

Figure 4: 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$

From figure 4,
$\sum \sigma_{X_1} = \sum \sigma_{X_2} \neq \sum \sigma_{X_3}$. And therefore $|\sum \sigma_{X_i} - \sum \sigma_{X_j}| < \epsilon$ i.e.., the sum of the membership of the nodes of the subset is more or less equal. The fuzzy graph $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$.
Case (ii)(b) The fuzzy graph $\underset{\sim}{G}_{k_{sp}} : (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ is a Strict 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$ for the same partitioned node set.

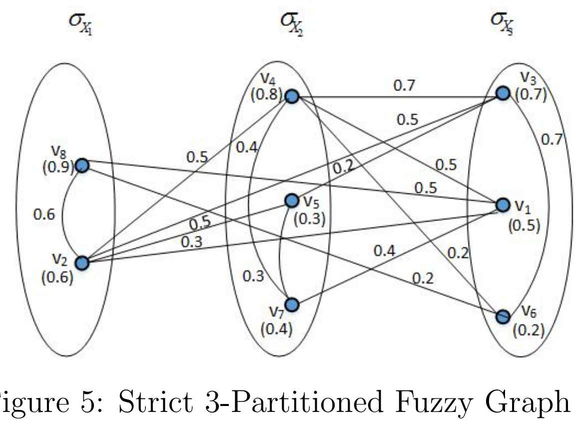

Figure 5: Strict 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$

Case (iii)(a) The node set σ can be partitioned into 4 subset.

$$\sigma_{X_1} = \{v_8, v_6\}$$
$$\sigma_{X_2} = \{v_4, v_5\}$$
$$\sigma_{X_3} = \{v_3, v_7\}$$
$$\sigma_{X_4} = \{v_1, v_2\}$$

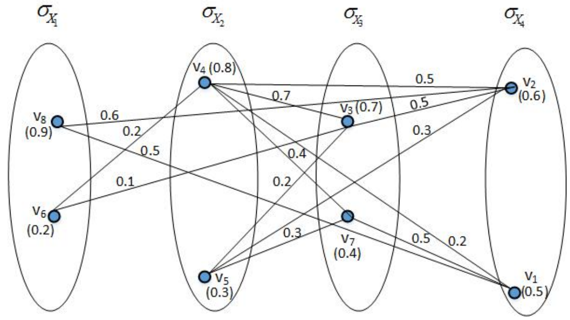

Figure 6: 4-Partitioned Fuzzy Graph $\underset{\sim}{G}$

From figure 6,
$\sum \sigma_{X_1} = \sum \sigma_{X_2} = \sum \sigma_{X_3} = \sum \sigma_{X_4}$. i.e.., the sum of the membership of the nodes of the subset is equal to 1.1. The fuzzy graph $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a 4-Partitioned Fuzzy Graph.
Case (iii)(b) The fuzzy graph $\underset{\sim}{G}_{k_{sp}} : (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ is a Strict 4-Partitioned Fuzzy Graph $\underset{\sim}{G}$ for the same partitioned node set.

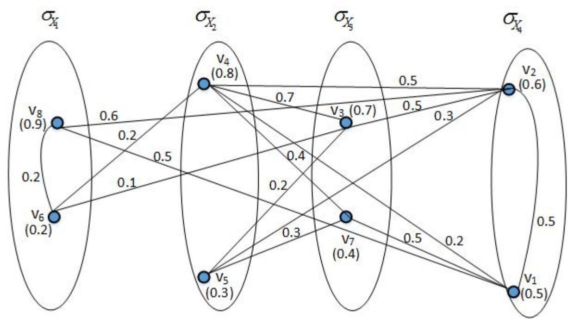

Figure 7: Strict 4-Partitioned Fuzzy Graph $\underset{\sim}{G}$

Further partitioning cannot be proceeded, since the sum of the membership value of the nodes in the subset is more or less equal to the maximum membership value of the nodes of the fuzzy graph $\underset{\sim}{G}(\sigma, \mu)$.

3.2. Another example for partitioning up to k=5

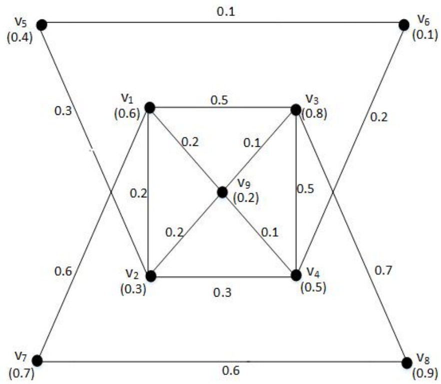

Figure 8: A Fuzzy Graph $\underset{\sim}{G}$

An array of membership values is partitioned into k-disjoint subsets that all have more or less equal sum of node memberships and they completely covers the node set of $\sigma_{k_p(\underset{\sim}{G})}$. Let the node set $\sigma(\underset{\sim}{G})$ of the fuzzy graph be $v_1 = 0.6, v_2 = 0.3, v_3 = 0.8, v_4 = 0.5, v_5 = 0.4, v_6 = 0.1, v_7 = 0.7, v_8 = 0.9$ and $v_9 = 0.2$ such that $\sum_i \sigma_{\underset{\sim}{G}}(v_i)$ is 4.5. Considering the above graph with $n = 9$ partitioning is as follows.
Case (i)(a) The node set σ can be partitioned into 2 subset.

$$\sigma_{X_1} = \{v_8, v_1, v_4, v_9\}$$
$$\sigma_{X_2} = \{v_3, v_2, v_6, v_5, v_7\}$$

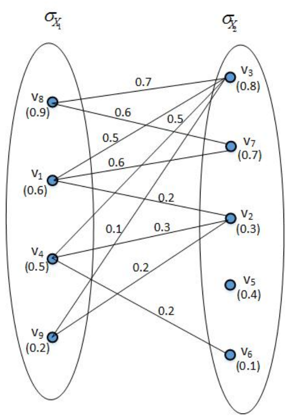

Figure 9: 2-Partitioned Fuzzy Graph $\underset{\sim}{G}$

From figure 9,
$\sum \sigma_{X_1} \neq \sum \sigma_{X_2}$ and therefore $|\sum \sigma_{X_i} - \sum \sigma_{X_j}| < \epsilon$ i.e.., the sum of the membership of the nodes of the subset is more or less equal. The fuzzy graph $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a 2-Partitioned Fuzzy Graph $\underset{\sim}{G}$.
Case (i)(b) According the definition of Strict k-partitioned fuzzy graph, the fuzzy graph $\underset{\sim}{G}_{k_{sp}} : (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ is a Strict 2-Partitioned Fuzzy Graph $\underset{\sim}{G}$ for the same partitioned node set.

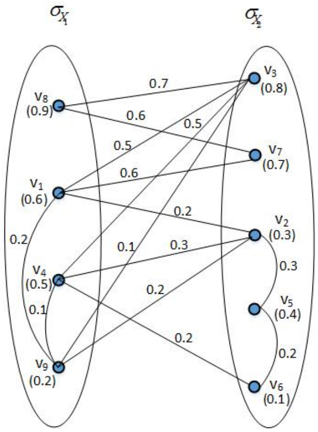

Figure 10: Strict 2-Partitioned Fuzzy Graph $\underset{\sim}{G}$

Case (ii)(a) The node set σ can be partitioned into 3 subset.

$$\sigma_{X_1} = \{v_4, v_6, v_8\}$$
$$\sigma_{X_2} = \{v_2, v_3, v_5\}$$
$$\sigma_{X_3} = \{v_1, v_7, v_9\}$$

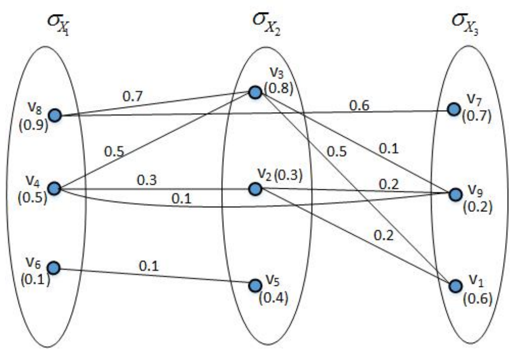

Figure 11: 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$

From figure 11,
$\sum \sigma_{X_1} = \sum \sigma_{X_2} = \sum \sigma_{X_3}$. i.e.., the sum of the membership of the nodes of the subset is equal to 1.5. The fuzzy graph $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$.

Case (ii)(b) The fuzzy graph $\underset{\sim}{G}_{k_{sp}} : (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ is a Strict 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$ for the same partitioned node set.

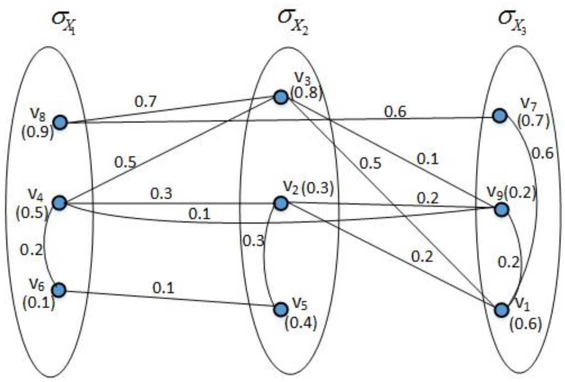

Figure 12: Strict 3-Partitioned Fuzzy Graph $\underset{\sim}{G}$

Case (iii)(a) The node set σ can be partitioned into 4 subset.

$$\sigma_{X_1} = \{v_8, v_9\}$$
$$\sigma_{X_2} = \{v_2, v_2, v_6\}$$
$$\sigma_{X_3} = \{v_5, v_7\}$$
$$\sigma_{X_4} = \{v_1, v_4\}$$

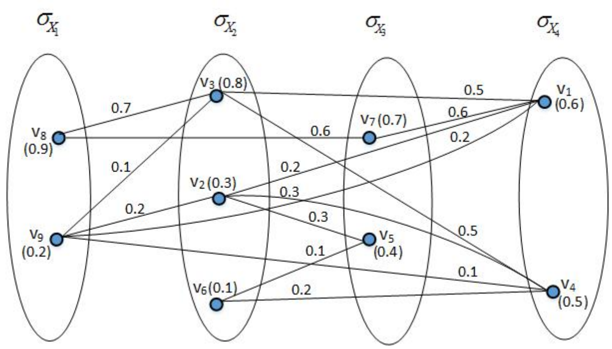

Figure 13: 4-Partitioned Fuzzy Graph $\underset{\sim}{G}$

From figure 13,
$\sum \sigma_{X_1} \neq \sum \sigma_{X_2} \neq \sum \sigma_{X_3} = \sum \sigma_{X_4}$. And therefore $|\sum \sigma_{X_i} - \sum \sigma_{X_j}| < \epsilon$ i.e.., the sum of the membership of the nodes of the subset is more or less equal. The fuzzy graph $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a 4-Partitioned Fuzzy Graph.

Remark 3.4. *The fuzzy graph in fig.13 is also a Strict 4-Partitioned Fuzzy Graph $\underset{\sim}{G}$ for the same partitioned node set. Since there is no edge within the nodes of the subset, Strict 4-Partitioned Fuzzy Graph and 4-partitioned fuzzy graph are same for this case.*

Case (iv) (a) The node set σ can be partitioned into 5 subset.

$$\sigma_{X_1} = \{v_8\}$$
$$\sigma_{X_2} = \{v_3, v_6\}$$
$$\sigma_{X_3} = \{v_7, v_9\}$$
$$\sigma_{X_4} = \{v_1, v_2\}$$
$$\sigma_{X_5} = \{v_4, v_5\}$$

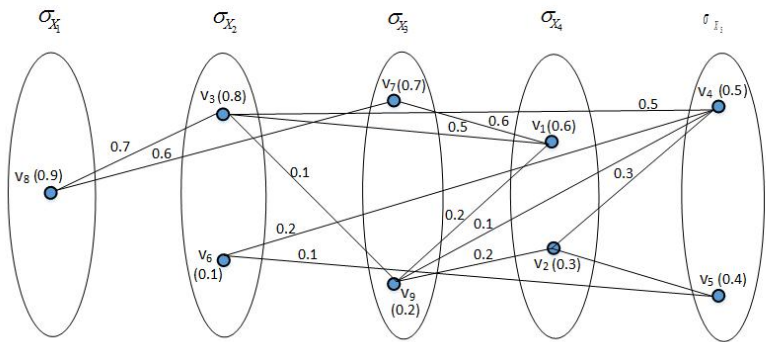

Figure 14: 5-Partitioned Fuzzy Graph $\underset{\sim}{G}$

From figure 14,
$\sum \sigma_{X_1} = \sum \sigma_{X_2} = \sum \sigma_{X_3} = \sum \sigma_{X_4} = \sum \sigma_{X_5}$. i.e., the sum of the membership of the nodes of the subset is 0.9. The fuzzy graph $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ is a 5-partitioned fuzzy graph.

Case (iv)(b) The fuzzy graph $\underset{\sim}{G}_{k_{sp}} : (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ is a Strict 5-partitioned fuzzy graph $\underset{\sim}{G}$ for the same partitioned node set.

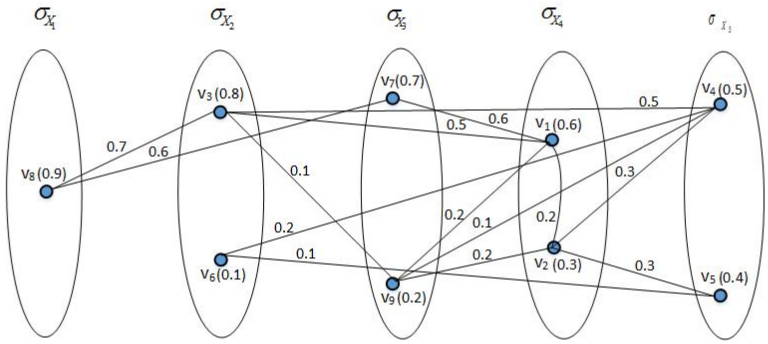

Figure 15: Strict 5-Partitioned Fuzzy Graph $\underset{\sim}{G}$

Further the fuzzy graph could not be partitioned, since the partitioning of the node set σ is allowed up to maximum among the membership value of the nodes of the fuzzy graph $\underset{\sim}{G}(\sigma, \mu)$.

4. Theoretical concepts

Theorem 4.1. *Let* $\underset{\sim}{G}_{k_p}$ *be a k-partitioned fuzzy graph then the* $Size(\underset{\sim}{G}) > Size(\underset{\sim}{G}_{k_p})$ *where*

$$Size(\underset{\sim}{G}_{k_p}) = \sum \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$$

for $u_i \in \sigma_{X_i}$ *and* $v_j \in \sigma_{X_j}$, $\forall\ i \neq j$.

Proof.

Let $\underset{\sim}{G}_{k_p} : (\sigma_{k_p(\underset{\sim}{G})}, \mu_{k_p(\underset{\sim}{G})})$ be a k-partitioned fuzzy graph.
The fuzzy relation
$\mu_{k_p(\underset{\sim}{G})}(u_i v_j) =$

$$\begin{cases} \mu_{\underset{\sim}{G}}(u_i v_j) & u_i \in \sigma_{X_i} \text{ and } v_j \in \sigma_{X_j}\ \forall i \neq j \\ 0 & otherwise \end{cases}$$

. Size ($\underset{\sim}{G}$) $\neq \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ uv \notin \sigma_{X_i} or \sigma_{X_j}}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$

From the definition 3.1 and 3.2 we have,
$\sum \mu_{(\underset{\sim}{G})}(u_i v_j) = \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i \neq j}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$
$+ \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i = j}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$
$= S(\underset{\sim}{G}_{k_p}) + \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i = j}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$.
Therefore, $\sum \mu_{(\underset{\sim}{G})}(u_i v_j)$
$- \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i = j}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j) = S(\underset{\sim}{G}_{k_p})$.

$\implies \sum \mu_{(\underset{\sim}{G})}(u_iv_j) > S(\underset{\sim}{G}_{k_p})$ and
hence Size$(\underset{\sim}{G}) > \sum \mu_{k_p(\underset{\sim}{G})}(u_iv_j)$
i.e.,Size$(\underset{\sim}{G})$ > Size$(\underset{\sim}{G})$.
Illustration 4.2. Let us consider the example of fig 1, We have Size$(\underset{\sim}{G}) = 5.9$.
From fig.2, Size of $(\underset{\sim}{G}_{2_p})$=3.8. i.e., $(5.9 - 2.1)$
From fig.3, Size of $(\underset{\sim}{G}_{3_p})$=3.9. i.e., $(5.9 - 2)$
From fig.4, Size of $(\underset{\sim}{G}_{4_p})$=3.9. i.e., $(5.9 - 0.7)$
From all the above we could say Size$(\underset{\sim}{G}) > \sum \mu_{k_p(\underset{\sim}{G})}(u_iv_j)$ where k=2,3,...

Theorem 4.3. *Let $\underset{\sim}{G}_{k_{sp}}$ be a Strict k-partitioned fuzzy graph then*
(i) O($\underset{\sim}{G}_{k_{sp}}$)= $k \sum_{v_i \in X_j} \sigma(v_i)$ for $\sigma_{X_i} = \sigma_{X_j}$
(ii) O($\underset{\sim}{G}_{k_{sp}}$)¿ $k\{\min\{\sum_{v_i \in X_j} \sigma(v_i)\}\}$ for $\sigma_{X_i} \neq \sigma_{X_j}$
Proof. Let $\underset{\sim}{G}_{k_{sp}} : (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ be a Strict k-partitioned fuzzy graph. As the node set of $(\underset{\sim}{G})_{k_{sp}}$ is $\sigma_{X_i} \cup \sigma_{X_j}$ and the fuzzy subset $\sigma_{k_p((\underset{\sim}{G}))}$ on $\sigma_{((\underset{\sim}{G}))}$ is defined as $|\sum \sigma_{X_i} - \sum \sigma_{X_j}| < \epsilon$ where $i, j = 1, 2, ...k$ and $i \neq j$
Case(i) For $\sum \sigma_{X_i} = \sum \sigma_{X_j}$,
Order($\underset{\sim}{G}_{k_{sp}}$)= $k \sum \sigma_{X_1}$
$=\sum_{v_i \in X_1} \sigma(v_i) + \sum_{v_i \in X_2} \sigma(v_i) + ... + \sum_{v_i \in X_k} \sigma(v_i)$
Hence O($\underset{\sim}{G}_{k_{sp}}$)=$\sum_{v_i \in X_1} \sigma(v_i)$
whenever $\sum \sigma_{X_i} = \sum \sigma_{X_j}$.
Case(ii) For $\sum \sigma_{X_i} \neq \sum \sigma_{X_j}$,
As we know by the definition 3.1
$|\sum \sigma_{X_i} - \sum \sigma_{X_j}| < \epsilon$, then
Order($\underset{\sim}{G}_{k_{sp}}$)$\neq k \sum \sigma_{X_1}$,
Therefore we take min $\left\{\sum_{v_i \in X_j} \sigma(v_i)\right\}$.
And hence

$$\sum_i \sigma_{X_i} > k\{\ min \left\{\sum \sigma_{X_i}\right\}\}$$

O($\underset{\sim}{G}_{k_{sp}}$) $> k\ \{\min\{\sum_{v_i \in Xj} \sigma(v_i)\}\}$
whenever $\sum \sigma_{X_i} \neq \sum \sigma_{X_j}$.

Remark 4.4. *The above theorem is true for k-partitioned fuzzy graph, since subset of Strict k-partitioned fuzzy graph and k-partitioned fuzzy graph have same partitioned node set.*

Illustration 4.5. (i) Let us consider the example of fig 3, We have O($\underset{\sim}{G}_{2_{sp}}$) = 4.4 and
$\sum \sigma_{X_1} = 2.2$, $\sum \sigma_{X_2} = 2.2$.

Then $4.4 = 2(2.2)$. Therefore $\mathrm{O}(\underset{\sim}{G}_{2_{sp}}) = k\left\{\sum \sigma_{X_1}\right\}$.
(ii)Let us consider the example of fig 5,
For $k = 3$, We have
$\mathrm{O}(\underset{\sim}{G}_{3_{sp}}) = 4.4$ and $\sum \sigma_{X_1} = 1.5$, $\sum \sigma_{X_2} = 1.5$, $\sum \sigma_{X_3} = 1.4$, where $\sum \sigma_{X_2} \neq \sum \sigma_{X_3}$.
Then $4.4 > 3min\left\{1.5, 1.5, 1.4\right\}$
$\implies 4.4 > 3(1.4)$
$\implies 4.4 > 4.2$.
Therefore $\mathrm{O}(\underset{\sim}{G}_{3_{sp}}) > k\ \{\min\{\sum_i \sigma(v_i)\}\}$.

Theorem 4.6. *Let $\underset{\sim}{G}_{k_{sp}}$ be a Strict k-partitioned fuzzy graph then the Size $(\underset{\sim}{G})$=Size $(\underset{\sim}{G}_{k_{sp}})$ where Size $(\underset{\sim}{G}) = \sum \mu_{(\underset{\sim}{G})}(u_i v_j)$ for $u_i \in \sigma_{X_i}$ and $v_j \in \sigma_{X_j}$.*
Proof. Let $\underset{\sim}{G}_{k_{sp}} : (\sigma_{k_{sp}(\underset{\sim}{G})}, \mu_{k_{sp}(\underset{\sim}{G})})$ be a Strict k-partitioned fuzzy graph and we have fuzzy relation
$\mu_{k_{sp}(\underset{\sim}{G})}(u_i v_j) =$

$$\begin{cases} \mu_{\underset{\sim}{G}}(u_i v_j) & u_i \in \sigma_{X_i} \text{ and } v_j \in \sigma_{X_j}\ \forall i \neq j \\ \mu_{\underset{\sim}{G}}(u_i v_j) & u_i v_j \in \sigma_{X_i} \text{ and } u_i v_j \in \sigma_{X_j}\ \forall i = j \end{cases}$$

We know that
$\sum \mu_{(\underset{\sim}{G})}(u_i v_j) = \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i \neq j}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$
$+ \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i = j}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$
Size$(\underset{\sim}{G})$= $S(\underset{\sim}{G}_{k_p}) + \sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i = j}} \mu_{k_p(\underset{\sim}{G})}(u_i v_j)$. Hence Size$(\underset{\sim}{G})$=$S(\underset{\sim}{G}_{k_{sp}})$

Illustration 4.7.
Let us consider the example of fig 10, We know $\sum \mu_{(\underset{\sim}{G})}(u_i v_j)$ =4.6,
$\sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i \neq j}} \mu_{2_{sp}(\underset{\sim}{G})}(u_i v_j)$=3.9 and
$\sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i = j}} \mu_{2_{sp}(\underset{\sim}{G})}(u_i v_j)$=0.7.
Therefore $4.6 = 3.9 + 0.7$. Hence Size$(\underset{\sim}{G})$=$S(\underset{\sim}{G}_{2_{sp}})$
By considering the example of fig 12,
$\sum \mu_{(\underset{\sim}{G})}(u_i v_j)$ =4.6,
$\sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i \neq j}} \mu_{3_{sp}(\underset{\sim}{G})}(u_i v_j)$=3.3 and
$\sum_{\substack{u_i \in \sigma_{X_i}, v_j \in \sigma_{X_j} \\ i = j}} \mu_{2_{sp}(\underset{\sim}{G})}(u_i v_j)$=1.3.
Therefore $4.6 = 3.3 + 1.3$. Hence Size$(\underset{\sim}{G})$=$S(\underset{\sim}{G}_{3_{sp}})$.

5. Conclusion

In this paper, we describe the partitioning of node set into k subsets in a way that the sum of the nodes in each partitioned sets is more or less equal. On partitioning the node set we get 2-partition, 3-partition and also 4-partition subsets to which we have drawn k-partitioned fuzzy graphs and Strict k-partitioned fuzzy graphs respectively. k-partition is possible for any 'n' number of nodes. Theorems related to size and order are studied. Its properties will be studied in the next paper. Further partition could be developed by increasing the number of nodes in a fuzzy graph.

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