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SOME RESULTS ON INJECTIVE CHROMATICS TOPOLOGICAL INDICES OF SOME GRAPHS

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Abstract: Graph coloring is an assignment of colors, labels or weights to the elements of graphs subject to certain conditions. Coloring the vertices of a graph G such that adjacent vertices possessing different colors is the notion of proper coloring. A proper coloring $\mathcal{C}$ of a graph G is called an injective coloring of G if any two vertices of G having the same neighbouring vertex have different colors in $\mathcal{C}$. As a coloring analogue to Zagreb indices and irregularity indices in the literature, chromatic Zagreb and irregularity indices have been introduced very recently. In this paper, we introduce the notion of injective chromatic Zagreb indices and injective chromatic irregularity indices and determine these indices for some standard classes of graphs.

Keywords and Phrases: Chromatic Zagreb indices, irregularity indices, injective Chromatic Zagreb indices, injective irregularity indices.

2010 Mathematics Subject Classification: 05C15, 05C30.

1. Introduction

A *topological index* is a real number connected with a molecular graph which remains invariant under graph isomorphisms. In chemical graph theory, topological indices as molecular descriptors, are mainly divided into degree based and distance based indices and among the degree based topological indices, Zagreb indices are the earliest and the mostly used. The chromatic version of these Zagreb indices are introduced in literature recently, in [6]. For terms and definitions which are not introduced in this paper, we refer the reader to [5, 2, 3, 9]. Throughout our study, we consider $G = (V, E)$ as a finite, non-trivial, undirected, simple and connected graph.

A *graph coloring* is an assignment of colors (or labels or weights) to the elements of a graph obeying certain norms and regulations. Unless mentioned otherwise, by coloring a graph, we mean coloring the vertices of a graph. A *proper vertex coloring* of a graph G is an assignment $\varphi : V(G) \to \mathcal{C}$of the vertices of G with colors in $\mathcal{C}$ such that adjacent vertices of G has different colors, where $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\}$ is a set of colors. The *chromatic number*, $\chi(G)$ of graphs is the minimum number of colors required in a proper coloring of the given graph.

An *injective coloring* of a graph G has been introduced in [10] as an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The injective coloring may not be always a proper coloring. But, in this paper, the notion injective coloring represents an injective coloring which is proper. The *injective chromatic number*, $\chi_i(G)$, is the minimum number of colors needed for an proper injective coloring.

The set of all vertices of G which have the color c_i is called the *color class* of that color c_i in G, denoted by $\mathcal{C}_i$. The cardinality of the color class of a color c_i is said to be the *strength* of that color in G and is denoted by $\theta(c_i)$. We can also define a function $\zeta_i : V(G) \to \{1, 2, 3, \ldots, \ell\}$ such that $\zeta_i(v_i) = s \iff \varphi_i(v_i) = c_s, c_s \in \mathcal{C}$. Also, we denote the number of edges with end points having colors c_t and c_s by η_{ts}, where $t < s, 1 \leq t, s \leq \chi_i(G)$.

Some studies on injective coloring parameters of certain graph classes have been conducted in (see [11] [12]). Motivated by the studies on different types of graph colorings, injective coloring parameters and chromatic Zagreb indices [6], in this paper, we define and discuss the concepts of injective chromatic Zagreb indices and injective chromatic irregularity indices of certain graph classes analogous to the definitions of Zagreb and irregularity indices of graphs (see [1] [4] [7] [8]).

2. Injective Chromatic Zagreb and Irregularity Indices of Graphs

Definition 2.1. *Let G be a graph and let $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\}$ be an injective coloring of G such that $\varphi_i(v_i) = c_s; 1 \leq i \leq n, 1 \leq s \leq \ell$. Then, for $1 \leq t \leq \ell\,!$,*

(i) The first injective chromatic Zagreb index of G, denoted by $M_1^{\varphi_i t}(G)$, is defined as $M_1^{\varphi_i t}(G) = \sum\limits_{i=1}^{n}(\zeta(v_i))^2$.

(ii) The second injective chromatic Zagreb index of G, denoted by $M_2^{\varphi_i t}(G)$*, is defined as* $M_2^{\varphi_i t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)\cdot\zeta(v_j)),\ v_iv_j \in E(G)$.

(iii) The injective chromatic irregularity index of G, denoted by $M_3^{\varphi_i t}(G)$*, is defined as* $M_3^{\varphi_i t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i)-\zeta(v_j)|,\ v_iv_j \in E(G)$.

(iv) The injective chromatic total irregularity index of G, denoted by $M_4^{\varphi_i t}(G)$*, is defined as* $M_4^{\varphi_i t}(G) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i)-\zeta(v_j)|,\ v_i, v_j \in V(G)$.

In accordance with the above notions, the minimum and maximum injective chromatic Zagreb indices and the corresponding irregularity indices are defined as follows.

$$
\begin{aligned}
M_1^{\varphi_i^-}(G) &= \min\{M_1^{\varphi_i t}(G) : 1 \le t \le \ell!\},\\
M_1^{\varphi_i^+}(G) &= \max\{M_1^{\varphi_i t}(G) : 1 \le t \le \ell!\},\\
M_2^{\varphi_i^-}(G) &= \min\{M_2^{\varphi_i t}(G) : 1 \le t \le \ell!\},\\
M_2^{\varphi_i^+}(G) &= \max\{M_2^{\varphi_i t}(G) : 1 \le t \le \ell!\},\\
M_3^{\varphi_i^-}(G) &= \min\{M_3^{\varphi_i t}(G) : 1 \le t \le \ell!\},\\
M_3^{\varphi_i^+}(G) &= \max\{M_3^{\varphi_i t}(G) : 1 \le t \le \ell!\}\\
M_4^{\varphi_i^-}(G) &= \min\{M_4^{\varphi_i t}(G) : 1 \le t \le \ell!\},\\
M_4^{\varphi_i^+}(G) &= \max\{M_4^{\varphi_i t}(G) : 1 \le t \le \ell!\}.
\end{aligned}
$$

Remark 1. As a remark, some working formulas for the injective chromatic Zagreb indices and injective chromatic irregularity indices are to be discussed here. We recall that η_{ts} denotes the number of edges with end points t, s where $t < s, 1 \le t, s \le \chi_i(G)$.

1. We know that, $M_1^{\varphi_i t}(G) = \sum_{i=1}^{n}(\zeta_i(v_i))^2,\ v_i \in V(G)$. The working principle for first injective chromatic Zagreb index is as follows:
   $M_1^{\varphi_i t}(G) = \sum_{j=1}^{\ell}\theta(c_j)\cdot j^2,\ c_j \in \mathcal{C}$.

2. We know that, $M_2^{\varphi_i t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta_i(v_i)\cdot\zeta_i(v_j)),\ v_iv_j \in E(G)$. The working

principle for second injective chromatic Zagreb index is as follows: $M_2^{\varphi_i t}(G) = \sum_{1\leq t,s\leq \chi_i(G)}^{t<s} ts\eta_{ts}$.

3. We know that, $M_3^{\varphi_i t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta_i(v_i) - \zeta_i(v_j)|,\ v_iv_j \in E(G)$. The working principle for injective chromatic irregularity index is as follows: $M_3^{\varphi_i t}(G) = \sum_{1\leq t,s\leq \chi_i(G)}^{t<s} \eta_{ts}|t-s|$.

4. We know that, $M_4^{\varphi_i t}(G) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta_i(v_i) - \zeta_i(v_j)|,\ v_i, v_j \in V(G)$. The working principle for total injective chromatic irregularity index is as follows: $M_4^{\varphi_i t}(G) = \frac{1}{2}\sum_{t,s\in\mathcal{C}}^{t<s} \theta(c_t)\cdot\theta(c_s)|t-s|$.

3. Injective Chromatic Topological Indices of Paths and Cycles

Motivated by the studies mentioned above, we study the injective chromatic Zagreb indices and injective chromatic irregularity indices of certain fundamental graph classes in the following discussion.

Theorem 3.1. *For a path P_n, we have*

(i) $$M_1^{\varphi_e^-}(P_n) = \begin{cases} \frac{14n}{3}; & n \equiv 0 \pmod 3\\ \frac{14n-11}{3}; & n \equiv 1 \pmod 3\\ \frac{14n-13}{3}; & n \equiv 2 \pmod 3; \end{cases}$$

(ii) $$M_2^{\varphi_e^-}(P_n) = \begin{cases} \frac{11n-18}{3}; & n \equiv 0 \pmod 3\\ \frac{11n-11}{3}; & n \equiv 1 \pmod 3\\ \frac{11n-16}{3}; & n \equiv 2 \pmod 3; \end{cases}$$

(iii) $$M_3^{\varphi_e^-}(P_n) = \begin{cases} \frac{4n-6}{3}; & n \equiv 0 \pmod 3\\ \frac{4n-4}{3}; & n \equiv 1 \pmod 3\\ \frac{4n-5}{3}; & n \equiv 2 \pmod 3; \end{cases}$$

(iv) $$M_4^{\varphi_e^-}(P_n) = \begin{cases} \frac{4n^2}{18}; & n \equiv 0 \pmod 3\\ \frac{4n^2+n-5}{18}; & n \equiv 1 \pmod 3\\ \frac{4n^2-n-5}{18}; & n \equiv 2 \pmod 3; \end{cases}$$

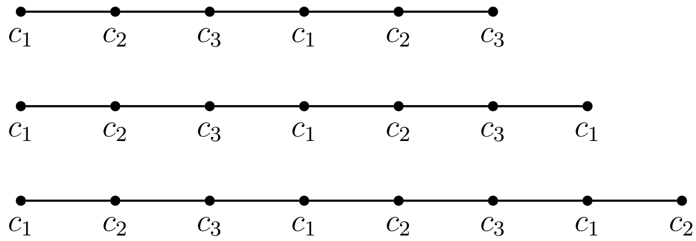

Figure 1: The injective coloring of the graph P_n with different values of n.

Proof. Consider a path P_n with vertex set $\{v_1, v_2, \ldots, v_n\}$. Then, P_n has $n-1$ edges. Now we consider the three separate cases. In all cases the injective chromatic number, $\chi_i(P_n) = 3$.

Case-1: Let $n \equiv 0 \pmod 3$ be assumed. When $n \equiv 0 \pmod 3$, obeying the rules of injective colouring φ_i^-, we can find three colour classes with same cardinality $\frac{n}{3}$ and we colour them with minimal colours c_1, c_2, c_3. In order to get the clear picture of the colour classes, we define 3 independent sets in P_n as $S_1 = \{v_1, v_4, \ldots, v_{n-2}\}, S_2 = \{v_2, v_5, \ldots, v_{n-1}\}$ and $S_3 = \{v_3, v_6, \ldots, v_n\}$.

Further, injectively colour S_i with colour c_i for $1 \leq i \leq 3$ to have the values $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{n}{3}$ and $\eta_{12} = \eta_{23} = \frac{n}{3}, \eta_{13} = \frac{n-3}{2}$.

Part (i): Adding the squares of the colours according to injective colouring and from the definition of the first injective chromatic Zagreb indices, the result follows as (Refer Remark [1]):

$$M_1^{\varphi_i^-}(P_n) = \sum_{i=1}^{n} (\zeta_i(v_i))^2 = 14\frac{n}{3}.$$

Part (ii): Here, for this particular part, S_1 is coloured with c_2, S_2 with c_1 and S_3 with c_3. So we will have $\eta_{12} = \eta_{13} = \frac{n}{3}$ and $\eta_{23} = \frac{n-3}{3}$. Thus the result is immediate from the definition of the second injective chromatic Zagreb index as (Refer Remark [1]):

$$M_2^{\varphi_i^-}(P_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} (\zeta_i(v_i) \cdot \zeta_i(v_j)) = \frac{11n - 18}{3}.$$

Part (iii): The definition of the injective chromatic irregularity indices of a graph, gives the result as follows (Refer Remark [1]):

$$M_3^{\varphi_i^-}(P_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta_i(v_i) - \zeta_i(v_j)| = \frac{4n - 6}{3}.$$

Part (iv): First we assign injective colouring to the vertices as S_i is coloured with c_i for $1 \leq i \leq 3$. In order to calculate the injective chromatic total irregularity of P_n, all the possible vertex pairs from P_n have to be considered and their possible colour distances are determined. The possibility of the vertex pairs which contribute to the colour distance are calculated considering all vertex pairs. Now from the definition of the injective chromatic total irregularity indices of a graph, the result follows as (Refer Remark [1]):

$$M_4^{\varphi_i^-}(P_n) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta_i(v_i) - \zeta_i(v_j)| = \frac{4n^2}{18}.$$

Case-2: Let $n \equiv 1 \pmod 3$ be assumed. When $n \equiv 1 \pmod 3$, obeying the rules of injective colouring, we can find three colour classes and we colour them with minimal colours c_1, c_2, c_3. For the easy flow of the proof we define 3 independent sets in P_n as $S_1 = \{v_1, v_4, \ldots, v_n\}, S_2 = \{v_2, v_5, \ldots, v_{n-2}\}$ and $S_3 = \{v_3, v_6, \ldots, v_{n-1}\}$.

Now we colour S_i with colour c_i for $1 \leq i \leq 3$ to have the values $\theta(c_1) = \frac{n+2}{3}, \theta(c_2) = \theta(c_3) = \frac{n-1}{3}$ and $\eta_{12} = \eta_{23} = \eta_{13} = \frac{n-1}{3}$.

Part (i): From the definition of the first injective chromatic Zagreb indices the result follows as (Refer Remark [1]):

$$M_1^{\varphi_i^-}(P_n) = \sum_{i=1}^{n}(\zeta_i(v_i))^2 = \frac{14n-11}{3}.$$

Part (ii): From the definition of the second injective chromatic Zagreb indices the result follows as (Refer Remark [1]):

$$M_2^{\varphi_i^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta_i(v_i) \cdot \zeta_i(v_j)) = \frac{11n-11}{3}.$$

Part (iii): From the definition of the injective chromatic irregularity indices of a graph, gives the result as follows (Refer Remark [1]):

$$M_3^{\varphi_i^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta_i(v_i) - \zeta_i(v_j)| = \frac{4(n-1)}{3}.$$

Part (iv): First we assign injective colouring to the vertices as S_i is coloured with c_i for $1 \leq i \leq 3$. In order to calculate the injective chromatic total irregularity of P_n, we calculate the colour distances of all the possible vertex pairs from P_n as in previous case. From the definition of the injective chromatic total irregularity indices of a graph, the result follows as (Refer Remark [1]):

$$M_4^{\varphi_i^-}(P_n) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta_i(v_i) - \zeta_i(v_j)| = \frac{4n^2+n-5}{18}.$$

Case-3: Let $n \equiv -1 \pmod 3$ be assumed. Apply injective colouring to the vertices to have the colour classes as $S_1 = \{v_1, v_4, \ldots, v_{n-1}\}, S_2 = \{v_2, v_5, \ldots, v_n\}$ and $S_3 = \{v_3, v_6, \ldots, v_{n-2}\}$.

Now we colour S_i with colour c_i for $1 \leq i \leq 3$ to have the values $\theta(c_1) = \theta(c_2) = \frac{n+1}{3}, \theta(c_3) = \frac{n-2}{3}$ and $\eta_{12} = \frac{n+1}{3}, \eta_{23} = \eta_{13} = \frac{n-2}{3}$.

Part (i): Adding the squares of the colours according to injective colouring and from the definition of the first injective chromatic Zagreb indices the result follows as (Refer Remark [1]):

$$M_1^{\varphi_i^-}(P_n) = \sum_{i=1}^{n}(\zeta_i(v_i))^2 = \frac{14n-13}{3}.$$

Part (ii): The result is concluded from the definition of the second injective chromatic Zagreb indices as (Refer Remark [1]):

$$M_2^{\varphi_i^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta_i(v_i)\cdot\zeta_i(v_j)) = \frac{11n-16}{3}.$$

Part (iii): The definition of the injective chromatic irregularity indices of a graph, gives the result thus (Refer Remark [1]):

$$M_3^{\varphi_i^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta_i(v_i) - \zeta_i(v_j)| = \frac{4n-5}{3}.$$

Part (iv): Just as in the previous cases, from the definition of the injective chromatic total irregularity indices of a graph, the result obtained is (Refer Remark [1]):

$$M_4^{\varphi_i^-}(P_n) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta_i(v_i) - \zeta_i(v_j)| = \frac{4n^2-n-5}{18}.$$

Using the notion of injective coloring, the results obtained by considering the maximum values are charted below as next theorem.

Theorem 3.2. *For a path P_n, we have*

$$\textit{(i)}\ M_1^{\varphi_i^+}(P_n)=\begin{cases}\frac{14n}{3}; & n\equiv 0 \pmod 3\\ \frac{14n+13}{3}; & n\equiv 1 \pmod 3\\ \frac{14n+11}{3}; & n\equiv -1 \pmod 3);\end{cases}$$

$$\textit{(ii)}\ M_2^{\varphi_i^+}(P_n)=\begin{cases}\frac{11n-9}{3}; & n\equiv 0 \pmod 3\\ \frac{11n-11}{3}; & n\equiv 1 \pmod 3\\ \frac{11n-4}{3}; & n\equiv -1 \pmod 3);\end{cases}$$

$$\textit{(iii)}\ M_3^{\varphi_i^+}(P_n)=\begin{cases}\frac{4n-3}{3}; & n\equiv 0 \pmod 3\\ \frac{4n-4}{3}; & n\equiv 1 \pmod 3\\ \frac{4n-5}{3}; & n\equiv -1 \pmod 3);\end{cases}$$

$$\textit{(iv)}\ M_4^{\varphi_i^+}(P_n)=\begin{cases}\frac{4n^2}{18}; & n\equiv 0 \pmod 3\\ \frac{4n^2+n-5}{18}; & n\equiv 1 \pmod 3\\ \frac{4n^2-n-5}{18}; & n\equiv -1 \pmod 3);\end{cases}$$

Proof. Here the proof is similar to that of the proof of Theorem 3.1.

Next, we discuss the injective chromatic Zagreb indices and injective chromatic irregularity indices of cycles in the following theorem.

Theorem 3.3. *For a cycle C_n, we have*

$$\textit{(i)}\ M_1^{\varphi_e^-}(C_n)=\begin{cases}\frac{14n}{3}; & n\equiv 0 \pmod 3\\ \frac{14n-11}{3}; & n\equiv 1 \pmod 3\\ \frac{14n+68}{3}; & n\equiv 2 \pmod 3);\end{cases}$$

$$\textit{(ii)}\ M_2^{\varphi_e^-}(C_n)=\begin{cases}\frac{11n}{3}; & n\equiv 0 \pmod 3\\ \frac{11n+28}{3}; & n\equiv 1 \pmod 3\\ \frac{11n+56}{3}; & n\equiv 2 \pmod 3);\end{cases}$$

$$\textit{(iii)}\ M_3^{\varphi_e^-}(C_n)=\begin{cases}\frac{4n}{3}; & n\equiv 0 \pmod 3\\ \frac{4n+2}{3}; & n\equiv 1 \pmod 3\\ \frac{4n+4}{3}; & n\equiv 2 \pmod 3);\end{cases}$$

$$\textit{(iv)}\ M_4^{\varphi_e^-}(C_n)=\begin{cases}\frac{4n^2}{18}; & n\equiv 0 \pmod 3\\ \frac{4n^2+10n-14}{18}; & n\equiv 1 \pmod 3\\ \frac{4n^2+20n-56}{18}; & n\equiv 2 \pmod 3);\end{cases}$$

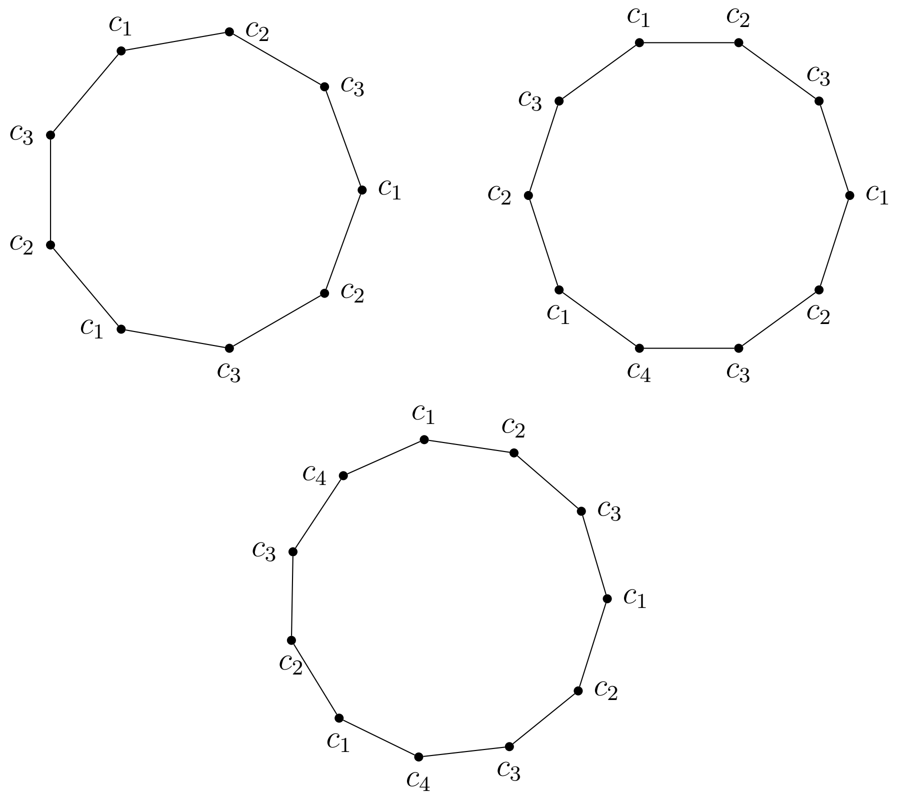

Figure 2: The injective coloring of the graph C_n with different values of n

Proof. Consider a cycle C_n with vertex set $\{v_1, v_2, \dots, v_n\}$. Then, C_n has n edges. Now we consider the three separate cases.

Case-1: Let $n \equiv 0 \pmod 3$ be assumed. When $n \equiv 0 \pmod 3$, obeying the rules of injective coloring, we can find three color classes with same cardinality $\frac{n}{3}$ and we color them with minimal colors c_1, c_2, c_3. In order to get the clear picture of the color classes, we define 3 independent sets in C_n as $S_1 = \{v_1, v_4, \dots, v_{n-2}\}, S_2 = \{v_2, v_5, \dots, v_{n-1}\}$ and $S_3 = \{v_3, v_6, \dots, v_n\}$.

Now, we color S_i with color c_i for $1 \leq i \leq 3$ to have the values $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{n}{3}$ and $\eta_{12} = \eta_{23} = \eta_{13} = \frac{n}{3}$.

Part (i): Adding the squares of the colors according to injective coloring and from the definition of the first injective chromatic Zagreb indices the result follows as:

$$M_1^{\varphi_e^{-}}(C_n) = \sum_{i=1}^{n} (\zeta(v_i))^2 = \frac{n}{3}(1+4+9) = 14\frac{n}{3}.$$

Part (ii): Here, for this particular part, S_1 is colored with c_2, S_2 with c_1 and S_3 with c_3. So we will have $\eta_{12} = \eta_{13} = \frac{n}{3}$ and$\eta_{23} = \frac{n-3}{3}$. Thus the result follows from the definition of the second injective chromatic Zagreb indices as:

$$M_2^{\varphi_e^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)\cdot\zeta(v_j)) = \frac{2n}{3} + 3n - 6 = \frac{11n-18}{3}.$$

Part (iii): The definition of the injective chromatic irregularity indices of a graph, gives the result as follows:

$$M_3^{\varphi_e^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i) - \zeta(v_j)| = \frac{2n}{3} + \frac{2(n-3)}{3} = \frac{4n-6}{3}.$$

Part (iv): First we assign injective coloring to the vertices as S_i is colored with c_i for $1 \le i \le 3$. In order to calculate the injective chromatic total irregularity of P_n, all the possible vertex pairs from P_n have to be considered and their possible color distances are determined. The possibility of the vertex pairs which contribute to the color distance are calculated considering all vertex pairs. Only the vertex pairs with different colors are to be considered. Now from the definition of the injective chromatic total irregularity indices of a graph, the result follows as:

$$M_4^{\varphi_e^-}(P_n) = \frac{1}{2}\sum_{u,v\in V(\check{P}_n)}|\varphi_e(u) - \varphi_e(v)| = \frac{1}{2}(\frac{n^2}{9}) = \frac{4n^2}{18}$$

Case-2: Let $n \equiv 1 \pmod 3$ be assumed. When $n \equiv 1 \pmod 3$, obeying the rules of injective coloring, we can find three color classes and we color them with minimal colors c_1, c_2, c_3. For the easy flow of the proof we define 3 independent sets in P_n as $S_1 = \{v_1, v_4, \ldots, v_n\}, S_2 = \{v_2, v_5, \ldots, v_{n-2}\}$ and $S_3 = \{v_3, v_6, \ldots, v_{n-1}\}$.

Now, we color S_i with color c_i for $1 \le i \le 3$ to have the values $\theta(c_1) = \frac{n+2}{3}, \theta(c_2) = \theta(c_3) = \frac{n-1}{3}$ and $\eta_{12} = \eta_{23} = \eta_{13} = \frac{n-1}{3}$.

Part (i): From the definition of the first injective chromatic Zagreb indices the result follows as:

$$M_1^{\varphi_e^-}(P_n) = \sum_{i=1}^{n}(\zeta(v_i))^2 = \frac{n+2}{3} + (4+9)\frac{n-1}{3} = \frac{14n-11}{3}$$

Part (ii): From the definition of the second injective chromatic Zagreb indices the result follows as:

$$M_2^{\varphi_e^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)\cdot\zeta(v_j)) = 11\frac{n-1}{3} = \frac{11n-11}{3}$$

Part (iii): From the definition of the injective chromatic irregularity indices of a graph, gives the result as follows:

$$M_3^{\varphi_e^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i) - \zeta(v_j)| = \frac{4(n-1)}{3} = \frac{4n-4}{3}.$$

Part (iv): First we assign injective coloring to the vertices as S_i is colored with c_i for $1 \leq i \leq 3$. In order to calculate the injective chromatic total irregularity of P_n, we calculate the color distances of all the possible vertex pairs from P_n as in previous case. From the definition of the injective chromatic total irregularity indices of a graph, the result follows as:

$$M_4^{\varphi_e^-}(P_n) = \frac{1}{2}\sum_{u,v\in V(\check{P}_n)}|\varphi_e(u) - \varphi_e(v)| = \frac{4n^2+n-5}{18}.$$

Case-3: Let $n \equiv -1 \pmod 3$ be assumed. Apply injective coloring to the vertices to have the color classes as $S_1 = \{v_1, v_4, \ldots, v_{n-1}\}, S_2 = \{v_2, v_5, \ldots, v_n\}$ and $S_3 = \{v_3, v_6, \ldots, v_{n-2}\}$.

Now we color S_i with color c_i for $1 \leq i \leq 3$ to have the values $\theta(c_1) = \theta(c_2) = \frac{n+1}{3}, \theta(c_3) = \frac{n-2}{3}$ and $\eta_{12} = \frac{n+1}{3}, \eta_{23} = \eta_{13} = \frac{n-2}{3}$.

Part (i): Adding the squares of the colors according to injective coloring and from the definition of the first injective chromatic Zagreb indices the result follows as:

$$M_1^{\varphi_e^-}(P_n) = \sum_{i=1}^{n}(\zeta(v_i))^2 = 5\frac{n+1}{3} + 9\frac{n-2}{3} = \frac{14n-13}{3}.$$

Part (ii): The result follows from the definition of the second injective chromatic Zagreb indices as:

$$M_2^{\varphi_e^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)\cdot\zeta(v_j)) = 2\frac{n+1}{3} + 9\frac{n-2}{3} = \frac{11n-16}{3}.$$

Part (iii): The definition of the injective chromatic irregularity indices of a graph, gives the result as follows:

$$M_3^{\varphi_e^-}(P_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i) - \zeta(v_j)| = \frac{n+1}{3} + \frac{3(n-2)}{3} = \frac{4n-5}{3}.$$

Part (iv): Just as in the previous cases, from the definition of the injective chromatic total irregularity indices of a graph, the result follows as:

$$M_4^{\varphi_e^-}(P_n) = \frac{1}{2}\sum_{u,v\in V(\check{P}_n)} |\varphi_e(u) - \varphi_e(v)| = \frac{4n^2 - n - 5}{18}.$$

4. Conclusion

In this article, we have discussed certain colouring related topological indices, called injective chromatic Zagreb and irregularity indices, of paths and cycles. The study seems to be promising for further studies as these parameters can be computed for many other graph classes and classes of derived graphs. These parameters can be determined for various graph operations, graph products and graph powers also. Various types of similar colouring related parameters with respect to different types of graph colourings offer much for further investigations. The concept can be extended to edge colourings and map colourings also.

These newly defined colouring related parameters can be found to have numerous applications in various fields like Mathematical Chemistry, Distribution Theory, Optimization Techniques etc. Similar studies are possible in various other fields. All these facts highlight the wide scope for further research in this area.

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