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FIXED POINTS OF COMPLEX VALUED A_b-METRIC SPACE

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Abstract: In this paper, we introduce the concept of complex valued A_b-metric space and prove some fixed point theorems. Examples are also given as a support of our results.

Keywords and Phrases: A_b-metric space, Complex valued metric space, Complex valued A_b-metric space, fixed point, Kannan's fixed point theorem.

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1. Introduction

Azam et al. [1] introduced the concept of complex valued metric space and proved some fixed point results for a pair of mappings for a contraction condition satisfying a rational expression. Moreover, Shin Min Kang et al.[11] introduced the notion of complex valued G-metric space and proved contraction principle in this space. Rao et al. [2] introduced the concept of complex valued b-metric space in 2013. In 2014, Nabil M. Mlaiki [8] introduced complex valued S-metric space and proved the existence and the uniqueness of a common fixed point of two self mappings in this space. Recently, Ozgur [10] introduced the concept of complex valued G_b-metric space and proved Banach contraction principle and Kannan's fixed point theorem in this new space. Priyobarta et al. [9] also extended the concept of complex valued metric space to complex valued S_b-metric space and proved some fixed point theorems. In this paper, we extend the concept of complex valued metric space to complex valued A_b-metric space and prove some fixed point theorems.

2. Preliminaries

In this section, we recall A_b-metric space.

Definition 2.1. *[5] Let X be a nonempty set and $b \geq 1$ be a given real number. A function $A : X^n \to [0, \infty)$ is called an A_b-metric on X if for any $x_i, a \in X,\ i = 1, 2, 3, ..., n$, the following conditions hold:*

(A_b1) $A(x_1, x_2, ..., x_{n-1}, x_n) \geq 0$,

(A_b2) $A(x_1, x_2, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = ... = x_{n-1} = x_n$,

(A_b3) $A(x_1, x_2, ..., x_{n-1}, x_n) \leq b[A(x_1, x_1, ..., (x_1)_{n-1}, a) + A(x_2, x_2, ..., (x_2)_{n-1}, a) ... + A(x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) + A(x_n, x_n, ..., (x_n)_{n-1}, a)]$.

The pair (X, A) is called an A_b-metric space.

Example 2.2. [5] Let $X = [1, \infty)$. Define $A_b : X^n \to [0, \infty)$ by

$$A(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2 \text{ for all } x_i \in X,\ i = 1, 2, 3, ..., n.$$

Then (X, A_b) is an A_b-metric space with $b = 2 > 1$.

3. Complex valued A_b-metric space

The concept of complex valued metric space was initiated by Azam et al. [1]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied :

(C_1) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,

(C_2) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,

(C_3) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,

(C_4) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

Particularly, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C_2), (C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:

1. If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \precsim bz$ for all $0 \precsim z \in \mathbb{C}$.

2. If $z_1 \precsim z_2$, then $az_1 \precsim az_2$ for all $0 \leq a \in \mathbb{R}$.

3. If $0 \precsim z_1 \precsim z_2$, then $|z_1| \leq |z_2|$.

4. If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.

5. If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Now, we give the definition of the complex valued A_b-metric space.

Definition 3.1. *Let X be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $A : X^n \to \mathbb{C}$ satisfies for all $x_i, a \in X$, $i = 1, 2, 3, ..., n$:*

(CA_b1) $0 \precsim A(x_1, x_2, ..., x_{n-1}, x_n)$,

(CA_b2) $A(x_1, x_2, ..., x_{n-1}, x_n) = 0 \ \Leftrightarrow \ x_1 = x_2 = ... = x_{n-1} = x_n$,

(CA_b3) $A(x_1, x_2, ..., x_{n-1}, x_n) \precsim b[A(x_1, x_1, ..., (x_1)_{n-1}, a) + A(x_2, x_2, ..., (x_2)_{n-1}, a) ... + A(x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) + A(x_n, x_n, ..., (x_n)_{n-1}, a)]$.

Then A is called a complex valued A_b-metric on X and the pair (X, A) is called a complex valued A_b-metric space.

Example 3.2. Let $X = \mathbb{R}$ and $A : X^n \to \mathbb{C}$ be such that

$$A(x_1, x_2, ..., x_{n-1}, x_n) = (\alpha + i\beta)A_*(x_1, x_2, ..., x_{n-1}, x_n)$$

where $\alpha, \beta \geq 0$ are constants and A_* is an A_b-metric on X. Then we show that A is a complex valued A_b-metric on X.

1. Since $A_*(x_1, x_2, ..., x_{n-1}, x_n) \geq 0$, $\forall x_i \in X$, $i = 1, 2, 3, ..., n$ and $\alpha, \beta \geq 0$, we have $A(x_1, x_2, ..., x_{n-1}, x_n) = (\alpha + i\beta)A_*(x_1, x_2, ..., x_{n-1}, x_n) \succsim 0 + i0 = 0$ i.e. $A(x_1, x_2, ..., x_{n-1}, x_n) \succsim 0, \forall x_i \in X, i = 1, 2, 3, ..., n$.

2. $A(x_1, x_2, ..., x_{n-1}, x_n) = 0 \Leftrightarrow (\alpha + i\beta)A_*(x_1, x_2, ..., x_{n-1}, x_n) = 0 \Leftrightarrow A_*(x_1, x_2, ..., x_{n-1}, x_n) = 0 \Leftrightarrow x_1 = x_2 = ... = x_{n-1} = x_n$.

3. $\alpha A_*(x_1, x_2, ..., x_{n-1}, x_n) \leq b[\alpha A_*(x_1, x_1, ..., (x_1)_{n-1}, a) + \alpha A_*(x_2, x_2, ..., (x_2)_{n-1}, a) + ... + \alpha A_*(x_n, x_n, ..., (x_n)_{n-1}, a)$ and $\beta A_*(x_1, x_2, ..., x_{n-1}, x_n) \leq b[\beta A_*(x_1, x_1, ..., (x_1)_{n-1}, a) + \beta A_*(x_2, x_2, ..., (x_2)_{n-1}, a) + ... + \beta A_*(x_n, x_n, ..., (x_n)_{n-1}, a)]$

Thus, we have

$$\begin{aligned}A(x_1, x_2, ..., x_{n-1}, x_n) &= (\alpha + i\beta)A_*(x_1, x_2, ..., x_{n-1}, x_n)\\ &= \alpha A_*(x_1, x_2, ..., x_{n-1}, x_n) + i\beta A_*(x_1, x_2, ..., x_{n-1}, x_n)\\ &\precsim b[A(x_1, x_1, ..., (x_1)_{n-1}, a) + A(x_2, x_2, ..., (x_2)_{n-1}, a)]\\ &\quad + ... + A(x_n, x_n, ..., (x_n)_{n-1}, a)].\end{aligned}$$

Therefore A is a complex valued A_b-metric on X.
As a particular case, we have the following example of complex valued A_b-metric on X.
The mapping $A : X^n \to \mathbb{C}$ defined by $A(x_1, x_2, ..., x_{n-1}, x_n) = (1 + i)\sum_{i=1}^{n}\sum_{i<j} |x_i - x_j|^2$ is a complex valued A_b-metric on $X = \mathbb{R}$ with $b = 2$.

Definition 3.3. *A complex valued A_b-metric space (X, A) is said to be symmetric if $A(x_1, x_1, ..., (x_1)_{n-1}, x_2) = A(x_2, x_2, ..., (x_2)_{n-1}, x_1)$, $\forall x_1, x_2 \in X$.*

Definition 3.4. *Let (X, A) be a complex valued A_b-metric space.*

(i) A sequence $\{x_p\}$ in X is said to be complex valued A_b-convergent to x if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $A(x_p, x_p, ..., x_p, x) \prec a$ or $A(x, x, ..., x, x_p) \prec a$ for all $p \geq k$ and is denoted by $\lim_{p\to\infty} x_p = x$ or $x_p \to x$ as $p \to \infty$.

(ii) A sequence $\{x_p\}$ in X is called complex valued A_b-Cauchy if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $A(x_p, x_p, ..., x_p, x_q) \prec a$ for each $p, q \geq k$.

(iii) If every complex valued A_b-Cauchy sequence in X is complex valued A_b-convergent in X, then (X, A) is said to be complex valued A_b-complete.

Lemma 3.5. *Let (X, A) be a complex valued A_b-metric space and let $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is complex valued A_b-convergent to x if and only if $|A(x_p, ..., x_p, x)| \to 0$ as $p \to \infty$ or $|A(x, ..., x, x_p)| \to 0$ as $p \to \infty$.*
Proof. ($\Rightarrow$) Let us assume that $\{x_p\}$ is complex valued A_b-convergent to x and let

$$a = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$$

where $\epsilon > 0$ is a real number. Then $0 \prec a \in \mathbb{C}$ and there is a natural number k such that $A(x_p, x_p, ..., x_p, x) \prec a$ for all $p \geq k$. Thus, $|A(x_p, x_p, ..., x_p, x)| < |a| = \epsilon$

for all $p \geq k$ and so $|A(x_p, x_p, ..., x_p, x)| \to 0$ as $p \to \infty$.
($\Leftarrow$) Let us suppose that $|A(x_p, x_p, ..., x_p, x)| \to 0$ as $p \to \infty$. For a given $a \in \mathbb{C}$ with $0 \prec a$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}$, $|z| < \delta$ implies $z \prec a$.
Considering δ, we have a natural number k such that $|A(x_p, x_p, ..., x_p, x)| < \delta$ for all $p \geq k$. This means that $A(x_p, x_p, ..., x_p, x) \prec a$ for all $p \geq k$ i.e. $\{x_p\}$ is complex valued A_b-convergent to x.
Similarly, we can prove for the other condition as

$$A(x, x, ..., x, x_p) \precsim bA(x_p, x_p, ..., x_p, x).$$

Lemma 3.6. *Let (X, A) be a complex valued A_b-metric space and $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is complex valued A_b-Cauchy sequence if and only if $|A(x_p, ..., x_p, x_q)| \to 0$ as $p, q \to \infty$.*
Proof. ($\Rightarrow$) Let $\{x_p\}$ be complex valued A_b-Cauchy sequence and

$$b = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$$

where $\epsilon > 0$ is a real number. Then $0 \prec b \in \mathbb{C}$ and there is a natural number k such that $A(x_p, x_p, ..., x_p, x_q) \prec b$ for all $p, q \geq k$. Therefore, $|A(x_p, x_p, ..., x_p, x_q)| < |b| = \epsilon$ for all $p, q \geq k$ and so $|A(x_p, x_p, ..., x_p, x_q)| \to 0$ as $p, q \to \infty$.
($\Leftarrow$) Let us assume that $|A(x_p, x_p, ..., x_p, x_q)| \to 0$ as $p, q \to \infty$. Then there exists a real number $r > 0$ such that for $z \in \mathbb{C}$, $|z| < r$ implies $z \prec b$, where $b \in \mathbb{C}$ with $0 \prec b$.
For this r there is a natural number k such that $|A(x_p, x_p, ..., x_p, x_q)| < r$ for all $p, q \geq k$. This means that $A(x_p, x_p, ..., x_p, x_q) \prec b$ for all $p, q \geq k$. Hence $\{x_p\}$ is complex valued A_b-Cauchy sequence.

Lemma 3.7. *Let (X, A) be a complex valued A_b-metric space.*
Then $A(x, x, ..., x, y) \precsim bA(y, y, ..., y, x)$, for all $x, y \in X$.

4. Main Results

We now state and prove our main results.
Theorem 4.1. *Let (X, A) be a complete complex valued A_b-metric space and $f : X \to X$ be a mapping satisfying*

$$A(fx_1, fx_2, ..., fx_n) \precsim kA(x_1, x_2, ..., x_n) \tag{1}$$

for all $x_1, x_2, ..., x_n \in X$, where $k \in \left[0, \frac{1}{b^2}\right)$. Then f has a unique fixed point in X.
Proof. Let $x_0 \in X$ be an arbitrary point. And let a sequence $\{x_p\}$ be defined by $x_p = f^p x_0$. From (1) we obtain

$$A(x_p, x_p, ..., x_p, x_{p+1}) \precsim kA(x_{p-1}, x_{p-1}, ..., x_{p-1}, x_p) \tag{2}$$

By repeatedly applying (2), we find

$$A(x_p, x_p, ..., x_p, x_{p+1}) \precsim k^p A(x_0, x_0, ..., x_0, x_1) \tag{3}$$

Using (CA_b3) and (3), for all $p, q \in \mathbb{N}$ with $p < q$, we have

$$\begin{aligned} A(x_p, x_p, ..., x_p, x_q) &\precsim (n-1)bA(x_p, x_p, ..., x_p, x_{p+1}) + b^2 A(x_{p+1}, x_{p+1}, ..., x_{p+1}, x_q) \\ &\precsim (n-1)bA(x_p, x_p, ..., x_p, x_{p+1}) + (n-1)b^3 A(x_{p+1}, x_{p+1}, ..., x_{p+1}, x_{p+2}) \\ &\quad + b^4 A(x_{p+2}, x_{p+2}, ..., x_{p+2}, x_q) \\ &\precsim (n-1)bA(x_p, x_p, ..., x_p, x_{p+1}) + (n-1)b^3 A(x_{p+1}, x_{p+1}, ..., x_{p+1}, x_{p+2}) \\ &+ ... + b^{2(q-p-1)} A(x_{q-1}, x_{q-1}, ..., x_{q-1}, x_q) \\ &\prec (n-1)b\{A(x_p, x_p, ..., x_p, x_{p+1}) + b^2 A(x_{p+1}, x_{p+1}, ..., x_{p+1}, x_{p+2}) \\ &+ b^4 A(x_{p+2}, x_{p+2}, ..., x_{p+2}, x_{p+3}) + ... + b^{2(q-p-1)} A(x_{q-1}, x_{q-1}, ..., x_{q-1}, x_q)\} \\ &\precsim (n-1)b(k^p + b^2 k^{p+1} + b^4 k^{p+2} + ... + b^{2(q-p-1)} k^{q-1}) A(x_0, x_0, ..., x_0, x_1) \\ &\precsim (n-1)bk^p (1 + b^2 k + b^4 k^2 + ... + b^{2(q-p-1)} k^{q-p-1}) A(x_0, x_0, ..., x_0, x_1) \\ &\precsim \frac{(n-1)bk^p}{1 - b^2 k} A(x_0, x_0, ..., x_0, x_1). \end{aligned}$$

Thus, we obtain

$$|A(x_p, x_p, ..., x_p, x_q)| \leq \frac{(n-1)bk^p}{1 - b^2 k} |A(x_0, x_0, ..., x_0, x_1)|.$$

Since, $k \in \left[0, \frac{1}{b^2}\right)$ where $b > 1$, taking limit as $p, q \to \infty$, we have

$$\frac{(n-1)bk^p}{1 - b^2 k} |A(x_0, x_0, ..., x_0, x_1)| \to 0.$$

This means that $|A(x_p, x_p, ..., x_p, x_q)| \to 0$ as $p, q \to \infty$.
So, by Lemma 3.6., $\{x_p\}$ is a complex valued A_b-Cauchy sequence. Completeness of (X, A) implies that there is an element $u \in X$ such that $\{x_p\}$ is complex valued A_b-convergent to u. We show that u is a fixed point of f.
Using (CA_b3) and (1), we have

$$\begin{aligned} A(fu, fu, ..., fu, u) &\precsim (n-1)bA(fu, fu, ..., fu, fx_p) + bA(u, u, ..., u, fx_p) \\ &\precsim (n-1)bkA(u, u, ..., u, x_p) + bA(u, u, ..., u, x_{p+1}) \\ \Rightarrow |A(fu, fu, ..., fu, u)| &\leq (n-1)bk|A(u, u, ..., u, x_p)| \\ &\qquad + b|A(u, u, ..., u, x_{p+1})| \to 0 \text{ as } p \to \infty \\ \Rightarrow |A(fu, fu, ..., fu, u)| &= 0 \\ &\Rightarrow fu = u. \end{aligned}$$

Finally, we prove the uniqueness of the fixed point of f. Let w be another fixed point of f. Using (1), we have

$$A(u,u,...,u,w) = A(fu,fu,...,fu,fw) \precsim kA(u,u,...,u,w)$$
$$\Rightarrow |A(u,u,...,u,w)| \leq k|A(u,u,...,u,w)|$$

Since, $k \in \left[0, \frac{1}{b^2}\right)$ and $b \geq 1$, we must have $|A(u,u,...,u,w)| = 0$. Thus $u = w$ and so u is the unique fixed point of f.

Corollary 4.2. *Let (X, A) be a complete complex valued A_b-metric space and $f : X \to X$ be a mapping satisfying*

$$A(fx,fx,...,fx,fy) \precsim kA(x,x,...,x,y)$$

for all $x, y \in X$, where $k \in \left[0, \frac{1}{b^2}\right)$. Then f has a unique fixed point in X.
Proof. One can prove it exactly in the same way as in the proof of Theorem 4.1.

Corollary 4.3. *Let (X, A) be a complete complex valued A_b-metric space and $f : X \to X$ be a mapping satisfying for some positive integer m,*

$$A(f^m x_1, f^m x_2, ..., f^m x_n) \precsim kA(x_1, x_2, ..., x_n)$$

for all $x_1, x_2, ..., x_n \in X$, where $k \in \left[0, \frac{1}{b^2}\right)$. Then f has a unique fixed point in X.
Proof. It follows from Theorem 4.1. that f^m has a unique fixed point u in X. And $f^m(fu) = f(f^m u) = fu$. Therefore, fu is a fixed point of f^m. Since u is the unique fixed point of f^m, we have $fu = u$ i.e. u is a fixed point of f. Also we see that a fixed point of f is also a fixed point of f^m since $fv = v$ implies $f^2 v = fv = v$ and so on giving $f^m v = v$. Hence the fixed point of f is unique.

Corollary 4.4. *Let (X, A) be a complete complex valued A_b-metric space and $f : X \to X$ be a mapping satisfying for some positive integer m,*

$$A(f^m x, f^m x, ..., f^m x, f^m y) \precsim kA(x,x,...,x,y)$$

for all $x, y \in X$, where $k \in \left[0, \frac{1}{b^2}\right)$. Then f has a unique fixed point in X.
We next prove Kannan's fixed point theorem for complex valued A_b-metric spaces.

Theorem 4.5. *Let (X, A) be a complete complex valued A_b-metric space and let $f : X \to X$ be a mapping satisfying the following condition for every $x, y \in X$:*

$$A(fx,fx,...,fx,fy) \precsim \alpha(A(x,x,...,x,fx) + A(y,y,...,y,fy)) \tag{4}$$

where $0 \leq \alpha < \min\left\{\frac{1}{2}, \frac{1}{(n-1)b^2}\right\}$. Then f has a unique fixed point in X.
Proof. Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_p\}$ by $x_{p+1} =$

$fx_p = f^{p+1}x_0$ for all $p \geq 0$. We shall show that $\{x_p\}$ is a complex valued A_b-Cauchy sequence. If $x_p = x_{p+1}$, then x_p is a fixed point of f. Thus, we suppose that $x_p \neq x_{p+1}$ for all $p \geq 0$. Setting $A(x_p, x_p, ..., x_p, x_{p+1}) = A_p$, it follows from (4) that

$$\begin{aligned} A(x_p, x_p, ..., x_p, x_{p+1}) &= A(fx_{p-1}, fx_{p-1}, ..., fx_{p-1}, fx_p) \\ &\precsim \alpha(A(x_{p-1}, x_{p-1}, ..., x_{p-1}, fx_{p-1}) + A(x_p, x_p, ..., x_p, fx_p)) \\ &= \alpha(A(x_{p-1}, x_{p-1}, ..., x_{p-1}, x_p) + A(x_p, x_p, ..., x_p, x_{p+1})) \\ &\precsim \alpha(A_{p-1} + A_p) \\ \Rightarrow A_p &\precsim \frac{\alpha}{1-\alpha}A_{p-1} = \beta A_{p-1} \end{aligned}$$

where $\beta = \frac{\alpha}{1-\alpha} < 1$ as $\alpha < \frac{1}{2}$. If we repeat this process, we get

$$A_p \precsim \beta^p A_0.$$

Using (4), we obtain

$$\begin{aligned} &A(f^p x_0, ..., f^p x_0, f^{p+q}x_0) \\ &\quad \precsim \alpha[A(f^{p-1}x_0, ..., f^{p-1}x_0, f^p x_0) + A(f^{p+q-1}x_0, ..., f^{p+q-1}x_0, f^{p+q}x_0)] \\ &\quad \precsim \alpha[\beta^{p-1}A(x_0, ..., x_0, fx_0) + \beta^{p+q-1}A(x_0, ..., x_0, fx_0)]. \\ &\text{So,} \quad |A(f^p x_0, ..., f^p x_0, f^{p+q}x_0)| = |A(x_p, ..., x_p, x_{p+q})| \to 0 \ \text{ as } \ p \to \infty. \end{aligned}$$

It implies that $\{x_p\}$ is a complex valued A_b-Cauchy sequence in X. By the completeness of X, there exists $u \in X$ such that $x_p \to u$ as $p \to \infty$. We show that u is a fixed point of f.
From (CA_b3), we get

$$\begin{aligned} A(fu, ..., fu, u) &\precsim (n-1)bA(fu, ..., fu, f^{p+1}x_0) + b^2A(f^{p+1}x_0, ..., f^{p+1}x_0, u) \\ &\precsim (n-1)b(\alpha(A(u, ..., u, fu) + A(f^p x_0, ..., f^p x_0, f^{p+1}x_0))) \\ &\quad + b^2A(f^{p+1}x_0, ..., f^{p+1}x_0, u) \\ &= (n-1)b\alpha A(u, ..., u, fu) + (n-1)b\alpha A(f^p x_0, ..., f^p x_0, f^{p+1}x_0) \\ &\quad + b^2A(f^{p+1}x_0, ..., f^{p+1}x_0, u) \\ &\precsim (n-1)b^2\alpha A(fu, ..., fu, u) + (n-1)b\alpha A(f^p x_0, ..., f^p x_0, f^{p+1}x_0) \\ &\quad + b^2A(f^{p+1}x_0, ..., f^{p+1}x_0, u) \\ \Rightarrow A(fu, ..., fu, u) &\precsim \frac{1}{1-(n-1)b^2\alpha}[(n-1)b\alpha A(f^p x_0, ..., f^p x_0, f^{p+1}x_0) \\ &\quad + b^2A(f^{p+1}x_0, ..., f^{p+1}x_0, u)] \end{aligned}$$

$$\Rightarrow |A(fu, ..., fu, u)| \leq \frac{1}{1-(n-1)b^2\alpha}[(n-1)b\alpha|A(f^p x_0, ..., f^p x_0, f^{p+1}x_0)| + b^2|A(f^{p+1}x_0, ..., f^{p+1}x_0, u)|] \to 0 \text{ as } p \to \infty$$
$$\Rightarrow |A(fu, ..., fu, u)| = 0 \Rightarrow fu = u.$$

Therefore, u is a fixed point of f. Now, we show that the fixed point of f is unique. For this, we assume that there exists another point v in X such that $fv = v$. Then, we have

$$\begin{aligned} A(v, ..., v, u) &= A(fv, ..., fv, fu) \\ &\precsim \alpha[A(v, ..., v, fv) + A(u, ..., u, fu)] \\ &\precsim \alpha[A(v, ..., v, v) + A(u, ..., u, u)] = 0 \\ \Rightarrow A(v, ..., v, u) = 0 &\Rightarrow u = v. \end{aligned}$$

Hence the fixed point of f is unique.

Corollary 4.6. *Let (X, A) be a complete complex valued A_b-metric space and let $f : X \to X$ be a mapping satisfying for some positive integer m, the following condition for every $x, y \in X$:*

$$A(f^m x, f^m x, ..., f^m x, f^m y) \precsim \alpha(A(x, x, ..., x, f^m x) + A(y, y, ..., y, f^m y))$$

where $0 \leq \alpha < \min\left\{\frac{1}{2}, \frac{1}{(n-1)b^2}\right\}$. Then f has a unique fixed point in X.

Theorem 4.7. *Let (X, A) be a complete complex valued A_b-metric space and let the mapping $f : X \to X$ satisfy the following condition for every $x, y \in X$:*

$$A(fx, fx, ..., fx, fy) \precsim \alpha(A(x, x, ..., x, fy) + A(y, y, ..., y, fx)) \tag{5}$$

where $\alpha \in \left[0, \frac{1}{b^2\{(n-1)b+1\}}\right)$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. And let a sequence $\{x_p\}$ be defined by $x_{p+1} = fx_p = f^{p+1}x_0$ for all $p \geq 0$. We show that $\{x_p\}$ is complex valued A_b-Cauchy sequence. If $x_p = x_{p+1}$, then x_p is a fixed point of f. Thus, we suppose that $x_p \neq x_{p+1}$ for any $p \geq 0$.
Setting $A(x_p, x_p, ..., x_p, x_{p+1}) = A_p$, we have, from (5),

$$\begin{aligned} A(x_p, x_p, ..., x_p, x_{p+1}) &= A(fx_{p-1}, fx_{p-1}, ..., fx_{p-1}, fx_p) \\ &\precsim \alpha(A(x_{p-1}, x_{p-1}, ..., x_{p-1}, fx_p) + A(x_p, x_p, ..., x_p, fx_{p-1})) \end{aligned}$$

$$
\begin{aligned}
&= \alpha(A(x_{p-1}, x_{p-1}, ..., x_{p-1}, x_{p+1}) \\
&\qquad + A(x_p, x_p, ..., x_p, x_p)) \\
&= \alpha A(x_{p-1}, x_{p-1}, ..., x_{p-1}, x_{p+1}) \\
&\precsim (n-1)\alpha b A(x_{p-1}, x_{p-1}, ..., x_{p-1}, x_p) + \alpha b A(x_{p+1}, x_{p+1}, ..., x_{p+1}, x_p)) \\
&\precsim (n-1)\alpha b A(x_{p-1}, x_{p-1}, ..., x_{p-1}, x_p) + \alpha b^2 A(x_p, x_p, ..., x_p, x_{p+1}))
\end{aligned}
$$

$$\Rightarrow A_p \precsim (n-1)\alpha b A_{p-1} + \alpha b^2 A_p$$

$$\Rightarrow (1-\alpha b^2) A_p \precsim (n-1)\alpha b A_{p-1}$$

$$\Rightarrow A_p \precsim \frac{(n-1)\alpha b}{1-\alpha b^2} A_{p-1}$$

$$\Rightarrow A_p \precsim k A_{p-1} \quad \text{where} \quad k = \frac{(n-1)\alpha b}{1-\alpha b^2} < 1.$$

Repeating this process, we get,

$$A_p \precsim k^p A_0 \quad \text{i.e.} \quad A(x_p, ..., x_p, x_{p+1}) \precsim k^p A(x_0, ..., x_0, x_1).$$

Note

$$
\begin{aligned}
\alpha < \frac{1}{b^2\{(n-1)b+1\}} &\Rightarrow \alpha b^2 < \frac{1}{(n-1)b+1} \\
&\Rightarrow 1 - \alpha b^2 > 1 - \frac{1}{(n-1)b+1} = \frac{(n-1)b}{(n-1)b+1} > 0 \\
&\Rightarrow 1 - \alpha b^2 > 0
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\alpha < \frac{1}{b^3(n-1)+b^2} &\Rightarrow \alpha b^3(n-1) + \alpha b^2 < 1 \\
&\Rightarrow \alpha b^3(n-1) < 1 - \alpha b^2 \\
&\Rightarrow \frac{\alpha b^3(n-1)}{1-\alpha b^2} < 1 \\
&\Rightarrow \frac{(n-1)\alpha b}{1-\alpha b^2} < \frac{1}{b^2} < 1 \Rightarrow k < 1.
\end{aligned}
$$

Now, for $q > p$, we have

$$
\begin{aligned}
&A(f^p x_0,..., f^p x_0, f^q x_0) \\
&\quad \precsim b[(n-1)A(f^p x_0, ..., f^p x_0, f^{p+1} x_0) + A(f^q x_0, ..., f^q x_0, f^{p+1} x_0)] \\
&\quad \precsim (n-1)bA(f^p x_0, ..., f^p x_0, f^{p+1} x_0) + b^2 A(f^{p+1} x_0, ..., f^{p+1} x_0, f^q x_0)
\end{aligned}
$$

$$\precsim (n-1)bA(f^px_0,...,f^px_0,f^{p+1}x_0)+(n-1)b^3A(f^{p+1}x_0,...,f^{p+1}x_0,f^{p+2}x_0)$$
$$+b^4A(f^{p+2}x_0,...,f^{p+2}x_0,f^qx_0)$$
$$\precsim (n-1)bA(f^px_0,...,f^px_0,f^{p+1}x_0)+(n-1)b^3A(f^{p+1}x_0,...,f^{p+1}x_0,f^{p+2}x_0)$$
$$+(n-1)b^5A(f^{p+2}x_0,...,f^{p+2}x_0,f^{p+3}x_0)+...+b^{2(q-p-1)}A(f^{q-1}x_0,...,f^{q-1}x_0,f^qx_0)$$
$$\prec (n-1)b[A(f^px_0,...,f^px_0,f^{p+1}x_0)+b^2A(f^{p+1}x_0,...,f^{p+1}x_0,f^{p+2}x_0)$$
$$+...+b^{2(q-p-1)}A(f^{q-1}x_0,...,f^{q-1}x_0,f^qx_0)]$$
$$\precsim (n-1)b[k^p+b^2k^{p+1}+b^4k^{p+2}+...+b^{2(q-p-1)}k^{q-1}]A(x_0,...,x_0,x_1)$$
$$=(n-1)bk^p[1+b^2k+(b^2k)^2+...+(b^2k)^{(q-p-1)}]A(x_0,...,x_0,x_1)$$
$$\precsim \frac{(n-1)bk^p}{1-b^2k}A(x_0,...,x_0,x_1)$$
$$\Rightarrow |A(f^px_0,...,f^px_0,f^qx_0)|\leq \frac{(n-1)bk^p}{1-b^2k}|A(x_0,...,x_0,x_1)|$$

Taking limit as $p\to\infty$ (and consequently $p,q\to\infty$), we have

$$|A(f^px_0,...,f^px_0,f^qx_0)|\to 0.$$

This implies that the sequence $\{x_p\}$ is complex valued A_b-Cauchy in X. By the completeness of X, there exists $u\in X$ such that $\lim_{p\to\infty} x_p=u$. We show that u is a fixed point of f.

We have

$$A(fu,...,fu,u)\precsim (n-1)bA(fu,...,fu,f^{p+1}x_0)+bA(u,...,u,f^{p+1}x_0)$$
$$\precsim (n-1)b[\alpha(A(u,...,u,f^{p+1}x_0)+A(f^px_0,...,f^px_0,fu))]$$
$$+bA(u,...,u,f^{p+1}x_0)$$
$$=((n-1)b\alpha+b)A(u,...,u,f^{p+1}x_0)+(n-1)b\alpha A(f^px_0,...,f^px_0,fu)$$
$$\precsim ((n-1)b\alpha+b)A(u,...,u,f^{p+1}x_0)+(n-1)b\alpha[(n-1)$$
$$bA(f^px_0,...,f^px_0,u)+bA(fu,...,fu,u)]$$
$$=((n-1)b\alpha+b)A(u,...,u,f^{p+1}x_0)+(n-1)^2b^2\alpha A(f^px_0,...,f^px_0,u)$$
$$+(n-1)b^2\alpha A(fu,...,fu,u)$$
$$\Rightarrow A(fu,...,fu,u)\precsim \frac{1}{1-(n-1)b^2\alpha}[((n-1)b\alpha+b)A(u,...,u,f^{p+1}x_0)$$
$$+(n-1)^2b^2\alpha A(f^px_0,...,f^px_0,u)]$$
$$\Rightarrow |A(fu,...,fu,u)|\leq \frac{1}{1-(n-1)b^2\alpha}[((n-1)b\alpha+b)|A(u,...,u,f^{p+1}x_0)|$$

$$+ (n-1)^2 b^2 \alpha |A(f^p x_0, ..., f^p x_0, u)|] \to 0 \quad \text{as} \quad p \to \infty$$
$$\Rightarrow A(fu, ..., fu, u) = 0 \Rightarrow fu = u.$$

Therefore u is a fixed point of f. Now, we show that the fixed point of f is unique. Let there be another point v in X such that $fv = v$. Then, we have

$$\begin{aligned} A(u, ..., u, v) &= A(fu, ..., fu, fv) \\ &\precsim \alpha[A(u, ..., u, fv) + A(v, ..., v, fu)] \\ &= \alpha[A(u, ..., u, v) + A(v, ..., v, u)] \\ &\precsim \alpha[A(u, ..., u, v) + bA(u, ..., u, v)] \\ &= \alpha(1+b)A(u, ..., u, v) \\ \Rightarrow A(u, ..., u, v) &\precsim \alpha(1+b)A(u, ..., u, v). \end{aligned}$$

But, $\alpha < \dfrac{1}{b^2\{(n-1)b+1\}} < \dfrac{1}{b^2(b+1)} \Rightarrow \alpha(b+1) < \dfrac{1}{b^2} < 1$. Therefore, we must have $A(u, ..., u, v) = 0 \Rightarrow u = v$. Hence the fixed point of f is unique.

Corollary 4.8. *Let (X, A) be a complete complex valued A_b-metric space and let $f : X \to X$ be a mapping satisfying for some positive integer m, the following condition for every $x, y \in X$:*

$$A(f^m x, ..., f^m x, f^m y) \precsim \alpha(A(x, ..., x, f^m y) + A(y, ..., y, f^m x))$$

where $\alpha \in \left[0, \dfrac{1}{b^2\{(n-1)b+1\}}\right)$. Then f has a unique fixed point in X.

We give an example illustrating the validity of our Theorem 4.1.

Example 4.9. Let $X = [-1, 1]$ and $A : X^n \to \mathbb{C}$ be defined as follows

$$A(x_1, x_2, ..., x_{n-1}, x_n) = (1+i) \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2, \quad \forall x_i \in X, \quad i = 1, 2, ..., n.$$

Then, (X, A) is a complete complex valued A_b-metric space with $b = 2$.
If we define $f : X \to X$ by $fx = \frac{x}{5}$, then f satisfies the following condition for all $x_i \in X$, $i = 1, 2, 3, ..., n$.

$$A(fx_1, fx_2, ..., fx_n) = A\left(\frac{x_1}{5}, \frac{x_2}{5}, ..., \frac{x_n}{5}\right) = \frac{1}{25} A(x_1, x_2, ..., x_n)$$

$$\precsim kA(x_1, x_2, ..., x_n),$$

where $k \in \left[\frac{1}{25}, \frac{1}{b^2}\right) \subset \left[0, \frac{1}{b^2}\right)$, $b = 2$. And $x = 0$ is the unique fixed point of f in X as asserted by Theorem 4.1. On the other hand, if $X = (0, 1]$, then (X, A) is not complete and f has no fixed point in X.

5. Conclusion

The complex valued A_b-metric space introduced in this paper further generalizes complex valued metric space and its other generalizations discussed by some authors. Indeed, complex valued metric space, complex valued b-metric space , complex valued S-metric space and complex valued S_b metric space etc. are special cases and the fixed point results proved in this paper generalize the corresponding fixed point results in these spaces.

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