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A NOTE ON ASYMMETRIC BILATERAL BAILEY TRANSFORM

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Abstract: In this paper, making use of some known summation formulas for bilateral q-series and asymmetric bilateral Bailey transform, certain transformations and identities have been established.

Keywords and Phrases: Basic hypergeometric series, bilateral basic hypergeometric series, Bailey transform, symmetric bilateral Bailey transform, asymmetric bilateral Bailey transform, summation formula, transformation formula, identities.

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1. Notations and Definitions

Let q be a complex number such that $0 < |q| < 1$, we define the q-shifted factorial for all integers k by

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i) \text{ and } (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{1.1}$$

For brevity, we employ the condensed notation,

$$(a_1,a_2,a_3,...,a_r;q)_k = (a_1;q)_k(a_2;q)_k...(a_r;q)_k. \tag{1.2}$$

Further, following [3; (1.2.22), p.4] we define the generalized basic hypergeometric series as,

$${}_r\Phi_s\left[\begin{array}{l} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_s \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n z^n}{(q,b_1,b_2,...,b_s;q)_n}\left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r}, \tag{1.3}$$

which is convergent for $|z| < \infty$ if $r \leq s$ and for $|z| < 1$, if $r = s+1$.
Following [3; (5.1.1), p. 125] the generalized basic bilateral hypergeometric series is defined as,

$${}_r\Psi_s\left[\begin{array}{l} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_s \end{array}\right] = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n z^n}{(b_1,b_2,...,b_s;q)_n}\left\{(-1)^n q^{n(n-1)/2}\right\}^{s-r}, \tag{1.4}$$

which is convergent for $|z| < \infty$ if $r \leq s$ and for $\left|\frac{b_1b_2...b_s}{a_1a_2...a_r}\right| < |z| < 1$ if $r = s$.

2. Introduction

Bailey [2] established a remarkable result which has become known as Bailey transform. It states,
If

$$\beta_n = \sum_{r=0}^{n}\alpha_r u_{n-r}v_{n+r} \tag{2.1}$$

and

$$\gamma_n = \sum_{r=0}^{\infty}\delta_{r+n}u_r v_{r+2n} \tag{2.2}$$

then

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\delta_n, \tag{2.3}$$

subject to conditions on four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the infinite series absolutely convergent.

Andrews and Warnaar [1] generalized the Bailey transform as the following two bilateral versions.

(a) Symmetric bilateral Bailey transform

If

$$\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.4}$$

and

$$\gamma_n = \sum_{r=|n|}^{\infty} \delta_r u_{r-n} v_{r+n} \tag{2.5}$$

then

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.6}$$

subject to conditions on four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all infinite series absolutely convergent.

(b) Asymmetric bilateral Bailey transform

Let $m = \max(n, -n-1)$.

If

$$\beta_n = \sum_{r=-n-1}^{n} \alpha_r u_{n-r} v_{n+r+1} \tag{2.7}$$

and

$$\gamma_n = \sum_{r=m}^{\infty} \delta_r u_{r-n} v_{r+n+1} \tag{2.8}$$

then

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.9}$$

subject to conditions on four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the infinite series absolutely convergent.

We shall make use of following summations in our analysis.

$${}_1\Psi_1 \left[\begin{matrix} a; q; z \\ b \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n z^n}{(b;q)_n} = \frac{\left(q, \dfrac{b}{a}, az, \dfrac{q}{az}; q\right)_\infty}{\left(b, \dfrac{q}{a}, z, \dfrac{b}{az}; q\right)_\infty} \tag{2.10}$$

[3; App. II (II.29), p.239]

Taking $a = q^{-n}$, $b = q^{2+n}$ and zq^n for z in (2.10) we have,

$$\sum_{r=-n-1}^{n} \frac{(q^{-n};q)_r(zq^n)^r}{(q^{2+n};q)_r} = \frac{(q,q^2;q)_n\left(1-\frac{q}{z}\right)\left(z,\frac{q^2}{z};q\right)_n}{(q^2;q)_{2n}}. \tag{2.11}$$

$${}_3\Psi_3\left[\begin{array}{c} b,c,d;q;\frac{q^2}{bcd} \\ \frac{q^2}{b},\frac{q^2}{c},\frac{q^2}{d} \end{array}\right] = \frac{\left(q,\frac{q^2}{bc},\frac{q^2}{bd},\frac{q^2}{cd};q\right)_\infty}{\left(\frac{q^2}{b},\frac{q^2}{c},\frac{q^2}{d},\frac{q^2}{bcd};q\right)_\infty}. \tag{2.12}$$

[3; Ex. (5.18) (II), p.137]

Taking $b = q^{-n}$ in (2.12) we have,

$$\sum_{r=-n-1}^{n} \frac{(q^{-n},c,d;q)_r\left(\frac{q^{2+n}}{cd}\right)^r}{\left(q^{2+n},\frac{q^2}{c},\frac{q^2}{d};q\right)_r} = \frac{(1-q)\left(q^2,\frac{q^2}{cd};q\right)_n}{\left(\frac{q^2}{c},\frac{q^2}{d};q\right)_n}. \tag{2.13}$$

We shall make use of (2.11) and (2.13) in order to establish certain transformation formulas.

3. Main Results

In this section we shall establish certain transformation formulas.

(a) Choosing $u_r = \frac{1}{(q;q)_r}$, $v_r = \frac{1}{(q;q)_r}$ and $\alpha_r = (-1)^r q^{r(r-1)/2} z^r$ in (2.7) we get,

$$\beta_n = \frac{1}{(q;q)_n(q;q)_{n+1}} \sum_{r=-n-1}^{n} \frac{(q^{-n};q)_r(zq^n)^r}{(q^{2+n};q)_r}, \tag{3.1}$$

Now making use of (2.11) we have

$$\beta_n = \frac{\left(1-\frac{q}{z}\right)\left(z,\frac{q^2}{z};q\right)_n}{(1-q)(q^2;q)_{2n}}. \tag{3.2}$$

Again, choosing $\delta_r = (\alpha,\beta;q)_r\left(\frac{q^2}{\alpha\beta}\right)^r$ and $m = n$ in equation (2.8) we have

$$\gamma_n = \frac{(\alpha,\beta;q)_n\left(\frac{q^2}{\alpha\beta}\right)^n}{(q;q)_{2n+1}}\ {}_2\Phi_1\left[\begin{array}{c} \alpha q^n,\beta q^n;q;\frac{q^2}{\alpha\beta} \\ q^{2+2n} \end{array}\right]. \tag{3.3}$$

Summing ${}_2\Phi_1$ series in (3.3) by making use of [3; App. II (II.8), p. 236] we obtain,

$$\gamma_n = \frac{\left(\frac{q^2}{\alpha}, \frac{q^2}{\beta}; q\right)_\infty (\alpha, \beta; q)_n \left(\frac{q^2}{\alpha\beta}\right)^n}{\left(q, \frac{q^2}{\alpha\beta}; q\right)_\infty \left(\frac{q^2}{\alpha}, \frac{q^2}{\beta}; q\right)_n}. \tag{3.4}$$

Putting these values in (2.9) we obtain the transformation,

$$\frac{\left(\frac{q^2}{\alpha}, \frac{q^2}{\beta}; q\right)_\infty}{\left(q, \frac{q^2}{\alpha\beta}; q\right)_\infty} {}_2\Psi_3 \left[\begin{matrix} \alpha, \beta; q; \frac{zq^2}{\alpha\beta} \\ \frac{q^2}{\alpha}, \frac{q^2}{\beta}, 0 \end{matrix} \right]$$

$$= \left(1 - \frac{q}{z}\right) {}_4\Phi_3 \left[\begin{matrix} z, \frac{q^2}{z}, \alpha, \beta; q; \frac{q^2}{\alpha\beta} \\ -q, q^{3/2}, -q^{3/2} \end{matrix} \right] \tag{3.5}$$

provided $\left|\frac{q^2}{\alpha\beta}\right| < 1$.

(b) Choosing $u_r = \frac{1}{(q;q)_r}$, $v_r = \frac{1}{(q;q)_r}$ and $\alpha_r = \frac{(-1)^r q^{r(r+1)/2}(c,d;q)_r}{(q^2/c, q^2/d;q)_r} \left(\frac{q}{cd}\right)^r$ in (2.7) we have,

$$\beta_n = \frac{1}{(q;q)_n (q;q)_{n+1}} \sum_{r=-n-1}^{n} \frac{(q^{-n};q)_r (c,d;q)_r \left(\frac{q^{2+n}}{cd}\right)^r}{\left(\frac{q^2}{c}, \frac{q^2}{d}; q\right)_r (q^{2+n};q)_r}. \tag{3.6}$$

Now using (2.13) we have,

$$\beta_n = \frac{\left(\frac{q^2}{cd}; q\right)_n}{\left(q, \frac{q^2}{c}, \frac{q^2}{d}; q\right)_n}. \tag{3.7}$$

Again, taking $m=n$, $\delta_r=(\alpha,\beta;q)_r\left(\frac{q^2}{\alpha\beta}\right)^r$ in (2.8) we find,

$$\gamma_n=\sum_{r=0}^{\infty}\frac{(\alpha,\beta;q)_{r+n}\left(\frac{q^2}{\alpha\beta}\right)^{r+n}}{(q;q)_r(q;q)_{r+2n+1}}$$
$$=\frac{(\alpha,\beta;q)_n\left(\frac{q^2}{\alpha\beta}\right)^n}{(q;q)_{2n+1}}\,{}_2\Phi_1\left[\begin{matrix}\alpha q^n,\beta q^n;q;\frac{q^2}{\alpha\beta}\\ q^{2+2n}\end{matrix}\right]. \tag{3.8}$$

Summing ${}_2\Phi_1$ series in (3.8) by using [3; App. II (II.8), p. 236] we get,

$$\gamma_n=\frac{\left(\frac{q^2}{\alpha},\frac{q^2}{\beta};q\right)_\infty(\alpha,\beta;q)_n\left(\frac{q^2}{\alpha\beta}\right)^n}{\left(q,\frac{q^2}{\alpha\beta};q\right)_\infty\left(\frac{q^2}{\alpha},\frac{q^2}{\beta};q\right)_n}. \tag{3.9}$$

Putting these values in (2.9) we have,

$$\frac{\left(\frac{q^2}{\alpha},\frac{q^2}{\beta};q\right)_\infty}{\left(q,\frac{q^2}{\alpha\beta};q\right)_\infty}\,{}_4\Psi_5\left[\begin{matrix}\alpha,\beta,c,d;q;\frac{q^3}{\alpha\beta cd}\\ \frac{q^2}{\alpha},\frac{q^2}{\beta},\frac{q^2}{c},\frac{q^2}{d},0\end{matrix}\right]$$
$$=\ {}_3\Phi_2\left[\begin{matrix}\alpha,\beta,\frac{q^2}{cd};q;\frac{q^2}{\alpha\beta}\\ \frac{q^2}{c},\frac{q^2}{d}\end{matrix}\right] \tag{3.10}$$

provided $\left|\frac{q^2}{\alpha\beta}\right|<1$.

4. Special Cases

In this section we shall discuss the special cases of (3.5) and (3.10).

(i) For $\alpha,\beta\to\infty$, (3.5) yields

$$\frac{1}{(q^2;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3}{2}n^2}(zq^{1/2})^n=\left(1-\frac{q}{z}\right)\sum_{n=0}^{\infty}\frac{\left(z,\frac{q^2}{z};q\right)_n q^{n^2+n}}{(q^2;q^2)_n(q^3;q^2)_n}. \tag{4.1}$$

Applying Jacobi's triple product identity, viz.,

$$\sum_{k=-\infty}^{\infty}q^{k^2}z^k=(q^2,-zq,-q/z;q^2)_\infty \tag{4.2}$$

[3; App. II (II. 28), p. 239]

on the left hand side of (4.1) we have

$$\sum_{n=0}^{\infty}\frac{(z,q^2/z;q)_n q^{n^2+n}}{(q;q)_{2n}}=\frac{(q^3,zq^2,q/z;q^3)_\infty}{(1-q/z)(q^2;q)_\infty}. \tag{4.3}$$

For $z=-q$, (4.3) gives,

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n^2 q^{n^2+n}}{(q;q)_{2n}}=\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty(q^2;q)_\infty}.$$

Taking $z=-1$ in (4.3) we get,

$$\sum_{n=0}^{\infty}\frac{(-1,-q^2;q)_n q^{n^2+n}}{(q;q)_{2n}}=\frac{(-q,-q^2;q^3)_\infty}{(q,q^2;q^3)_\infty}. \tag{4.4}$$

Taking $z=1$ in (3.5) we get,

$$_2\Psi_3\left[\begin{array}{c}\alpha,\beta;q;\dfrac{q^2}{\alpha\beta}\\ \dfrac{q^2}{\alpha},\dfrac{q^2}{\beta},0\end{array}\right]=\frac{\left(q,\dfrac{q^2}{\alpha\beta};q\right)_\infty}{\left(\dfrac{q^2}{\alpha},\dfrac{q^2}{\beta};q\right)_\infty}. \tag{4.5}$$

For $\alpha,\beta\to\infty$, (4.5) yields

$$\sum_{n=-\infty}^{\infty}(-)^n q^{\frac{3}{2}n^2+\frac{1}{2}n}=(q;q)_\infty, \tag{4.6}$$

which is Euler's identity.
Taking $\alpha,\beta\to\infty$ in (3.10) we get,

$$\sum_{n=0}^{\infty}\frac{\left(\dfrac{q^2}{cd};q\right)_n q^{n^2+n}}{\left(q,\dfrac{q^2}{c},\dfrac{q^2}{d};q\right)_n}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(c,d;q)_n(-1)^n q^{\frac{3}{2}n(n-1)}}{\left(\dfrac{q^2}{c},\dfrac{q^2}{d};q\right)_n}\left(\frac{q^3}{cd}\right)^n. \tag{4.7}$$

Taking $c,d\to\infty$ in (4.7) we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5}{2}n^2+\frac{1}{2}n}. \tag{4.8}$$

Applying (4.2) on the right hand side of (4.8) we get,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{4.9}$$

which is Rogers-Ramanujan second identity.
Taking $c = d = -q$ in (4.7) we get,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3}{2}n^2+\frac{3}{2}n} = (q;q)_\infty, \tag{4.10}$$

which is Euler's identity.

References

[1] Andrews, G.E. and Warnaar, S.O., The Bailey transform and false theta functions, The Ramanujan Journal, Vol. 14, no. 1, (2007), pp. 173-188.

[2] Bailey, W.N., Identities of the Rogers-Ramanujan type, Proc. London Math. SOc. (2) 50, (1949), pp. 1-10.

[3] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University, Press (1991).