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TRANSFORMATION FORMULAE FOR POLY-BASIC HYPERGEOMETRIC SERIES

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Abstract: In this paper, an identity has been established by making use of Bailey's transform. Using certain known summation formulae and the identity established herein, interesting transformation formulae for poly-basic hypergeometric series have been established.

Keywords and Phrases: Identity, Bailey transform, summation formula, transformation formula, basic hypergeometric series, poly-basic hypergeometric series.

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1. Introduction, Notations and Definitions

Here, we shall adopt the following notations and definitions. The q-rising factorial is defined as, for $|q| < 1$,

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \qquad n \geq 1, \tag{1.1}$$

$$(a;q)_0 = 1, \tag{1.2}$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r) \tag{1.3}$$

and

$$(a_1, a_2, a_3, ..., a_r; q)_n = (a_1;q)_n(a_2;q)_n...(a_r;q)_n. \tag{1.4}$$

A basic hypergeometric series (q-series) is defined by

$$ {}_r\Phi_s \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r}, \quad (1.5) $$

[3; (1.2.22) p. 4]

A poly-basic hypergeometric series is defined as,

$$ \Phi \left[\begin{matrix} a_1, a_2, ..., a_r : c_{1,1}, ..., c_{1,r_1}; ...; c_{m,1}, ...c_{m,r_m}; q, q_1, ..., q_m; z \\ b_1, b_2, ..., b_s : d_{1,1}, ..., d_{1s_1}; ...; d_{m,1}, ..., d_{m,s_m} \end{matrix} \right] $$

$$ = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r} \times $$

$$ \times \prod_{j=1}^{m} \frac{(c_{j,1}, ..., c_{j,r_j}; q_j)_n}{(d_{j,1}, ..., d_{j,s_j}; q_j)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{s_j - r_j}. \quad (1.6) $$

[3; (3.9.1), (3.9.2) p. 95]

A truncated basic hypergeometric series defined by,

$$ {}_{r+1}\Phi_r \left[\begin{matrix} a_1, a_2, ..., a_{r+1}; q; z \\ b_1, b_2, ..., b_r \end{matrix} \right]_N = \sum_{n=0}^{N} \frac{(a_1, a_2, ..., a_{r+1}; q)_n z^n}{(q, b_1, b_2, ..., b_r; q)_n}. \quad (1.7) $$

We shall make use of following summation formulae of truncated basic hypergeometric series in our analysis.

$$ {}_2\Phi_1 \left[\begin{matrix} a, b; q; q \\ abq \end{matrix} \right]_n = \frac{(aq, bq; q)_n}{(q, abq; q)_n}. \quad (1.8) $$

[Agarwal 1; (2.1) p. 389]

$$ {}_3\Phi_2 \left[\begin{matrix} a, b, q; q; q \\ e, abq^2/e \end{matrix} \right]_n = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)} \left[1 - \frac{(a, b; q)_{n+1}}{(e/q, abq/e; q)_{n+1}} \right]. \quad (1.9) $$

[Agarwal 2; p. 79]

In the summation formula [Gasper and Rahman 3; App. II (II.21)] if we take $c = aq^{n+1}$, we find the following sum,

$$ {}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{matrix} \right]_n = \frac{(aq, bq; q)_n}{(q, aq/b; q)_n b^n}, \quad |b| > 1. \tag{1.10} $$

Taking $a = bcd$ in [Gasper and Rahman 3; App. II (II. 22)] we find

$$ {}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix} \right]_n = \frac{(aq, bq, cq, dq; q)_n}{(q, aq/b, aq/c, aq/d; q)_n}. \tag{1.11} $$

Gasper's indefinite bibasic sum,

$$ \sum_{k=0}^{n} \frac{(1 - ap^k q^k)(a; p)_k (c; q)_k}{(1 - a)(q; q)_k (ap/c; p)_k} c^{-k} = \frac{(ap; p)_n (cq; q)_n}{(q; q)_n (ap/c; p)_n c^n}. \tag{1.12} $$

[Gasper and Rahman 3; App. II (II. 34)]

We shall also use the following identity,

$$ \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r + \sum_{n=0}^{\infty} \alpha_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r + \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r \tag{1.13} $$

Proof of the identity (1.13)

In Bailey's transform [Slater 4; (2.4.1)-(2.4.3) p. 59] if we take $u_r = v_r = 1$, it takes the following form,

If

$$ \beta_n = \sum_{r=0}^{n} \alpha_r \tag{1.14} $$

and

$$ \gamma_n = \sum_{r=n}^{\infty} \delta_r = \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \tag{1.15} $$

Then under suitable convergence conditions

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.16} $$

(1.16) can be expressed as,

$$ \sum_{n=0}^{\infty} \alpha_n \left\{ \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \right\} = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r, \tag{1.17} $$

which on simplifications gives (1.13).

2. Transformation Formulae

In this section we shall establish certain transformation formulae for poly-basic series.

(i) Choosing $\alpha_n = \dfrac{(a,b;q)_n q^n}{(q,abq;q)_n}$ and $\delta_n = \dfrac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}$ in (1.13) and using (1.8) we get,

$$\frac{(aq,bq;q)_\infty(\alpha p,\beta p;p)_\infty}{(q,abq;q)_\infty(p,\alpha\beta p;p)_\infty} + \Phi\left[\begin{array}{l} a,b:\alpha,\beta;q,p;pq \\ abq:p,\alpha\beta p \end{array}\right]$$

$$= \Phi\left[\begin{array}{l} a,b:\alpha p,\beta p;q,p;q \\ abq:p,\alpha\beta p \end{array}\right] + \Phi\left[\begin{array}{l} aq,bq:\alpha,\beta;q,p;p \\ abq:p,\alpha\beta p \end{array}\right]. \tag{2.1}$$

(ii) Taking $p=q$ in (2.1) we get,

$$\frac{(aq,bq,\alpha q,\beta q;q)_\infty}{(q,q,abq,\alpha\beta q;q)_\infty} + {}_4\Phi_3\left[\begin{array}{l} a,b,\alpha,\beta;q;q^2 \\ abq,q,\alpha\beta q \end{array}\right]$$

$$= {}_4\Phi_3\left[\begin{array}{l} a,b,\alpha q,\beta q;q;q \\ abq,q,\alpha\beta q \end{array}\right] + {}_4\Phi_3\left[\begin{array}{l} aq,bq,\alpha,\beta;q;q \\ abq,q,\alpha\beta q \end{array}\right]. \tag{2.2}$$

(iii) Taking $\alpha_n = \dfrac{(a,q\sqrt{a},-q\sqrt{a},b;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b;q)_n b^n}$ and $\delta_n = \dfrac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}$ in (1.13) and using (1.10) and (1.8) we obtain,

$$\Phi\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b:\alpha,\beta;q,p;p/b \\ \sqrt{a},-\sqrt{a},aq/b:p,\alpha\beta p \end{array}\right]$$

$$= \Phi\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b:\alpha p,\beta p;q,p;1/b \\ \sqrt{a},-\sqrt{a},aq/b:p,\alpha\beta p \end{array}\right] + \Phi\left[\begin{array}{l} aq,bq:\alpha,\beta;q,p;p/b \\ aq/b:p,\alpha\beta p \end{array}\right], \quad \left|\frac{1}{b}\right| < 1. \tag{2.3}$$

For $p=q$, (2.3) yields

$${}_6\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b,\alpha,\beta;q;q/b \\ q,\sqrt{a},-\sqrt{a},aq/b,\alpha\beta q \end{array}\right]$$

$$= {}_6\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b,\alpha q,\beta q;q;1/b \\ q,\sqrt{a},-\sqrt{a},aq/b,\alpha\beta q \end{array}\right] + {}_4\Phi_3\left[\begin{array}{l} aq,bq,\alpha,\beta;q;q/b \\ q,aq/b,\alpha\beta q \end{array}\right], \quad |b| > 1. \tag{2.4}$$

As $b \to \infty$ in (2.4) we get,

$${}_5\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},\alpha,\beta;q;q \\ q,\sqrt{a},-\sqrt{a},\alpha\beta q,0 \end{array}\right]$$

$$= {}_5\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \alpha q, \beta q; q; 1 \\ q, \sqrt{a}, -\sqrt{a}, \alpha\beta q, 0 \end{matrix}\right] + {}_3\Phi_3\left[\begin{matrix} aq, \alpha, \beta; q; q^2 \\ q, \alpha\beta q, 0 \end{matrix}\right]. \tag{2.5}$$

(iv) Taking $\alpha_n = \dfrac{(a, q\sqrt{a}, -q\sqrt{a}, b; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/b; q)_n b^n} \quad |b| > 1$,

and $\delta_n = \dfrac{(\alpha, p\sqrt{\alpha}, -p\sqrt{\alpha}, \beta; p)_n}{(p, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha p/\beta; p)_n \beta^n}, \quad |\beta| > 1$ in (1.13) and making use of (1.10) we find,

$$\Phi\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b : \alpha, p\sqrt{\alpha}, -p\sqrt{\alpha}, \beta; q, p; \dfrac{1}{b\beta} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b} : p, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha p}{\beta} \end{matrix}\right]$$

$$= \Phi\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b : \alpha p, \beta p; q, p; \dfrac{1}{b\beta} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b} : p, \dfrac{\alpha p}{\beta} \end{matrix}\right]$$

$$+\Phi\left[\begin{matrix} aq, bq : \alpha, p\sqrt{\alpha}, -p\sqrt{\alpha}, \beta; q, p; \dfrac{1}{b\beta} \\ \dfrac{aq}{b} : p, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha p}{\beta} \end{matrix}\right]. \tag{2.6}$$

For $p = q$, (2.6) gives

$${}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta; q; \dfrac{1}{b\beta} \\ q, \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b}, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{\beta} \end{matrix}\right]$$

$$= {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, \alpha q, \beta q; q; \dfrac{1}{b\beta} \\ q, \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b}, \dfrac{\alpha q}{\beta} \end{matrix}\right]$$

$$+ {}_6\Phi_5\left[\begin{matrix} aq, bq, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta; q; \dfrac{1}{b\beta} \\ q, \dfrac{aq}{b}, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{\beta} \end{matrix}\right]. \tag{2.7}$$

For $b, \beta \to \infty$, (2.7) yields

$${}_6\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}; q; q \\ q, \sqrt{a}, -\sqrt{a}, \sqrt{\alpha}, -\sqrt{\alpha}, 0, 0 \end{matrix}\right]$$

$$= {}_4\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \alpha q; q; q \\ q, \sqrt{a}, -\sqrt{a}, 0, 0 \end{matrix}\right] + {}_4\Phi_5\left[\begin{matrix} aq, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}; q; q \\ q, \sqrt{\alpha}, -\sqrt{\alpha}, 0, 0 \end{matrix}\right]. \tag{2.8}$$

(v) Choosing $\alpha_n = \frac{(a,b;q)_n q^n}{(e,abq^2/e;q)_n}$, $\delta_n = \frac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}$ in (1.13) and using (1.9) and (1.8) we find,

$$\frac{(\alpha p,\beta p;p)_\infty}{(p,\alpha\beta p;p)_\infty}\left\{1-\frac{(a,b;q)_\infty}{(e/q,abq/e;q)_\infty}\right\}\frac{(q-e)(e-abq)}{(aq-e)(e-bq)}+\Phi\left[\begin{array}{l}\alpha,\beta : a,b;p,q;pq\\ \alpha\beta p : e,abq^2/e\end{array}\right]$$

$$=\Phi\left[\begin{array}{l}\alpha p,\beta p : a,b;p,q;q\\ \alpha\beta p : e,abq^2/e\end{array}\right]+\frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}\left\{1-\frac{(a,b;q)_{n+1}}{(e/q,abq/e;q)_{n+1}}\right\}. \tag{2.9}$$

For $p=q$, (2.9) yields

$${}_4\Phi_3\left[\begin{array}{l}\alpha,\beta,a,b;q;q^2\\ \alpha\beta q,e,abq^2/e\end{array}\right]=\frac{eq(1-a)(1-b)(\alpha q,\beta q,aq,bq;q)_\infty}{(aq-e)(e-bq)(q,\alpha\beta q,e,abq^2/e;q)_\infty}$$

$$+\,{}_4\Phi_3\left[\begin{array}{l}\alpha q,\beta q,a,b;q;q\\ \alpha\beta q,e,abq^2/e\end{array}\right]-\frac{eq(1-a)(1-b)}{(aq-e)(e-bq)}\,{}_4\Phi_3\left[\begin{array}{l}\alpha,\beta,aq,bq;q;q\\ \alpha\beta q,e,abq^2/e\end{array}\right]. \tag{2.10}$$

If we put $e=q$ in (2.10) we get (2.2) again.

(vi) Taking $\alpha_n = \frac{(a,q\sqrt{a},-q\sqrt{a},b,c,d;q)_n q^n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d;q)_n}$,

and $\delta_n = \frac{(\alpha,p\sqrt{\alpha},-p\sqrt{\alpha},\beta,\gamma,\delta;p)_n p^n}{(p,\sqrt{\alpha},-\sqrt{\alpha},\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta;p)_n}$, in (1.13) and making use of (1.11) we have,

$$\frac{(aq,bq,cq,dq;q)_n(\alpha p,\beta p,\gamma p,\delta p;p)_n}{(q,aq/b,aq/c,aq/d;q)_n(p,\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta;p)_n}$$

$$+\Phi\left[\begin{array}{l}a,q\sqrt{a},-q\sqrt{a},b,c,d : \alpha,p\sqrt{\alpha},-p\sqrt{\alpha},\beta,\gamma,\delta;q,p;pq\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d : p,\sqrt{\alpha},-\sqrt{\alpha},\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta\end{array}\right]$$

$$=\Phi\left[\begin{array}{l}a,q\sqrt{a},-q\sqrt{a},b,c,d : \alpha p,\beta p,\gamma p,\delta p;q,p;q\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d : p,\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta\end{array}\right]$$

$$+\Phi\left[\begin{array}{l}aq,bq,cq,dq : \alpha,p\sqrt{\alpha},-p\sqrt{\alpha},\beta,\gamma,\delta;q,p;p\\ aq/b,aq/c,aq/d : p,\sqrt{\alpha},-\sqrt{\alpha},\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta\end{array}\right]. \tag{2.11}$$

For $p=q$, (2.11) yields

$$\frac{(aq,bq,cq,dq,\alpha q,\beta q,\gamma q,\delta q;q)_\infty}{(q,q,aq/b,aq/c,aq/d,\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q)_\infty}$$

$$+ {}_{12}\Phi_{11}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta; q; q^2 \\ q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta \end{matrix}\right]$$

$$= {}_{10}\Phi_{9}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, \alpha q, \beta q, \gamma q, \delta q; q; q \\ q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta \end{matrix}\right]$$

$$+ {}_{10}\Phi_{9}\left[\begin{matrix} aq, bq, cq, dq, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta; q, q \\ q, aq/b, aq/c, aq/d, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta \end{matrix}\right]. \tag{2.12}$$

If we take $\alpha = a, \beta = b, \gamma = c, \delta = d$ in (2.12) we obtain

$$\left\{\frac{(aq, bq, cq, dq; q)_\infty}{(q, aq/b, aq/c, aq/d; q)_\infty}\right\}^2 + \sum_{n=0}^{\infty}\left\{\frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q)_n q^n}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d; q)_n}\right\}^2$$

$$+2\ {}_{10}\Phi_{9}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, aq, bq, cq, dq; q; q \\ q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/b, aq/c, aq/d \end{matrix}\right]. \tag{2.13}$$

Taking $c = a, d = b$ in (2.13) we find,

$$\left\{\frac{(aq, bq; q)_\infty}{(q, aq/b; q)_\infty}\right\}^4 + \sum_{n=0}^{\infty}\left\{\frac{(a, b; q)_n}{(q, aq/b; q)_n}\right\}^4\left\{\frac{(1-aq^{2n})}{(1-a)}\right\}^2 q^{2n}$$

$$= 2\sum_{n=0}^{\infty}\frac{\{(a, b; q)_{2n}\}^2(1-aq^{2n})q^n}{\{(q, aq/b; q)_n\}^4(1-a)}. \tag{2.14}$$

For $b = a$, (2.14) yields,

$$\left\{\frac{(aq; q)_\infty}{(q; q)_\infty}\right\}^8 + \sum_{n=0}^{\infty}\left\{\frac{(a; q)_n}{(q; q)_n}\right\}^8\left\{\frac{1-aq^{2n}}{1-a}\right\}^2 q^{2n}$$

$$= 2\sum_{n=0}^{\infty}\frac{(a; q)_{2n}^4}{(q; q)_n^8}\left(\frac{1-aq^{2n}}{1-a}\right)q^n. \tag{2.15}$$

(vii) Choosing $\alpha_r = \dfrac{(1-ap^rq^r)(a; p)_r(c; q)_r}{(1-a)(q; q)_r(ap/c; p)_r c^r}$, $\quad |c| > 1$, and $\delta_n = \dfrac{(\alpha, \beta; q_1)_n q_1^n}{(q_1, \alpha\beta q_1; q_1)_n}$ in (1.13) and using (1.8) and (1.12) we find,

$$\sum_{n=0}^{\infty}\frac{(1-ap^nq^n)(a; p)_n(c; q)_n(\alpha, \beta; q_1)_n}{(1-a)(q; q)_n(ap/c; p)_n(q_1, \alpha\beta q_1; q_1)_n}\left(\frac{q_1}{c}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1-ap^nq^n(a; p)_n(c; q)_n}{(1-a)(q; q)_n(ap/c; p)_n}\frac{(\alpha q_1, \beta q_1; q_1)_n}{(q_1, \alpha\beta q_1; q_1)_n}\frac{1}{c^n}$$

$$+\sum_{n=0}^{\infty}\frac{(ap;p)_n(cq;q)_n}{(q;q)_n(ap/c;p)_n}\frac{(\alpha,\beta;q_1)_n}{(q_1,\alpha\beta q_1;q_1)_n}\left(\frac{q_1}{c}\right)^n. \quad (2.16)$$

Taking $p = q_1 = q$ in (2.16) we find (2.4).
A number of similar transformations can also be scored.

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