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AN EXTENSION OF SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract: In this paper we study some growth properties of composite functions formed with entire and meromorphic functions and their derivatives to generalise some earlier results of Banerjee and Adhikary.

Keywords and Phrases: Entire Function, Meromorphic Function, Growth, Composition.

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1. Introduction and Definitions

Let f and g be two transcendental entire functions in the open complex plane $\mathbb{C}$. In [6], Clunie showed that $\lim_{r\to\infty} \frac{T_{f\circ g}(r)}{T_f(r)} = \infty$ and $\lim_{r\to\infty} \frac{T_{f\circ g}(r)}{T_g(r)} = \infty$. In 1991, Singh and Baloria [12] investigated some comparative growth properties of $\log T_{f\circ g}(r)$ and $T_f(r)$ and raised the question for comparative growth of $\log T_{f\circ g}(r)$ and $T_g(r)$. After this, some results on comparative growth of $\log T_{f\circ g}(r)$ and $T_g(r)$ are closely investigated in [9] and [5]. In 2018, Banerjee and Adhikary [1] studied on comparative growth of composite function of the form $\psi \circ g$, where ψ is defined in [1] and g is an entire function. Very recently Banerjee and Adhikary [2] made close investigation on comparative growth properties of the functions $\psi \circ \phi$ with g, where ψ and ϕ formed by the functions f and g and their derivatives respectively.

In this paper, first we construct n functions $\psi_1, \psi_2, \cdots, \psi_n$ formed from the functions $f_1, f_2, \cdots, f_n$ and $a_{1i}, a_{2i}, \cdots, a_{ni}$, where the later functions are small

functions of $f_1, f_2, \cdots, f_n$ respectively as follows.
Let

$$\Psi_1(z) = \Sigma_{i=0}^{l_1} a_{1i}(z) f_1^{(i)}(z)$$
$$\Psi_2(z) = \Sigma_{i=0}^{l_2} a_{2i}(z) f_2^{(i)}(z)$$
$$\vdots$$
$$\Psi_n(z) = \Sigma_{i=0}^{l_n} a_{ni}(z) f_n^{(i)}(z),$$

where $f_k^{(i)}(z)$ is the i-th derivative of $f_k(z)$ and $f_k^{(0)}(z) = f_k(z)$ $(k = 1, 2, \cdots, n)$.
In [2], Banerjee and Adhikary proved some results on comparative growth properties of $\log T_{\psi \circ \phi}(r)$ and $T_g(r)$. In this paper it therefore seems reasonable to study some comparative growth properties of $\log T_{\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n}(r)$ and $T_{f_n}(r)$ to generalise the results of Banerjee and Adhikary [2].
Now we introduce the following definitions which we shall frequently use throughout the paper.

Definition 1.1. *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log r}$$

and

$$\lambda_f = \liminf_{r \to \infty} \frac{\log T_f(r)}{\log r}.$$

If f is entire then for all large values of r, since $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ [7] so we can easily obtain

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}$$

and

$$\lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

In [11], Sato defined generalised order of f as

$$\rho_k = \limsup_{r \to \infty} \frac{\log^{[k-1]} T_f(r)}{\log r}.$$

2. Lemmas

In this section we present some known results in the form of lemmas which will be needed in the sequel.

Lemma 2.1. [10] *Let $f(z)$ be an entire function of finite lower order. If there exist entire functions $b_i (i = 1, 2, ..., n; n \leq \infty)$ satisfying $T(r, b_i) = o\{T(r, f)\}$ and $\sum_{i=1}^{n} \delta(b_i, f) = 1$, then*

$$\lim_{r \to \infty} \frac{T_f(r)}{\log M_f(r)} = \frac{1}{\pi}.$$

Lemma 2.2. [4] *If $f(z)$ is meromorphic and $g(z)$ is entire, then for all large values of r*

$$T_{f \circ g}(r) \leq \{(1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 2.3. [13] *Let f and g be two entire functions. Then for all large values of r*

$$T_{f \circ g}(r) \geq \frac{1}{3} \log M_f(\frac{1}{9} M_g(\frac{r}{4})).$$

Lemma 2.4. [8] *If $f(z)$ be an entire function then for $r > 0$*

$$\frac{M_f(r)}{2r} \leq M_{f'}(r) \leq \frac{M_f(2r)}{r}.$$

In particular for all large values of r

$$T_{f'}(r) \leq \log M_{f'}(r) \leq \log M_f(2r) \leq 3T_f(4r).$$

Lemma 2.5. [3] *Let $f_1, f_2, \cdots, f_n$ be entire functions such that $M_{f_i}(r) > \frac{2+\epsilon}{\epsilon} |f_i(0)|$ for $i = 2, 3, \cdots, n$ and for any $\epsilon > 0$. Then for all large values of r*

$$T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq (1 + \epsilon)^{(n-1)} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r))).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let $f_1(z)$ be a non-constant meromorphic function and $f_2(z)$, $f_3(z)$, $\cdots$, $f_n(z)$ be entire functions. Then*

$$\liminf_{r \to \infty} \frac{\log^{[n]} T_{\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n}(r)}{\log T_{f_n}(r)} \leq 3(l_n + 1) \frac{\rho_{f_n}}{\lambda_{f_n}}.$$

Proof. If $\rho_{f_n} = \infty$ then the theorem is obvious. So we suppose that $\rho_{f_n} < \infty$.

We have for all large values of r and arbitrary ϵ (>0) from Lemma 2.2

$$\begin{aligned}
T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\leq \{1+o(1)\}\, T_{\psi_1}(M_{\psi_2\circ\psi_3\circ\cdots\circ\psi_n}(r))\\
&\leq \{1+o(1)\}\, T_{\psi_1}(R) \quad \text{where } R = M_{\psi_2\circ\cdots\circ\psi_n}(r)\\
&\leq \{1+o(1)\}\, T_{a_{10}f_1+a_{11}f_1^{(1)}+a_{12}f_1^{(2)}+\cdots+a_{1l_1}f_1^{(l_1)}}(R)\\
&\leq \{1+o(1)\}\, [\{T_{a_{10}}(R)+T_{f_1}(R)\}+\{T_{a_{11}}(R)+T_{f_1^{(1)}}(R)\}+\cdots\\
&+\{T_{a_{1l_1}}(R)+T_{f_1^{(l_1)}}(R)\}]+O(1)\\
&\leq \{1+o(1)\}\, [o\{T_{f_1}(R)\}+T_{f_1}(R)+o\{T_{f_1}(R)\}+T_{f_1^{(1)}}(R)+\cdots\\
&+o\{T_{f_1}(R)\}+T_{f_1^{(l_1)}}(R)]+O(1)\\
&\leq \{1+o(1)\}\, [T_{f_1}(R)+T_{f_1^{(1)}}(R)+\cdots+T_{f_1^{(l_1)}}(R)+o\{T_{f_1}(R)\}]+O(1)\\
&\leq \{1+o(1)\}\, [l_1+1+o(1)]R^{\rho_{f_1}+\epsilon}+O(1). \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\text{Now,}\quad R = M_{\psi_2\circ\cdots\circ\psi_n}(r) &\leq M_{\psi_2}(M_{\psi_3\circ\cdots\circ\psi_n}(r))\\
&\leq M_{\psi_2}(R_1) \quad \text{where } R_1 = M_{\psi_3\circ\cdots\circ\psi_n}(r)\\
&\leq M_{a_{20}}(R_1)M_{f_2}(R_1)+M_{a_{21}}(R_1)M_{f_2^{(1)}}(R_1)\\
&\quad +\cdots+M_{a_{2l_2}}(R_1)M_{f_2^{(l_2)}}(R_1).
\end{aligned}$$

Then for all large value of r, we get

$$\begin{aligned}
\log R &\leq \log M_{a_{20}}(R_1)+\log M_{f_2}(R_1)+\log M_{a_{21}}(R_1)+\log M_{f_2^{(1)}}(R_1)+\cdots\\
&+ \log M_{a_{2l_2}}(R_1)+\log M_{f_2^{(l_2)}}(R_1)\\
&\leq 3T_{a_{20}}(2R_1)+3T_{f_2}(2R_1)+3T_{a_{21}}(2R_1)+3T_{f_2^{(1)}}(2R_1)\\
&\quad +\cdots+3T_{a_{2l_2}}(2R_1)+3T_{f_2^{(l_2)}}(2R_1)\\
&\leq 3[T_{f_2}(2R_1)+T_{f_2^{(1)}}(2R_1)+\cdots+T_{f_2^{(l_2)}}(2R_1)]+o(1)T_{f_2}(2R_1)\\
&\leq 3[l_2+1+o(1)](2R_1)^{\rho_{f_2}+\epsilon}.
\end{aligned}$$

Taking logarithm on both sides we get

$$\log^{[2]} R \leq (\rho_{f_2}+\epsilon)\log R_1 + O(1). \qquad (3.2)$$

$$\begin{aligned}
\text{Again,}\quad R_1 = M_{\psi_3\circ\psi_4\circ\cdots\circ\psi_n}(r) &\leq M_{\psi_3}(M_{\psi_4\circ\cdots\circ\psi_n}(r))\\
&\leq M_{\psi_3}(R_2) \quad \text{where } R_2 = M_{\psi_4\circ\cdots\circ\psi_n}(r).
\end{aligned}$$

Proceeding similarly as above we can easily obtain

$$\log R_1 \leq 3[l_3 + 1 + o(1)](2R_2)^{\rho_{f_3}+\epsilon}$$

$$\log^{[2]} R_1 \leq (\rho_{f_3} + \epsilon)\log R_2 + O(1). \tag{3.3}$$

Now from (3.2) and (3.3) we get

$$\log^{[3]} R \leq (\rho_{f_3} + \epsilon)\log R_2 + O(1).$$

Taking (n-1) times logarithm in (3.1), we get

$$\begin{aligned}\log^{[n-1]} T_{\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n}(r) &\leq \log^{[n-1]} R + O(1) \\ &\leq (\rho_{f_{n-1}} + \epsilon)\log M_{\psi_n}(r) + O(1).\end{aligned} \tag{3.4}$$

Now for all large values of r

$$\begin{aligned}\log M_{\psi_n}(r) &\leq \log M_{a_{n0}}(r) + \log M_{f_n}(r) + \log M_{a_{n1}}(r) + \log M_{f_n^{(1)}}(r) + \cdots \\ &\quad + \log M_{a_{nl_n}}(r) + \log M_{f_n^{(l_n)}}(r) \\ &\leq 3T_{a_{n0}}(2r) + 3T_{f_n}(2r) + 3T_{a_{n1}}(2r) + 3T_{f_n^{(1)}}(2r) \\ &\quad + \cdots + 3T_{a_{nl_n}}(2r) + 3T_{f_n^{(l_n)}}(2r) \\ &\leq 3\{1 + o(1)\}[T_{f_n}(2r) + T_{f_n^{(1)}}(2r) + \cdots + T_{f_n^{(l_n)}}(2r)].\end{aligned} \tag{3.5}$$

Hence from (3.4) and (3.5) we have

$$\begin{aligned}\log^{[n-1]} T_{\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n}(r) &\leq (\rho_{f_{n-1}} + \epsilon)\log M_{\psi_n}(r) + O(1) \\ &\leq 3(\rho_{f_{n-1}} + \epsilon)\{1 + o(1)\}[T_{f_n}(2r) + T_{f_n^{(1)}}(2r) \\ &\quad + \cdots + T_{f_n^{(l_n)}}(2r)] + O(1)\end{aligned} \tag{3.6}$$

i.e,

$$\log^{[n]} T_{\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n}(r) \quad \leq \quad [\log T_{f_n}(2r) + \log T_{f_n^{(1)}}(2r) + \cdots + \log T_{f_n^{(l_n)}}(2r)] + O(1).$$

So for all large values of r, using Lemma 2.4

$$\begin{aligned}
\frac{\log^{[n]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{\log T_{f_n}(r)} &\leq \frac{[\log T_{f_n}(2r) + \log T_{f_n^{(1)}}(2r) + \cdots + \log T_{f_n^{(l_n)}}(2r)] + O(1)}{\log T_{f_n}(r)} \\
&\leq \{\frac{\log T_{f_n}(2r)}{\log T_{f_n}(r)} + \frac{\log T_{f_n^{(1)}}(2r)}{\log T_{f_n}(r)} + \cdots + \frac{\log T_{f_n^{(l_n)}}(2r)}{\log T_{f_n}(r)}\} + O(1) \\
&\leq [\frac{\log T_{f_n}(2r)}{\log T_{f_n}(r)} + \frac{3\log T_{f_n}(8r)}{\log T_{f_n}(r)} + \cdots + \frac{3\log T_{f_n}(2^{l_n+2}r)}{\log T_{f_n}(r)}] + O(1), \\
&\leq 3(l_n+1)\frac{\log T_{f_n}(2^{l_n+2}r)}{\log T_{f_n}(r)} + O(1) \\
&\leq 3(l_n+1)\frac{(\rho_{f_n}+\epsilon)\log(2^{l_n+2}r)}{(\lambda_{f_n}-\epsilon)\log r} + O(1), \quad \text{using Definition 1.1.}
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so

$$\limsup_{r\to\infty} \frac{\log^{[n]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{\log T_{f_n}(r)} \leq 3(l_n+1)\frac{\rho_{f_n}}{\lambda_{f_n}}.$$

Corollary 3.1. *In the above theorem if we take f_1 as an entire function instead of meromorphic function then we can also get same result.*

Proof. As $f_1, f_2, \cdots, f_n$ are entire functions we use Lemma 2.5 instead of Lemma 2.2, then for all large values of r we get

$$\begin{aligned}
T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\leq (1+\epsilon)^{(n-1)} T_{\psi_1}(M_{\psi_2}(\cdots M_{\psi_n}(r))) \\
&\leq (1+\epsilon)^{(n-1)} T_{\psi_1}(R) \quad \text{where } R = M_{\psi_2}(\cdots M_{\psi_n}(r)) \\
&\leq (1+\epsilon)^{(n-1)} T_{a_{10}f_1 + a_{11}f_1^{(1)} + a_{12}f_1^{(2)} + \cdots + a_{1l_1}f_1^{(l_1)}}(R) \\
&\leq (1+\epsilon)^{(n-1)} [\{T_{a_{10}}(R) + T_{f_1}(R)\} + \{T_{a_{11}}(R) + T_{f_1^{(1)}}(R)\} + \cdots \\
&+ \{T_{a_{1l_1}}(R) + T_{f_1^{(l_1)}}(R)\}] \\
&\leq (1+\epsilon)^{(n-1)} [o\{T_{f_1}(R)\} + T_{f_1}(R) + o\{T_{f_1}(R)\} + T_{f_1^{(1)}}(R) + \cdots \\
&+ o\{T_{f_1}(R)\} + T_{f_1^{(l_1)}}(R)] \\
&\leq (1+\epsilon)^{(n-1)} [T_{f_1}(R) + T_{f_1^{(1)}}(R) + \cdots + T_{f_1^{(l_1)}}(R) + o\{T_{f_1}(R)\}] \\
&\leq (1+\epsilon)^{(n-1)} [l_1 + 1 + o(1)] R^{\rho_{f_1}+\epsilon} + O(1).
\end{aligned}$$

Now we proceed as in Theorem 3.1 and come to the conclusion.

Remark 3.1. *If in Theorem 3.1, $a_{1i}, a_{2i}, \cdots, a_{ni}$ are meromorphic functions of order zero instead of small functions then we have the same result.*

Theorem 3.2. *Let $f_1(z), f_2(z), \cdots, f_n(z)$ be entire functions such that ψ_1, ψ_2, $\cdots$, ψ_n are of finite non-zero lower order with $\psi_n = f_n$, then for all large values of r*

$$\liminf_{r\to\infty} \frac{\log^{[n]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{\log T_{f_n}(r)} \geq \frac{\rho_{f_n}}{\lambda_{f_n}}.$$

Proof. For all large values of r using Lemma 2.3, we have

$$\begin{aligned} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\geq \frac{1}{3}\log M_{\psi_1}(\frac{1}{9}M_{\psi_2\circ\cdots\circ\psi_n}(\frac{r}{4})) \\ &\geq \frac{1}{3}[\frac{1}{9}M_{\psi_2\circ\psi_3\circ\cdots\circ\psi_n}(\frac{r}{4})]^{\lambda_{\psi_1}-\epsilon}. \end{aligned}$$

Then

$$\begin{aligned} \log T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\geq (\lambda_{\psi_1}-\epsilon)\log M_{\psi_2\circ\cdots\circ\psi_n}(\frac{r}{4}) + O(1) \\ &\geq (\lambda_{\psi_1}-\epsilon)(T_{\psi_2\circ\psi_3\circ\cdots\circ\psi_n}(\frac{r}{4}) + O(1). \end{aligned}$$

Again applying Lemma 2.3, we get

$$\begin{aligned} \log T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\geq (\lambda_{\psi_1}-\epsilon)\frac{1}{3}\log M_{\psi_2}(\frac{1}{9}\log M_{\psi_3\circ\cdots\circ\psi_n}(\frac{r}{4^2})) + O(1) \\ &\geq \frac{1}{3}(\lambda_{\psi_1}-\epsilon)[\frac{1}{9}\log M_{\psi_3\circ\cdots\circ\psi_n}(\frac{r}{4^2})]^{(\lambda_{\psi_2}-\epsilon)} + O(1). \end{aligned}$$

Therefore

$$\log^{[2]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) \geq (\lambda_{\psi_2}-\epsilon)\log M_{\psi_3\circ\cdots\circ\psi_n}(\frac{r}{4^2}) + O(1).$$

Proceeding as before, we get

$$\begin{aligned} \log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\geq (\lambda_{\psi_{n-1}}-\epsilon)\log M_{\psi_n}(\frac{r}{4^{n-1}})) + O(1) \\ &\geq (\lambda_{\psi_{n-1}}-\epsilon)T_{\psi_n}(\frac{r}{4^{n-1}})) + O(1). \end{aligned}$$

Therefore

$$\begin{aligned} \log^{[n]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\geq \log T_{f_n}(\frac{r}{4^{n-1}}) + O(1) \\ &\geq (\lambda_{f_n}-\epsilon)\log r + O(1). \end{aligned}$$

So for all large values of r

$$\begin{aligned}\frac{\log^{[n]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{\log T_{f_n}(r)} &\geq (\lambda_{f_n}-\epsilon)\frac{\log r}{\log T_{f_n}(r)}+O(1)\\ &\geq \frac{\lambda_{f_n}-\epsilon}{\rho_{f_n}+\epsilon}+O(1).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$\limsup_{r\to\infty}\frac{\log^{[n]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{\log T_{f_n}(r)} \geq \frac{\lambda_{f_n}}{\rho_{f_n}}.$$

Theorem 3.3. *Let $f_1(z)$ be a non-constant meromorphic function and $f_2(z)$, $f_3(z)$, $\cdots$, $f_n(z)$ be entire functions. Also let there exist entire functions $b_i(i = 1, 2 \ldots, k; k \leq \infty)$ such that $T_{b_i}(r) = o\{T_{f_n}(r)\}$ with $\Sigma_{i=1}^{k}\delta(b_i, f_n) = 1$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{T_{f_n}(2^{l_n+1}r)} \leq 3(l_n+1)\pi\rho_{f_{n-1}}.$$

Proof. For large values of r and arbitrary ϵ (> 0) we get from (3.5), using Lemma 2.4

$$\begin{aligned}\log M_{\psi_n}(r) &\leq 3\{1+o(1)\}[T_{f_n}(2r)+T_{f_n^{(1)}}(2r)+\cdots+T_{f_n^{(l_n)}}(2r)]\\ &\leq 3\{1+o(1)\}[\log M_{f_n}(2r)+\log M_{f_n^{(1)}}(2r)+\cdots+\log M_{f_n^{(l_n)}}(2r)]\\ &\leq 3\{1+o(1)\}[\log M_{f_n}(2r)+\log M_{f_n}(4r)+\cdots+\log M_{f_n}(2^{l_n+1}r)],\\ &\leq 3\{1+o(1)\}(l_n+1)[\log M_{f_n}(2^{l_n+1}r)].\end{aligned}$$

So from (3.4)

$$\begin{aligned}\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\leq (\rho_{f_{n-1}}+\epsilon)\log M_{\psi_n}(r)+O(1)\\ &\leq 3(\rho_{f_{n-1}}+\epsilon)(l_n+1)\{1+o(1)\}[\log M_{f_n}(2^{l_n+1}r)]+O(1)\end{aligned}$$

i.e.,

$$\frac{\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{T_{f_n}(2^{l_n+1}r)} \leq \frac{3(\rho_{f_{n-1}}+\epsilon)(l_n+1)\{1+o(1)\}[\log M_{f_n}(2^{l_n+1}r)]+O(1)}{T_{f_n}(2^{l_n+1}r)}.$$

Since ϵ (> 0) is arbitrary, so we have by using Lemma 2.1.

$$\limsup_{r\to\infty}\frac{\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{T_{f_n}(2^{l_n+1}r)} \leq 3(l_n+1)\pi\rho_{f_{n-1}}.$$

Corollary 3.2. *In the above theorem if we take f_1 as an entire function instead of meromorphic function then we also get the same result.*

Remark 3.2. *If in Theorem 3.3, $a_{1i}, a_{2i}, \cdots, a_{ni}$ are meromorphic functions of order zero instead of small functions then also we have the same result.*

Theorem 3.4. *Let $f_1(z)$ be a non-constant meromorphic function of finite order in finite complex plane and $f_2(z), f_3(z), \cdots, f_n(z)$ be entire functions such that $0 < \lambda_{f_n} \leq \rho_{f_n} < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{T_{f_n^k}(\exp r)} = 0,$$

for $k = 0, 1, 2, \ldots$.
Proof. For all large values of r and arbitrary ϵ (> 0) we have from (3.6)

$$\begin{aligned}\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r) &\leq 3(\rho_{f_{n-1}} + \epsilon)\{1 + o(1)\}[T_{f_n}(2r) + T_{f_n^{(1)}}(2r) \\ &\quad + \cdots + T_{f_n^{(l_n)}}(2r)] + O(1) \\ &\leq 3(\rho_{f_{n-1}} + \epsilon)\{1 + o(1)\}(l_n + 1)(2r)^{\rho_{f_n}+\epsilon} + O(1). \end{aligned} \tag{3.7}$$

Also for all large values of r

$$T_{f_n^{(k)}}(\exp r) > (\exp r)^{\lambda_{f_n}-\epsilon}. \tag{3.8}$$

Therefore from (3.7) and (3.8) we have

$$\frac{\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{T_{f_n^{(k)}}(\exp r)} < \frac{3(\rho_{f_{n-1}} + \epsilon)\{1 + o(1)\}(l_n + 1)(2r)^{\rho_{f_n}+\epsilon} + O(1)}{(\exp r)^{\lambda_{f_n}-\epsilon}}.$$

Hence

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T_{\psi_1\circ\psi_2\circ\cdots\circ\psi_n}(r)}{T_{f_n^{(k)}}(\exp r)} = 0$$

for $k = 0, 1, 2, \ldots$.
Note. The condition $\rho_{f_1} < \infty$ is necessary in Theorem 3.4. Which follows from the following example.

Example 3.1. Let $f_1(z) = \exp^{[3]}(z)$ and $f_2(z) = f_3(z) = f_4(z) = \exp(z)$ and also let $a_1(z) = a_2(z) = a_3(z) = a_4(z) = z$. Then clearly $\rho_{f_1} = \infty$.

We construct the functions ψ_1, ψ_2, ψ_3 and ψ_4 as follows:

$$\begin{aligned}
\psi_1(z) &= a_1(z) f_1(z) = z \exp^{[3]}(z); \\
\psi_2(z) &= a_2(z) f_2^{(1)}(z) = z \exp(z); \\
\psi_3(z) &= a_3(z) f_3^{(2)}(z) = z \exp(z); \\
\psi_4(z) &= a_4(z) f_n^{(3)}(z) = z \exp(z).
\end{aligned}$$

So

$$\begin{aligned}
\psi(z) = \psi_1 \circ \psi_2 \circ \psi_3 \circ \psi_4(z) &= \psi_1 \circ \psi_2 \circ \psi_3(z \exp z) \\
&= \psi_1 \circ \psi_2(z e^z e^{z e^z}) \\
&= \psi_1 \circ \psi_2(z e^{z(1+e^z)}) \\
&= \psi_1(z e^{z(1+e^z)} . e^{z e^{z(1+e^z)}}) \\
&= \psi_1(z e^{z(1+e^z) + z e^{z(1+e^z)}}) \\
&= \psi_1(z e^{z[(1+e^z) + e^{z(1+e^z)}]} \\
&= z e^{z[(1+e^z)+e^{z(1+e^z)}]} . e^{e^{e^{z e^{z[(1+e^z)+e^{z(1+e^z)}]}}}} \\
&= z e^{z[(1+e^z)+e^{z(1+e^z)}] + e^{e^{z e^{z[(1+e^z)+e^{z(1+e^z)}]}}}}.
\end{aligned}$$

Then clearly

$$M_\psi(r) = r e^{r[(1+e^r)+e^{r(1+e^r)}] + e^{e^{r e^{r[(1+e^r)+e^{r(1+e^r)}]}}}}.$$

Again we know that

$$\begin{aligned}
3T_\psi(2r) &\geq \log M_\psi(r) \\
&\geq \log r + r[(1+e^r) + e^{r(1+e^r)}] + e^{e^{r e^{r[(1+e^r)+e^{r(1+e^r)}]}}} \\
&\geq e^{e^{r e^{r[(1+e^r)+e^{r(1+e^r)}]}}} + O(1).
\end{aligned}$$

So,

$$\log T_\psi(2r) \geq e^{r e^{r[(1+e^r)+e^{r(1+e^r)}]}} + O(1).$$

Therefore,

$$\log^{[2]} T_\psi(2r) \geq r e^{r[(1+e^r)+e^{r(1+e^r)}]} + O(1)$$

i.e,

$$\begin{aligned}
\log^{[3]} T_\psi(2r) &\geq \log r + r[(1+e^r) + e^{r(1+e^r)}] + O(1) \\
&\geq e^{r(1+e^r)} + O(1).
\end{aligned}$$

Since $T_{f_4}(r) = \frac{r}{\pi}$, so $T_{f_4}(\exp r) = \frac{\exp r}{\pi}$.
Therefore

$$T_{f_4^{(k)}}(\exp r) = \frac{\exp r}{\pi},$$

for $k = 0, 1, 2 \ldots$.
Hence

$$\lim_{r\to\infty} \frac{\log^{[3]} T_{\psi_1\circ\psi_2\circ\psi_3\circ\psi_4}(r)}{T_{f_4^{(k)}}(\exp r)} = \infty.$$

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References

[1] Banerjee D. and Adhikary M., On growth of composite entire and meromorphic functions, Bulletin of the Calcutta Mathematical Society, 110 (4) (2018), 323–332.

[2] Banerjee D. and Adhikary M., Further results on growth of composite entire and meromorphic functions, Ganita, 70 (1) (2020), 85-94.

[3] Banerjee D. and Adhikary M., On relative order of iterated functions with respect to iterated functions of entire and meromorphic functions, Asian Journal of Mathematics and Applications, Volume 2021 (2021), 1-14.

[4] Bergweiler W., On the Nevanlinna characteristic of a composite function, Complex Variables and Elliptic Equations, 10 (2-3) (1988), 225-236.

[5] Bhoosnurmath S. and Prabhaiah V., On the generalized growth properties of composite entire and meromorphic functions, Journal of Indian Acad Math, 29 (2) (2007), 343-369.

[6] Clunie J., The composition of entire and meromorphic functions, Mathematical essays dedicated to AJ Macintyre, (1970), 75-92.

[7] Hayman W. K., Meromorphic functions, Volume 78, Oxford Clarendon Press, 1964.

[8] Lahiri B. K. and Banerjee D., Generalised relative order of entire functions, Proceedings-National Academy of Sciences India Section A, 4 (2002), 351-372.

[9] Lahiri I., Growth of composite integral functions, Indian J. Pure Appl. Math, 20 (9) (1989), 899-907.

[10] Lin Q. and Dai C. J., On a conjecture of Shah concerning small function, Kexue Tongbao, 31 (4) (1986), 220-224.

[11] Sato D., On the rate of growth of entire functions of fast growth, Bulletin of the American Mathematical Society, 69 (3) (1963), 411-414.

[12] Singh A. P. and Baloria M., On maximum modulus and maximum term of composition of entire functions, Indian J. Pure Appl. Math, 22 (12) (1991), 1019-1026.

[13] Tu J., Chen Z.-X. and Zheng X.-M, Composition of entire functions with finite iterated order, Journal of mathematical analysis and applications, 353 (1) (2009), 295-304.